LOCAL NEWFORMS FOR THE GENERAL LINEAR GROUPS

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1. Introduction

Let us recall the theory of newforms from [1]. For integers $k, N \geq 1$, let $S_k(\Gamma_0(N))$ denote the space of elliptic cusp forms of weight $k \geq 1$ on the congruence subgroup $\Gamma_0(N)$. For integers $N', d \geq 1$ satisfying $N'|N$ and $d|(N/N')$, let us consider the map

$$\iota^N_{N',d} \colon S_k(\Gamma_0(N')) \to S_k(\Gamma_0(N))$$

which sends $f(\tau) \in S_k(\Gamma_0(N'))$ to $f(d\tau)$. In [1], Atkin and Lehner introduced the subspace $C^+(N)$ of $S_k(\Gamma_0(N))$ as the orthogonal complement of sum

$$\sum_{(N',d)} \operatorname{Image} \iota^N_{N',d}$$

where (N', d) runs over the pair of positive integers satisfying $N'|N$, $N' \neq N$ and $d|(N/N')$. A newform on $\Gamma_0(N)$ is a function $f \in C^+(N)$ which is not identically zero and is an eigenform with respect to the Hecke operators T_p for any prime number $p \nmid N$.

Let f be an elliptic cusp form of weight $k \geq 1$ on $\Gamma_0(M)$ for some integer $M \geq 1$ such that f is an eigenform with respect to the Hecke operators T_p for any prime number $p \nmid M$. In [1, Lemma 22, Theorem 4], Atkin and Lehner proved the following: there exists an integer $N|M$ and a newform g on $\Gamma_0(N)$ such that f and g have the same eigenvalue with respect to T_p for any prime number $p \nmid M$, and an integer N and the one-dimensional vector space $\mathbb{C}g$ are uniquely determined by f.

In [4], Casselman gave an interpretation of the theory of Atkin and Lehner from the viewpoint of the representation theory of $\mathrm{GL}_2(F)$, where F is a non-archimedean local field. Let $\mathfrak{o} \subset F$ denote the ring of integers in F. A local analogue of the congruence subgroup $\Gamma_0(N)$ for $\mathrm{GL}_2(F)$ is the subgroup $\mathbb{K}_0(I)$ of $\mathrm{GL}_2(\mathfrak{o})$ whose $(2,1)$ entry belongs to I, where $I \subset \mathfrak{o}$ is a non-zero ideal. Let π be an infinite dimensional irreducible smooth representation π of $\mathrm{GL}_2(F)$ with central character ω_π. For a non-zero ideal $I \subset \mathfrak{o}$, we let $V(I)$ denote the space of vectors v in the representation space of π such that for any $g \in K_0(I)$ we have $gv = \omega_\pi(a)v$ where a is the $(1,1)$ entry of g. One can check that $V(I) \neq 0$ for some non-zero ideal $I \subset \mathfrak{o}$. In [4] Casselman showed that, if we denote by $I_0 \subset \mathfrak{o}$ the largest ideal satisfying $V(I_0) \neq 0$, then $V(I_0)$ is one-dimensional. A non-zero vector in $V(I_0)$ is called a local newform for π. Jacquet, Piatetskii-Shapiro, and Shalika [8] extends the theory of Casselman [4] to the generic representations of $\mathrm{GL}_n(F)$.

Up to now, theories of local newforms have been constructed for a lot of classical groups over a non-archimedean local field: Roberts and Schmidt [19], [**?**], for $\mathrm{PGSp}_4(F)$ and the double cover $\widetilde{\mathrm{SL}}_2(F)$ of $\mathrm{SL}_2(F)$, Lansky and Raghuram[11] for $U(1,1)$, Miyauchi [14], [15], [16], [17] for $U(2,1)$, Okazaki [18] to GSp_4, and (Gross-)Tsai [20] for SO_{2n+1}.

In the theories of local newforms mentioned above, all representations π are assumed to be generic. The aim of this article is to give a survey of the author's joint work [2] with Hiraku Atobe and Satoshi Kondo for $GL_n(F)$, in which the assumption of genericity is removed.

2. Notation

We let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let F be a non-archimedean local field. Let $\mathfrak{o} \subset$ denote its ring of integers, and $\mathfrak{p} \subset \mathfrak{o}$ the maximal ideal of $\mathfrak{o}$. We fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$ of order 0.

