NOTE ON THE UNRAMIFIED COMPUTATION OF RANKIN-SELBERG INTEGRALS FOR QUASI-SPLIT CLASSICAL GROUPS OF BESSEL MODEL CASE

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1. Introduction

This note summarizes the content of the author's talk at RIMS conference 2022 "Automorphic form, automorphic L-functions and related topics". The goal of this note is to illustrate the strategy to compute the local Rankin-Selberg integrals with unramified data over p-adic fields, arising from the Bessel models for quasi-split special orthogonal groups and unitary groups, which is a joint work with with Dihua Jiang (University of Minnesota) and David Soudry (Tel Aviv University). The main results in our preprint are a generalization of Soudry's works for split special orthogonal groups in [6, 7]. The author is grateful to Kazuki Morimoto and Tadashi Miyazaki for inviting him to deliver a talk at this conference.

Let F be a number field and ς be a non-square element in $F^\times$. Denote E to be either F or the quadratic field extension $F[\sqrt{\varsigma}]$ of F, to be $\iota_{E/F}$ the nontrivial Galois element in $\mathrm{Gal}(F[\sqrt{\varsigma}]/F)$ if $E = F[\sqrt{\varsigma}]$ and to be the identity if $E = F$. Write $\mathbb{A}$ (resp. $\mathbb{A}_E$) to be the ring of adeles of F (resp. E). Let (V, b) be a non-degenerate space of dimension m over E, which is a quadratic space when $E = F$ and a Hermitian space when $E = F[\sqrt{\varsigma}]$. Define $G = \mathrm{Isom}(V, b)^\circ$ to be the identity connected component of the isometry group of the space V, which is a special orthogonal group or unitary group. More precisely, under a suitable choice of a basis for V, let us take the symmetric matrix J_{ℓ,S_0} defining (V, b)

$$J_{\ell,S_0} = \begin{pmatrix} & & w_\ell \\ & S_0 & \\ w_\ell & & \end{pmatrix}_{m\times m} \quad \text{where } w_\ell = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}_{\ell\times\ell}$$

and $S_0 = \mathrm{diag}\{a_0, a_1, \dots, a_{d_0-1}\}$. Then we may identify

$$\mathrm{Isom}(V, b) = \{g \in \mathrm{End}_F(V) \colon \bar{g}^t J_{\ell,S_0} g = J_{\ell,S_0}\},$$

where $\bar{g}$ is the Galois conjugate of g.

Denote by N_k the unipotent subgroup of G consisting of elements of form

$$n_k(z, x, c) := \begin{pmatrix} z & x & c \\ & I_{m-2k} & x' \\ & & z^* \end{pmatrix} \in G,$$

where $z \in Z_k$, the unipotent subgroup of all upper triangular unipotent matrices of size k and x' is the matrix determined by x. We may denote the k-dimensional totally isotropic and polarized subspaces $X_k^\pm$ such that N_k stabilized X_k^+.

Fix a nontrivial character ψ of $F\backslash\mathbb{A}$, which can be extended to a character of $E\backslash\mathbb{A}_E$ via composing with $\frac{1}{2}\mathrm{tr}_{E/F}$ in case $E \neq F$. Denote by ψ_{Z_k} the standard Whittaker character of $Z_k(\mathbb{A}_E)$,

$$\psi_{Z_k}(z) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{k-1,k}).$$

For any anisotropic vector $y_0 \in (X_k^+ \oplus X_k^-)^\perp$, define the character ψ_{k,y_0} of N_k by

$$\psi_{k,y_0}(n_k(z,x,c)) = \psi_{Z_k}(z)\psi(b({}^t x_k, J_{\ell-k,S_0}^{-1} y_0)) = \psi_{Z_k}(z)\psi(x_{k,\ell-k+1}),$$

where x_k is the bottom row of x and $x_{k,\ell-k+1}$ is the entry of x in the k-th row and $(\ell-k+1)$-th column. We may consider y_0 as a column vector of size $m-2k$. Denote by G_{k,y_0} the stablizer of $\mathrm{Isom}((X_k^+ \oplus X_k^-)^\perp, b)^\circ$ acting on ψ_{k,y_0} via conjugation.