For an integer $n \geq 1$, we set $G_n = \mathrm{GL}_n(F)$, and denote by $\mathrm{Irr}(G_n)$ the set of isomorphism classes of irreducible smooth representations π of G_n,

3. Local L-factors and local ε-factors

For $\pi \in \mathrm{Irr}(G_n)$, one can define the local L-factor $L(s,\pi)$ and the local ε-factor $\varepsilon(s,\pi,\psi)$. The local ε-factor is of the form $\varepsilon(s,\pi,\psi) = cq^{-as}$ for some non-zero constant c and some non-negative integer a. The integer a is called the exponent of the conductor of π. We denote a by $\mathrm{cond}(\pi)$.

We note that, when π corresponds to a Weil-Deligne representation $((\sigma,V),N)$ via the local Langlands correspondence, then $\mathrm{cond}(\pi)$ is equal to the sum of the Artin conductor of (σ,V) and the rank of $N : V^{I_F} \to V^{I_F}(1)$, where I_F denotes the inertia group.

Let us recall the local zeta integral of Godement-Jacquet [5]. For a matrix coefficient f of π, and for a Schwartz-Bruhat function Φ of $M_n(F)$, we set

$$Z(\Phi,s,f) = \int_{G_n} \Phi(g)|\det g|^s f(g)dg.$$

Then we have

$$\gamma(s,\pi,\psi)Z(\Phi,s,f) = Z(\widehat{\Phi}, 1-s+\frac{n-1}{2}, f^\vee),$$

where

$$\widehat{\Phi}(x) = \int_{M_n(F)} \Phi(y)\psi(xy)dy$$

and $f^\vee(g) = f(g^{-1})$. Then we have

$$\gamma(s, \pi, \psi) = \frac{\varepsilon(s, \pi, \psi) L(1-s, \pi^\vee)}{L(s, \pi)}.$$

If $\pi \in \mathrm{Irr}(G_n)$ is generic and $a = \mathrm{cond}(\pi)$, then $\pi^{\mathbb{K}_{n,a}}$ is one-dimensional and $\pi^{\mathbb{K}_{n,a'}} = 0$ for any integer a' with $0 \leq a' < a$. Here $\mathbb{K}_{n,a}$ denotes the group of matrices $g \in \mathrm{GL}_n(\mathfrak{o})$ whose n-th row is congruent to $(0, \dots, 0, 1)$ modulo $\mathfrak{p}^a$.

4. Some open compact subgroups

Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. For $n \geq 1$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{Z}_{\geq 0})^n$, we let $\mathbb{K}_{n,\lambda} \subset G_n$ denote the subset of $g \in \mathrm{GL}_n(\mathfrak{o})$ such that for $i = 1, \dots, n$, the i-th row of $g - 1_n$ is congruent to zero modulo $\mathfrak{p}^{\lambda_i}$. Then $\mathbb{K}_{n,\lambda}$ is a compact open subgroup of G_n.

5. The highest derivative

Let us recall the notion of highest derivative of $\pi \in \mathrm{Irr}(G_n)$ from Bernstein and Zelevinsky [3]. For $k = 1, \dots, n$, we let $N_k \subset G_k$ denote the subgroup of upper triangular unipotent matrices and $\psi_k \colon N_k \to \mathbb{C}^\times$ the character that sends $(n_{i,j}) \in N_k$ to $\psi(n_{1,2} + \dots + n_{k-1,k})$. We set

$$U_{n,k} = \begin{pmatrix} 1_{n-k} & * \\ 0 & N_k \end{pmatrix}.$$

Let $\theta \colon U_{n,k} \to \mathbb{C}^\times$ denote the character given by the composite of the surjection $U_{n,k} \to N_k$ and ψ_k.

For $\pi \in \mathrm{Irr}(G_n)$, let us consider the maximal quotient $\pi_{U_{n,k},\theta}$ of the representation space of π on which the group $U_{n,k}$ acts as θ. We regard $\pi_{U_{n,k},\theta}$ as a smooth representation of G_{n-k}. We set $\pi^{(k)} = \pi_{U_{n,k},\theta} \otimes |\det|^{-k/2}$. Let k_1 be the maximal integer satisfying $\pi^{(k_1)} \neq 0$. The smooth representation $\pi^{(k_1)}$ of G_{n-k_1} is called the highest derivative of π.

It is known that the highest derivative $\pi^{(k_1)}$ belongs to $\mathrm{Irr} G_{n-k_1}$.

We note that π is generic if and only if $k_1 = n$, and in this case we have $\pi^{(n)}$ is the trivial representation of G_0.

6. Main results

From now on we assume that F is of characteristic zero. For $\pi \in \mathrm{Irr}(G_n)$ and for $i = 0, 1, \dots$, we introduce non-negative integers $n^{\langle 0 \rangle}, n^{\langle 1 \rangle}, \dots$ and $\pi^{\langle i \rangle} \in \mathrm{Irr}(G_{n^{\langle i \rangle}})$ for $i = 0, 1, \dots$ as follows. We set $n^{\langle 0 \rangle} = n$ and $\pi^{\langle 0 \rangle} = \pi$. For $i \geq 1$, let $\pi^{\langle i \rangle} \in \mathrm{Irr}(G_{n^{\langle i \rangle}})$

denote the highest derivative of $\pi^{\langle i-1\rangle}$ in the sense of [3]. Then we have $n^{\langle 0\rangle} \geq n^{\langle 1\rangle} \geq \cdots \geq 0$ and $n^{\langle i\rangle} > n^{\langle i+1\rangle}$ if $n^{\langle i\rangle} \neq 0$. We note that π is generic if and only if $n^{\langle 1\rangle} = 0$.

For $j = 1, \ldots, n$, we set

$$\lambda_{\pi,j} = \begin{cases} \mathrm{cond}(\pi^{\langle i\rangle}) - \mathrm{cond}(\pi^{\langle i+1\rangle}), & \text{if } j = n^{\langle i\rangle} \text{ for some } i \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\lambda_\pi = (\lambda_{\pi,1}, \ldots, \lambda_{\pi,n}) \in (\mathbb{Z}_{\geq 0})^n.$$

For $\lambda \in (\mathbb{Z}_{\geq 0})^n$, we let $\vec{\lambda} = (\vec{\lambda}_1, \ldots, \vec{\lambda}_n)$ denote the element of $(\mathbb{Z}_{\geq 0})^n$ obtained by permuting the entries of λ so that $\vec{\lambda}_1 \leq \cdots \leq \vec{\lambda}_n$.

Theorem 6.1 (Atobe, Kondo, Y. [2]). *Let $\pi \in \mathrm{Irr}(G_n)$. Then $\pi^{\mathbb{K}_{n,\lambda_\pi}}$ is one-dimensional. For any $\lambda \in (\mathbb{Z}_{\geq 0})^n$ satisfying $\vec{\lambda} < \vec{\lambda}_\pi$ with respect to the lexicographical ordering, we have $\pi^{\mathbb{K}_{n,\lambda}} = 0$.*

Theorem 6.2 (Atobe, Kondo, Y. [2]). *For $\lambda = (c_1, \ldots, c_n) \in (\mathbb{Z}_{\geq 0})^n$, we set $|\lambda| = c_1 + \cdots + c_n$. Then for any $\lambda \in (\mathbb{Z}_{\geq 0})^n$ with $|\lambda| < |\lambda_\pi| = \mathrm{cond}(\pi)$, we have $\pi^{\mathbb{K}_{n,\lambda}} = 0$.*

Theorem 6.3 (Kondo, Y. [10], in preparation). *Let $\varpi \in \mathfrak{p}$ be a uniformizer. Let $m = \sharp\{j \mid c_{\pi,j} = 0\}$. For $i = 0, \ldots, m$, we let T_i denote the characteristic function of*

$$\mathbb{K}_{n,\vec{\lambda}_\pi} \mathrm{diag}(\overbrace{\varpi, \ldots, \varpi}^{i\text{-times}}, \overbrace{1, \ldots, 1}^{(n-i)\text{-times}}) \mathbb{K}_{n,\vec{\lambda}_\pi}.$$

Then T_i is an element of the Hecke algebra $\mathcal{H}(G_n, \mathbb{K}_{n,\vec{\lambda}_\pi})$ in which the characteristic function of $\mathbb{K}_{n,\vec{\lambda}_\pi}$ is a unit element. Let t_i denote the eigenvalue of T_i acting on the one dimensional vector space $\pi^{\mathbb{K}_{n,\vec{\lambda}_\pi}}$. Then we have

$$L(s,\pi)^{-1} = \sum_{i=0}^{m} (-1)^i t_i q^{\binom{i}{2}} q^{-i(\frac{n-1}{2}+s)}.$$

7. Zelevinsky classification

Before moving to explanations of the proof of main theorems, let us recall the classification of $\mathrm{Irr}(G_n)$ due to Zelevinsky [21].