For an automorphic function φ on $G(\mathbb{A})$, define its **Bessel-Fourier** coefficient along N_k with respect to ψ_{k,y_0} by

$$\varphi^{\psi_{k,y_0}}(g) = \int_{N_k(F)\backslash N_k(\mathbb{A})} \varphi(ng)\psi_{k,y_0}^{-1}(n)\,\mathrm{d}n, \tag{1.1}$$

which is $G_{k,y_0}(F)$-left invariant.

Let π and σ be irreducible automorphic representations of $G(\mathbb{A})$ and $G_{k,y_0}(\mathbb{A})$, respectively. *The global ψ_{k,y_0}-Bessel model of π with respect to σ* is defined by

$$\mathcal{P}^{\psi_{k,y_0}}(\varphi_\pi, \xi_\sigma) := \int_{G_{k,y_0}(F)\backslash G_{k,y_0}(\mathbb{A})} \varphi_\pi^{\psi_{k,y_0}}(g)\xi_\sigma(g)\,\mathrm{d}g, \tag{1.2}$$

where φ_π and ξ_σ are in π and σ, respectively.

To guarantee the convergence of the above period integral, we assume that one of φ_π and ξ_σ is rapidly decay such as cuspidal automorphic forms and the other is of moderate growth such as Eisenstein series. Thus, we have the following two families of the the **global zeta integral** as defined in [3]:

$$\mathcal{Z}(s, \varphi_\pi, \phi_{\tau\otimes\sigma}, \psi_{\ell,y_0}) = \begin{cases} \mathcal{P}^{\psi_{\ell,y_0}}(E(\phi_{\tau\otimes\sigma}, s), \varphi_\pi) & \textbf{Case 1} \\ \mathcal{P}^{\psi_{\ell,y_0}}(\varphi_\pi, E(\phi_{\tau\otimes\sigma}, s)) & \textbf{Case 2}, \end{cases} \tag{1.3}$$

where τ is an isobaric sum of cuspidal automorphic representations of $\mathrm{GL}(\mathbb{A}_E)$, π and σ are cuspidal. Then $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,y_0})$ converges absolutely and hence is holomorphic at s where the Eisenstein series $E(h, \phi, s)$ has no poles.

Next, we unfold the Eisenstein series $E(\phi_{\tau\otimes\sigma}, s)$ in the global zeta integrals. Due to the cuspidality of π, by the uniqueness of Bessel models for (π, σ) and the uniqueness of Whittaker models of τ, we obtain the Eulerian product

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,y_0}) = \prod_{v\in|F|} \mathcal{Z}_v(s, \phi_{\tau_v\otimes\sigma_v}, \pi_v, \psi_{\ell,y_0}). \tag{1.4}$$

The main goal of our work is to compute the above local zeta integral $\mathcal{Z}_v(s,\cdot)$ with unramified data. Thus we make the following **assumptions** on the local places:

- E_v is the unramified quadratic extension of F_v or $E_v = F_v$.
- $G(F_v)$ and $G_{k,y_0}(F_v)$ are quasi-split over F_v;
- π_v, σ_v, and τ_v are spherical representations;
- the conductor of ψ_v is the ring of integers $\mathcal{O}_v$.

Then we obtain the following identity

Theorem 1.1. *For the unramified data, one has*

$$\mathcal{Z}_v(s, \phi_{\tau_v\otimes\sigma_v}, \pi_v, \psi_{\ell,y_0}) = \frac{L(\pi_v \times \tau_v, s)}{L(\sigma_v \times \tau_v, s+\frac{1}{2})L(\tau_v, \rho, 2s)},$$

where ρ is the representation of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_r$ defined by

$$\rho = \begin{cases} \text{Asai} & \textit{if } \sigma \textit{ is on an even unitary,} \\ \text{Asai} \otimes \omega_{\mathrm{E/F}} & \textit{if } \sigma \textit{ is on an odd unitary,} \\ \Lambda^2 & \textit{if } \sigma \textit{ is on an even orthogonal,} \\ \mathrm{Sym}^2 & \textit{if } \sigma \textit{ is on an odd orthogonal.} \end{cases}$$

Following the unramified assumptions, we have following cases for $G \times G_{k,y_0}$:

- $G \times G_{k,y_0}$ is split:
  (1) $G \times G_{k,y_0} = (\mathrm{SO}_m, \mathrm{SO}_{m-2r-1})$
  (2) $G \times G_{k,y_0} = (\mathrm{GL}_m, \mathrm{GL}_{m-2r-1})$;
- $G \times G_{k,y_0}$ is quasi-split and non-split:
  (3) $G \times G_{k,y_0} = (\mathrm{SO}_m, \mathrm{SO}_{m-2r-1})$ and m is even;
  (4) $G \times G_{k,y_0} = (\mathrm{SO}_m, \mathrm{SO}_{m-2r-1})$ and m is odd;
  (5) $G \times G_{k,y_0} = (\mathrm{U}_m, \mathrm{U}_{m-2r-1})$.