For $\rho \in \mathrm{Irr}(G_n)$ and for $x \in \mathbb{R}$, we write $\rho(a) \in \mathrm{Irr}(G_n)$ for $\rho \otimes |\det(\ )|^x$. A segment is a non-empty finite subset Δ of $\coprod_{n\geq 1} \mathrm{Irr}(G_n)$ of the form $[\rho(x), \rho(y)] = \{\rho(x), \rho(x+1), \ldots, \rho(y)\}$ where ρ is an irreducible cuspidal representation of G_n for some $n \geq 1$ and x, y are integers satisfying $x \leq y$. A finite multiset of segments is called a multisegment. In [21], Zelevinsky construct a bijection Z from the set of multisegments to the set $\coprod_{n\geq 0} \mathrm{Irr}(G_n)$.

For a segment $\Delta = [\rho(x), \rho(y)]$, let us write $\ell(\Delta) = y - x + 1$ and $\Delta^e = [\rho(y), \rho(y)]$. For a segment $\Delta = [\rho(x), \rho(y)]$ with $\ell(\Delta) \geq 2$, we set $\Delta^- = [\rho(x), \rho(y-1)]$. For a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_t$, we set $\mathfrak{m}^e = \Delta_1^e + \cdots + \Delta_t^e$, and $\mathfrak{m}^- = \sum_i \Delta_i^-$, where i runs over the integers satisfying $1 \leq i \leq t$ and $\ell(\Delta_i) \geq 2$.

Let $\pi \in \mathrm{Irr}(G_n)$ for some $n \geq 0$ and let $\mathfrak{m}$ denote the multisegment satisfying $\pi = Z(\mathfrak{m})$. It then follows from [21] that the highest derivative of π is equal to $Z(\mathfrak{m}^-)$. We also note that π is generic if and only if $\mathfrak{m}^-$ is empty, i.e., any segment Δ in $\mathfrak{m}$ is a singleton.

We also set $\pi^e = Z(\mathfrak{m}^e)$. Then π^e is generic, and we have $\pi = \pi^e$ if and only π is generic.

8. Essential vectors

In this paragraph, we introduce the notion of essential vector for $\pi \in \mathrm{Irr}(G_n)$.

It is not hard to deduce Theorem 6.1 for π from the existence of an essential vector for π. However, an existence of an essential vector is unknown for general π at the present time, and in the proof of Theorem 6.1 in [2], some extra arguments are used to deduce the statement to the case where one can prove the existence of an essential vector for π. In spite of this current situation, the author believes that Theorem 6.1 should be proved as a consequence of the existence of an essential vector for any π.

Let $\pi \in \mathrm{Irr}(G_n)$. Let $n_{\langle 0\rangle}, n_{\langle 1\rangle}, \ldots$ and $\pi^{\langle i\rangle} \in \mathrm{Irr}(G_{n^{\langle i\rangle}})$ be as in Section 6. Let $m \geq 1$ be the smallest integer such that $\pi^{(m)}$ is the trivial representation of G_0. Then we have $n = n^{\langle 0\rangle} > n^{\langle 1\rangle} > \cdots > n^{(m)} = 0$. For $i = 1, \ldots, m$, we set $k_i = n^{\langle i\rangle} - n^{(i-1)}$. Let $P \subset G_n$ denote the standard parabolic subgroup of block upper triangular matrices corresponding to the partition $n = k_m + \cdots + k_1$ of n. Let $U \subset P$ denote the unipotent radical and $L \subset P$ the Levi subgroup of block diagonal matrices.

Let $N \subset G_n$ denote the group of upper triangular unipotent matrices. Let us regard N as a subgroup of P. Let $\theta\colon N \to \mathbb{C}^\times$ denote the character which sends $n = (n_{ij})_{1 \leq i,j \leq m} \in N$, where n_{ij} denotes the (i,j)-block of n, to $\prod_{i=1}^m \psi_{k_i}(n_i)$.