Remark 1.2. In [6, 7], Soudry computed the unramified local zeta integrals for **Case (1)** and obtain Theorem 1.1. In our joint preprint, we extend Soudry's results to all the remaining cases. In addition, although the tensor product L-factors of general linear groups are also given by Rankin-Selberg integrals, **Case (2)** provides its different integral representation, which still needs to be justified.

Remark 1.3. In Theorem 1.1, π and σ are *arbitrary* cuspidal automorphic representations, which are not necessary of global tempered Arthur parameters.

Remark 1.4. When σ defines on the trivial groups, the global zeta integral $\mathcal{Z}(s, \cdot)$ is the Rankin-Selberg integrals extensively studied in the *theory of automorphic descent*. See [2] for instance. To emphasize the integral in this special case, we will denote it by $\mathcal{Z}_{GRS}(s, \cdot)$.

2. Analytic properties of local zeta integrals over finite places

For convenience, denote by $H = G$ if $\mathcal{Z}_v(s, \cdot)$ is arisen from **Case 1** in (1.3) and $H = G_{k,y_0}$ if $\mathcal{Z}_v(s, \cdot)$ is arisen from **Case 2** in (1.3). We free the notation G to denote the group on which π defines. In other words, the Eisenstein series $E(s, \cdot)$ in (1.3) always defines on $H(\mathbb{A}_E)$ and π is always a cuspidal automorphic representation of $G(\mathbb{A}_E)$. Write H' to be the group on which σ defines.

We sometime use $\mathbb{U}_m$ to denote the special orthogonal group or unitary group, which preserves an m-dimensional Hermitian (or quadratic) space. To present the local integrals clearly, according to the matrices size of involved groups, we separate into three cases: **Case 1a** and **Case 1b**, arisen from **Case 1** in (1.3), and **Case 2** in (1.3). Let us tabulate the classical groups of Hermitian type on which groups τ, σ and π define in each case.

	H	H' for σ	GL_a for τ	G for π	ℓ
Case 1a	$\mathbb{U}_{m+2r+1}$	$\mathbb{U}_{m-2k-1}$	GL_{r+k+1}	$\mathbb{U}_m$	r
Case 1b	$\mathbb{U}_{m+2r}$	$\mathbb{U}_m$	GL_r	$\mathbb{U}_{m-2k-1}$	$r+k$
Case 2	$\mathbb{U}_{2r+m-2k-1}$	$\mathbb{U}_{m-2k-1}$	GL_r	$\mathbb{U}_m$	$k-r$

By default, $0 \le k \le \frac{m-1}{2}$ for all cases and $r \le k$ for **Case 2**. Remark that there are a pair of Bessel modes in each case as below, corresponding to the characters ψ_{ℓ,y_0} and $\psi_{k,w_0'}$ for certain anisotropic vectors w_0 and w_0', respectively.

	$(E(s,\cdot),\pi)$	(π,σ)
Case 1a	$\mathbb{U}_{m+2r+1} \times \mathbb{U}_m$	$\mathbb{U}_m \times \mathbb{U}_{m-2k-1}$
Case 1b	$\mathbb{U}_{m+2r} \times \mathbb{U}_{m-2k-1}$	$\mathbb{U}_{m-2k-1} \times \mathbb{U}_m$
Case 2	$\mathbb{U}_{2r+m-2k-1} \times \mathbb{U}_m$	$\mathbb{U}_m \times \mathbb{U}_{m-2k-1}$

More precisely, the two corresponding Bessel linear functionals lie in the following isomorphic Hom-spaces, due to Mœglin and Waldspurger in [5] and Gan and Ichino [1] when τ is generic,

$$\mathcal{Z}_v(s,\cdot) \in \mathrm{Hom}_{\mathbb{U}}((\mathrm{Ind}_P^H \tau_v|\cdot|^s \otimes \sigma_v) \otimes \pi_v, \psi_{\ell,y_0}) \cong \mathrm{Hom}_{\mathbb{U}}(\pi_v \otimes \sigma_v', \psi_{k,w_0'}) \ni c_{\psi_{k,w_0'}} \quad (2.1)$$

where σ_v' is a twist of σ_v by the outer automorphism of the even orthogonal group. Note that we omit the unipotent subgroups in the above Hom-spaces. The relation between this two linear functionals will be given in Part (2) of Proposition 2.1.

From now on, we drop the subscript v for simplicity. Let F be a non-archimedean field of characteristic 0. Denote ϖ to be a uniformizer, $val(\cdot)$ to be the normalized valuation of F, and $|\cdot| = q^{-val(\cdot)}$ to be the absolute value of F, where $q = |\mathcal{O}/\varpi\mathcal{O}|$.

Following the similar arguments in Sections 3 and 4 in [6], we may establish the analytic properties of local zeta integrals over finite places for all cases.

Proposition 2.1. *Assume that τ, σ and π are irreducible smooth representations and τ is generic.*

Then we have

1. *The local integrals $\mathcal{Z}_v(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0})$ converge absolutely in a right half plane, which depends only on the representations π, σ and τ.*
2. *For each $v_\pi \in \pi$, there exists $f_{\mathcal{W}(\tau),\sigma}$ such that*

$$\mathcal{Z}_v(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0}) = c_{\psi_{k,w_0'}}(v_\pi, v_\sigma) W_\tau(e),$$

where $c_{\psi_{k,w_0'}}$ is a local Bessel model associated to (π,σ) and $W_\tau(e)$ is the evaluation of a Whittaker function of τ at identity.

3. *$\mathcal{Z}_v(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0})$ continue to meromorphic functions in the whole complex plan and are rational functions of q^{-s}.*

Here $\mathcal{W}(\tau)$ is the Whittaker module of τ and $f_{\mathcal{W}(\tau),\sigma}$ is a holomorphic section in the induced representation $\mathrm{Ind}_{P(F)}^{H(F)} \mathcal{W}(\tau)|\det|^s \otimes \sigma$.

Remark that the above analytic properties guarantee our local integrals well-defined and not identically vanishing.

3. Local zeta integrals with unramified data

From now on, we assume that all data are unramified in sense of Section 1. Then $\mathbb{U}_m$ is a quasi-split group SO_m if $E = F$, a quasi-split unitary group U_m if E is the unramified quadratic extension of F, or a general linear group GL_m if $E = F \times F$. As in Remark 1.2, we only consider the quasi-split and non-split special orthogonal group $\mathbb{U}_m = \mathrm{SO}_m$ in our work when $E = F$ and m is even. The split even orthogonal group case has been studied in **Case (1)**. See Remark 1.2.

Denote π, σ, τ to be the unramified constituents of

$$\pi \prec \mathrm{Ind}_{B_G}^G \chi_\pi, \qquad \sigma \prec \mathrm{Ind}_{B_{H'}}^{H'} \chi_\sigma, \qquad \pi \prec \mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_a} \chi_\tau,$$

where B_X is a Borel subgroup of the quasi-split reductive group X and χ_θ is the unramified characters of the torus, corresponding to the Satake parameter of unramified representation θ. Note that if θ is generic, then $\theta = \mathrm{Ind}_{B_X}^X \chi_\theta$ is irreducible.

Furthermore, we can rewrite

$$\mathrm{Ind}_P^H \tau|\det|^s \otimes \sigma = \mathrm{Ind}_{B_H}^H \chi_\tau|\cdot|^s \otimes \chi_\sigma = \mathrm{Ind}_{\tilde{P}}^H \tilde{\tau},$$

where $\tilde{\tau} = \mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{\mathrm{rk}H}} \chi_\tau|\cdot|^s \otimes \chi_\sigma$, and $\mathrm{rk}H$ is the rank of H.