Then it follows from Zelevinsky [21] that there exists an injective homomorphism $\pi \in \mathrm{Ind}_N^{G_n} \theta$ and such a homomorphism is unique up to scalar. Let $\mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)$ denote the image of the injective homomorphism.

Let us write $G = G_n$ and $G' = G_{n-m}$. We regard G' as a subgroup of G via the injective homomorphism $G_{n-m} \hookrightarrow G_n$ that sends $h \in G_{n-m}$ to the matrix $g \in G_n$ whose submatrix obtained by removing the $n^{(m-1)}$-th, $\cdots$, and $n^{\langle 0\rangle}$-th rows and columns is equal to h, and the removed entries are equal to the entries in the same places of the identity matrix. We set $P' = G' \cap P$, $L' = G' \cap L$, and $U' = G' \cap U$. Then P'

is a parabolic subgroup of G' and $P' = L'U'$ is the Levi decomposition of P'. Let $K' = \mathrm{GL}_{n-m}(\mathfrak{o})$. Then we have $G' = P'K'$. Note that we have a canonical isomorphism $L' \cong G_{k_m-1} \times \cdots \times G_{k_1-1}$. For $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_m) \in \prod_{i=1}^m (\mathbb{C}^\times)^{k_{m+1-i}-1}$, let $W_{\mathrm{Ze}}^{\psi}((\underline{x})) : G' \to \mathbb{C}$ denote the function which sends $g' = \ell' u' k' \in G'$, where $\ell' \in L'$, $u' \in U'$ and $k' \in K'$, to $\delta_{P'}^{1/2}(\ell') \prod_{i=1}^m \mathrm{Wh}^{\psi^{-1}}(\underline{x}_i)(\ell_i)$, where $\delta_{P'}$ denotes the modulus character of P' and $\mathrm{Wh}^{\psi^{-1}}(\underline{x}_i)$ denote the unramified Whittaker function with Satake parameter $\underline{x}_i$ normalized as $\mathrm{Wh}^{\psi^{-1}}(\underline{x}_i)(1) = 1$. Here $\ell_i \in G_{k_{m+1-i}-1}$ denotes the i-th diagonal block of ℓ.

For $W \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)$, we set

$$I(s, W, \underline{x}) = \int_{N' \backslash G'} W(h) W_{\mathrm{Ze}}^{\psi}(\underline{x})(h) |\det h|^{s-\frac{m}{2}} dh \in \mathbb{C}((q^{-s})),$$

where q denotes the number of elements of $\mathfrak{o}/\mathfrak{p}$.

Definition 8.1. *We say that $W^{\mathrm{ess}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)$ is an essential vector for π if it is $\mathbb{K}'$-invariant and for any $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_m) \in \prod_{i=1}^m (\mathbb{C}^\times)^{k_{m+1-i}-1}$, we have*

$$I(s, W^{\mathrm{ess}}, \underline{x}) = \prod_{i=1}^{m} \prod_{j=1}^{k_{m+1-i}-1} L(s + s_{i,j} - i + 1, (\pi^{\langle m-i \rangle})^e)$$

where $s_{i,j}$ is the complex number satisfying $\underline{x}_i = (q^{-s_{i,1}}, \ldots, q^{-s_{i,k_{m+1-i}-1}})$.

Theorem 8.2 (Atobe, Kondo, Y. [2])**.** *An essential vector for π exists and is unique when π is the Speh representations $\pi = \mathrm{Sp}(\rho, m)$, with ρ tempered.*

In the proof of Theorem 8.2, we heavily use the Shalika model and the argument of Lapid and Mao [12]. When $L(s, \rho) \neq 1$, we also use Theorem 6.1 for ρ in the proof of Theorem 8.2 in [2]. For the proof of Theorem 6.1 in [2], we only need Theorem 8.2 in the case where ρ is a cuspidal ramified representation.

9. A sketch of the proof of Theorem 6.1 in [2]

By using the Mackey decomposition, we are reduced to the case where one of the following conditions is satisfied:

1. $L(s, \pi) = 1$,
2. There exists an unramified character χ of $F^\times$ such that $\pi = Z(\Delta_1 + \cdots + \Delta_t)$, where for $i = 1, \ldots, t$, the segment Δ_i is of the form $\Delta_i = [\chi|\ |^{x_i}, \chi|\ |^{y_i}]$ for some $x_i, y_i \in \mathbb{Z}$ satisfying $x_i \leq y_i$.