Over a general position, one has π and $\mathrm{Ind}_P^H \tau|\det|^s \otimes \sigma$ are generic and the local zeta integrals are absolutely convergent for all vectors. Then we have two Bessel linear functionals arisen corresponding to Bessel models in the tables of Section 2 and the theory of automorphic descent. That is,

$$\mathcal{Z}(s,\cdot) \in \mathrm{Hom}_{\mathbb{U}}((\mathrm{Ind}_P^H \tau|\cdot|^s \otimes \sigma) \otimes \pi, \psi_{\ell,y_0}) = \mathrm{Hom}_{\mathbb{U}}((\mathrm{Ind}_P^H \widetilde{\tau}) \otimes \pi, \psi_{\mathrm{rk}H,y_0}) \ni \mathcal{Z}_{GRS}(s,\cdot),$$

where $\mathcal{Z}_{GRS}$ is discussed in Remark 1.4. Furthermore, we have

$$\mathcal{Z}_{GRS}(s,\cdot) \in \mathrm{Hom}_{\mathbb{U}}((\mathrm{Ind}_P^H \widetilde{\tau}) \otimes \pi, \psi_{\mathrm{rk}H,y_0}) \cong \mathrm{Hom}_1(\mathcal{W}_{\psi_n}(\pi), \mathbb{C}),$$

and $\mathcal{Z}_{GRS}(s,\cdot)$ has been explicitly calculated in the theory of automorphic descent, due to a series of works by Ginzburg, Soudry, Kaplan, etc.

Finally, due to the Multiplicity One Theorem of Bessel models, one expects that the two linear functionals $\mathcal{Z}(s,\cdot)$ and $\mathcal{Z}_{GRS}(s,\cdot)$ are proportional to each other over a general position, i.e.,

$$\mathcal{Z}(s, f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}) = c_{s,\pi,\tau,\sigma} \mathcal{Z}_{GRS}(s, f^\circ_{\mathcal{W}(\tilde{\tau})}, v^\circ_\pi), \tag{3.1}$$

where $f^\circ_{\mathcal{W}(\tau),\sigma}$ and $f^\circ_{\mathcal{W}(\tilde{\tau})}$ are a spherical sections in $\mathrm{Ind}_P^H \tau|\cdot|^s \otimes \sigma$ and $\mathrm{Ind}_P^H \widetilde{\tau}$, respectively, and v°_π is a spherical vector in π.

After we normalize $f^\circ_{\mathcal{W}(\tau),\sigma}$, $f^\circ_{\mathcal{W}(\tilde{\tau})}$ and v°_π, then the factor $c_{s,\pi,\tau,\sigma}$ is unique. Following Proposition 2.1, $c_{s,\pi,\tau,\sigma}$ is defined for $\mathrm{Re}(s)$ sufficiently large and continues to meromorphic function in the whole complex plan, which is a rational function of q^{-s}. Our proof of Theorem 1.1 is reduced to compute $c_{s,\pi,\tau,\sigma}$ over a general position.

4. Reduction to the Whittaker models

In this section, we will explicate the local zeta integral $\mathcal{Z}(s,\cdot)$ and establish the identity between $\mathcal{Z}(s,\cdot)$ and $\mathcal{Z}_{GRS}(s,\cdot)$.

First, let us introduce the local integral informally for each case.

$$\begin{aligned} &\mathcal{Z}(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0}) \\ &= \begin{cases} \int_{H'N_k\backslash G} \int_{\hat{N}_{r,k}} c_{\psi_{k,w_0'}}(\pi(g)v_\pi, f_{\mathcal{W}(\tau),\sigma',s}(\varepsilon_0 u j(g)))\psi^{-1}_{r,y_0}(u)\,\mathrm{d}u\,\mathrm{d}g & \textbf{Case1a} \\ \int_{\hat{U}_r} c_{\psi_{k,w_0'}}(f_{\mathcal{W}(\tau),\pi_r,s}(\varepsilon_0 u), \xi_\sigma)\psi^{-1}_{r+k,y_0}(u)\,\mathrm{d}u & \textbf{Case1b} \\ \int_{H'\hat{N}_r\backslash G} \int_{\mathcal{L}_{r,k-r}} c_{\psi_{k,w_0'}}(\pi(\lambda \hat{w}_{r,k-r} h)v_\pi, f_{\mathcal{W}(\tau),\sigma,s}(h))\,\mathrm{d}h & \textbf{Case2}, \end{cases} \end{aligned} \tag{4.1}$$

where $c_{\psi_{k,w_0'}}$ belongs to the Hom-space in (2.1). One of the key normalization to fix $\mathcal{Z}(s,\cdot)$ is to normalize

$$c_{\psi_{k,w_0'}}(v^\circ_\pi, v^\circ_\sigma) = 1. \tag{4.2}$$

After normalizing the spherical vectors, one may obtain the unique local zeta integral $\mathcal{Z}(s,\cdot)$.

To reduce the length of this note, we will leave the notation without explanation, where all unexplained notations are certain unipotent subgroups occurring in the unfolding process. The main goal of (4.1) is to demonstrate how $c_{\psi_{k,w_0'}}$ occurs the local zeta integral.

The key ingredient for the proof of Theorem is to introduce a specific Bessel linear functional

$$c'_{\psi_{k,w'_0}} \in \mathrm{Hom}_{\mathbb{U}}(\mathcal{W}(\pi)\otimes\mathcal{W}(\sigma'), \psi_{k,w'_0})$$

factoring through the Whittaker models $\mathcal{W}(\pi)$ and $\mathcal{W}(\sigma')$ of π and σ, respectively.

Similar to Proposition 2.1, one can prove that

Lemma 4.1. *Assume that π and σ are generic.*

1. *Then $c'_{\psi_{k,w'_0}}$ is not identically zero, converges absolutely on a right half plane, and continues to meromorphic functions of χ_π and χ_σ;*
2. *If π and σ are unramified, then $c'_{\psi_{k,w'_0}}$ is a rational function in χ_π and χ_σ;*
3. *If π and σ are unramified, then $c'_{\psi_{k,w'_0}}$ is a Whittaker-Shintani function for the relevant pair (G, H') in sense of [4].*

Due to the uniqueness of Whittaker-Shintani functions, $c'_{\psi_{k,w'_0}}$ is proportional to $\mathrm{WS}_{\psi_{k,w'_0},\pi,\sigma'}$ defined in [4], that is,

$$\mathrm{WS}_{\psi_{k,w'_0},\pi,\sigma'}(g) = a(\chi_\pi,\chi_{\sigma'})\times\begin{cases} c'_{\psi_{k,w'_0}}(\pi(g)v^\circ_\pi, v^\circ_\sigma) & \textbf{Case 1a} \text{ and } \textbf{Case 2} \\ c'_{\psi_{k,w'_0}}(v^\circ_\pi, \pi(g)v^\circ_\sigma) & \textbf{Case 1b}, \end{cases}$$

where $a(\chi_\pi,\chi_{\sigma'})$ is a rational function in χ_π and χ_σ, and v°_π and v°_σ are spherical vectors of π and σ.

In general, the functional $c'_{\psi_{k,w'_0}}$ we introduced has poles of variables χ_π and χ_σ. However, $\mathrm{WS}_{\psi_{k,w'_0},\pi,\sigma'}$ is a polynomial of χ_π and χ_σ. Then we may use the rational function $a(\chi_\pi,\chi_{\sigma'})$ to normalize $c'_{\psi_{k,w'_0}}$ and give an explicit construction of the abstract Bessel linear functional $c_{\psi_{k,w'_0}}$

$$c_{\psi_{k,w'_0}} = a(\chi_\pi,\chi_{\sigma'})c'_{\psi_{k,w'_0}}. \tag{4.3}$$

Finally, we obtain a holomorphic linear functional $c_{\psi_{k,w'_0}}$.

Proposition 4.2. *If π and σ are unramified, then the linear functional $c_{\psi_{k,w'_0}}$ defined in* (4.3) *is a polynomial of χ_π and χ_σ.*

5. The identity

In the last section, we will plug $c_{\psi_{k,w'_0}}$ in (4.3) into $\mathcal{Z}(s,\cdot)$ and establish the identity between $\mathcal{Z}(s,\cdot)$ and $\mathcal{Z}_{GRS}(s,\cdot)$ to obtain the main identity in Theorem 1.1.