We say that π is of type χ if π satisfied the condition (2) above.

When the condition (1) is satisfied, we are reduced, by using the Mackey decomposition, to the case where $t = 1$, i.e., $\pi = Z(\Delta)$ for some segment Δ. In this case the assertion follows from Theorem 8.2.

When the (2), by using the Mackey decomposition and the result of Knight and Zelevinsky [9], we are reduced to the case where $x_1 > \cdots > x_t$ and $y_1 > \cdots > y_t$. According to Lapid and Mínguez [12], we refer to the latter case as the case where π is a ladder representation of type χ.

When χ is a ladder representation of type χ, we have the following Tadić determinant formula obtained by Lapid and Mínguez [12]: in the Grothendieck group of smooth representations of G_n, we have

$$\pi = \sum_{\sigma} \mathrm{sgn}(\sigma) Z([\chi|\,|^{x_{\sigma(1)}}, \chi|\,|^{y_1}]) \times \cdots \times Z([\chi|\,|^{x_{\sigma(1)}}, \chi|\,|^{y_1}])$$

where σ runs over the permutations of $\{1, \ldots, t\}$ such that the inequality $x_{\sigma(i)} \leq y_i$ holds for $i = 1, \ldots, t$, and $\pi_1 \times \cdots \times \pi_t$ denotes the normalized parabolic induction. of $\pi_1 \boxtimes \cdots \boxtimes \pi_t$.

We note that

$$\vec{\lambda}_\pi = \sum_{i=1}^{t-1} (0, \ldots, 0, \overbrace{1, \ldots, 1}^{\max(y_{i+1} - x_i + 2, 0)}).$$

Let $\pi' \in \mathrm{Irr}(G_n)$ and suppose that π' of type χ. We have in mind the case where π' appears in the right hand side of Tadić determinant formula for π. Let us write $\pi' = Z(\mathfrak{m}')$ and $\mathfrak{m}' = \Delta'_1 + \cdots + \Delta'_{t'}$. Then for $i = 1, \ldots, t'$, the segment Δ'_i is of the form $\Delta'_i = [\chi|\,|^{x'_i}, \chi|\,|^{y'_i}]$ with $x'_i, y'_i \in \mathbb{Z}$, $x'_i \leq y'_i$. Let $\lambda' = (\lambda'_1 \ldots, \lambda'_n) \in (\mathbb{Z}_{\geq 0})^{n'}$ and set $M_{\lambda'} = \bigoplus_{i=1}^{n'} \mathfrak{o}/\mathfrak{p}^{\lambda'_i}$. Then the Mackey decomposition shows the following: the dimension of $\mathbb{K}_{n,\lambda'}$-invariant part of π' is equal to the number of increasing filtrations

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{t'} = M_{\lambda'}$$

of $M_{\lambda'}$ by $\mathfrak{o}$-submodules such that for $i = 1, \ldots, t'$, the graded quotient F_i/F_{i-1} is generated, as an $\mathfrak{o}$-module, by at most $\ell(\Delta'_i)$ elements.

By using this formula, we are reduced to showing a formula on an alternating sum of the numbers of some filtrations on M_{λ_π}.

10. Proofs of Theorem 6.2 and Theorem 6.3

The proof of Theorem 6.2 is relatively easy, and follows from the functional equation of local zeta integrals in Godement and Jacquet [5].

We will give a very rough sketch of proof of Theorem 6.3. We are reduced to the case where π is a ladder representation of type χ for some unramified character χ of $F^\times$. Let us write $\pi = Z(\Delta_1 + \cdots + \Delta_t)$ so that the inequalities $x_1 > \cdots > x_t$ and $y_1 > \cdots > y_t$ are satisfied. Then π is a unique irreducible quotient of the normalized parabolic induction $Z(\Delta_m) \times \cdots \times Z(\Delta_1)$. Then a key point is an explicit construction of the non-trivial linear form

$$(Z(\Delta_1) \times \cdots \times Z(\Delta_m))^{\mathbb{K}_{n,\lambda_\pi}} \to \mathbb{C}$$

that factor through the surjective homomorphism

$$(Z(\Delta_1) \times \cdots \times Z(\Delta_m))^{\mathbb{K}_{n,\lambda_\pi}} \pi^{\mathbb{K},\lambda_\pi}.$$

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