First, replacing $c_{\psi_{k,w'_0}}$ by $c'_{\psi_{k,w'_0}}$ in (4.1) when χ_τ and χ_σ over a general position, we obtain a new local zeta integral $\mathcal{Z}'(s,\cdot)$ lying in $\mathrm{Hom}_{\mathbb{U}}((\mathrm{Ind}_P^H\tau|\cdot|^s\otimes\sigma)\otimes\pi, \psi_{\ell,y_0})$, equal to $\mathcal{Z}(s,\cdot)$ up to a rational function of χ_π, χ_τ and χ_σ. Then by combing integrations and changing variables, one can obtain

$$\mathcal{Z}'(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0}) = \mathcal{Z}_{GRS}(s, f_{\mathcal{W}(\tilde{\tau})}, v_\pi).$$

For instance, in **Case 1a**, over a general position, we have

$$\begin{aligned}
&\mathcal{Z}'(s, f_{\mathcal{W}(\tau),\sigma}, v_\pi, \psi_{\ell,y_0}) \\
=&\int_{H'N_k\backslash G}\int_{\hat{N}_{r,k}} c'_{\psi_{k,w'_0}}(\pi(g)v_\pi, f_{\mathcal{W}(\tau),\sigma',s}(\varepsilon_0 uj(g)))\psi^{-1}_{r,y_0}(u)\,\mathrm{d}u\,\mathrm{d}g \\
=&\int_{H'N_k\backslash G}\int_{\hat{N}_{r,k}}\int_{H'_\gamma\backslash H'}\int_{\mathcal{N}'} W_{v_\pi}(\gamma' n' hg) f_{\mathcal{W}(\tau)\otimes\mathcal{W}(\sigma')}(\gamma h\varepsilon_0 ug)\psi^{-1}_{r,y_0}(u)\,\mathrm{d}n'\,\mathrm{d}h\,\mathrm{d}u\,\mathrm{d}g \\
=&\mathcal{Z}_{GRS}(s, v_\pi, f_{\mathcal{W}(\tilde{\tau})}).
\end{aligned}$$

The key idea is to combine all the integrations since the integrals are absolutely convergent and transform the integral $\mathcal{Z}'(s,\cdot)$ to the integral $\mathcal{Z}_{GRS}(s,\cdot)$, which is known due to the theory of automorphic descent.

Therefore, if we take the Bessel linear functional $c_{\psi_{k,w'_0}}$ as in (4.3), for all cases we have

$$\begin{aligned}
&\mathcal{Z}_{ex}(s, f^\circ_{\mathcal{W}(\tau),v^\circ_\pi,\sigma}, \psi_{\ell,y_0}) \\
=&a(\chi_\pi, \chi_{\sigma'})\mathcal{Z}'_v(s, f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}) \\
=&\frac{L(s+\frac{1}{2}, \tau\times\pi)}{L(s+1,\tau\times\sigma)L(2s+1,\tau,\rho)}.
\end{aligned}$$

Remark that $\mathcal{Z}_{ex}(s,\cdot)$ is the zeta integral dependent of the choice of $c_{\psi_{k,w'_0}}$, which is defined *explicitly*. And we have not verify that the choice $c_{\psi_{k,w'_0}}$ in (4.3) satisfying (4.2).

Finally, $\mathcal{Z}_{ex}(s,\cdot)$ has the same analytic properties as Lemma 4.1. That is, if π, σ and τ are unramified and lying in a general position,

(1) $\mathcal{Z}_{ex}(s,\cdot)$ is not identically zero, converges absolutely on a right half plane , and continues to meromorphic functions;
(2) $\mathcal{Z}_{ex}(s,\cdot)$ is a rational function in s, χ_τ, χ_π and χ_σ;
(3) $\mathcal{Z}_{ex}(s,\cdot)$ is a Whittaker-Shintani function for the relevant pair (H,G).

Similarly,

$$\mathcal{Z}_{ex}(s, \rho(h)\cdot f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}) = c(s,\chi_\pi,\chi_\tau\otimes\chi_{\sigma'})\mathrm{WS}_{\psi_{\ell,y_0},\pi,\tau\times\sigma}(h).$$

By evaluating at the identity on the both sides separately, we obtain

$$\mathcal{Z}_{ex}(s, f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}) = \mathcal{Z}(s, f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}).$$

That is, $c_{\psi_{k,w'_0}}$ in (4.3) also satisfies (4.2), exactly which we normalize. Hence we conclude

$$\mathcal{Z}(s, f^\circ_{\mathcal{W}(\tau),\sigma}, v^\circ_\pi, \psi_{\ell,y_0}) = \frac{L(s+\frac{1}{2},\tau\times\pi)}{L(s+1,\tau\times\sigma)L(2s+1,\tau,\rho)}.$$

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