On definable topology – locally o-minimal case

海上保安大学校 * 藤田雅人
Masato Fujita, Japan Coast Guard Academy
和歌山大学 † 川上智博
Tomohiro Kawakami, Wakayama University

概要

This paper proposes an open problem on affineness of a topology definable in a (locally) o-minimal structure. It also introduces a partial solution in the o-minimal case by Peterzil and Rosel and announces our partial result in the case in which the structure is uniformly locally o-minimal of the second kind. They give solutions when definable topological spaces are of dimension one and bounded.

1 Introduction

O-minimal structures and their relatives are one of the central themes in the studies of model theory. Roughly speaking, when we consider a model-theoretic structure, we fix a set M and a family of subsets of the Cartesian products of M closed under several basic operations such as Boolean algebra of sets and projection image. A subset in the family is called a *definable set*. We concentrate on the case in which M has a dense linear order without endpoints $<$ and the set $\{(x,y) \in M^2 \mid x < y\}$ is definable. Under this condition, a structure is *o-minimal* if each definable subset of M is a finite union of points and open intervals. We call that the structure is *locally o-minimal* if each definable subset of M is locally a finite union of points and open intervals. The above explanations on o-minimality and local o-minimality are not

* 〒737-8512 広島県呉市若葉町 5-1
† 〒640-8510 和歌山県和歌山市栄谷 930

precise. See the definitions of o-minimality and local o-minimality in Section 2.

A lot of works on o-minimality have been done since 1980's. We do not review them here. Local o-minimality is relatively a new comer, and it was proposed in the last of 2000's [21]. Basic properties of definably complete locally o-minimal structures have been investigated in [6, 7, 8, 9, 11, 14] since 2010's.

The main topic of this paper is definable topology. We first recall its definition.

Definition 1.1. Consider an expansion of a dense linear order without endpoints and a definable set X. A topology τ on X is *definable* when τ has a basis of the form $\{B_y \subseteq X\}_{y \in Y}$, where $\bigcup_{y \in Y} \{y\} \times B_y$ is definable. We call the family $\{B_y\}_{y \in Y}$ a *definable basis* of τ. The pair (X, τ) of a definable set and a definable topology on it is called a *definable topological space*.

Since X is a subset of a Cartesian product M^n, X has the topology induced from the product topology of M^n. It is definable and called the *affine topology*. The notation τ^{af} denotes the affine topology.

To the best of the authors' knowledge, a few studies have been done on definable topology. However, the following problem is a main theme of these studies.

Open problem 1.2. Find a necessary and sufficient condition for a definable topology to be affine.

A definable topological space is *affine* if it is definably homeomorphic to a definable set with the affine topology.

Metric spaces with metrics definable in o-minimal structures have been investigated in [19, 22, 23]. (Vallette considered another type of problem different from the open problem.) Topology definable in an o-minimal structure was already studied in [17] when the definable set X is of dimension one. Gurrero et al. studied directed sets definable in o-minimal structures and, as an application, they found necessary and sufficient conditions for definable topologies to be definably compact [12]. No studies have been done for the case in which X is of dimension greater than one. We could not find studies on topologies definable in a locally o-minimal structure, neither. The authors get a partial result on the open problem when the structure is locally o-minimal and X is of dimension one [10]. This paper announces our result.

2 On o-minimality and local o-minimality

The definition of a structure given here is slightly different from the original definition in model theory. A reader who has interest in model theory should consult textbooks such as [1, 13, 15, 18, 20].

The notation $\mathbb{N}$ denotes the set of positive integers. In this paper, a *structure* is a pair $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$ of a set M and the collection $\mathfrak{S}$ of families $\mathfrak{S}_n$ of subsets of M^n satisfying the following conditions:

(i) The empty set and M^n are members of $\mathfrak{S}_n$ for all $n \in \mathbb{N}$. The set $\{(x, y) \in M^2 \mid x = y\}$ is also a member of $\mathfrak{S}_2$.

(ii) The families $\mathfrak{S}_n$ are closed under the boolean algebra for all $n \in \mathbb{N}$.

(iii) The Cartesian product $S_1 \times S_2$ belongs to $\mathfrak{S}_{m+n}$ if S_1 and S_2 are members of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, respectively.

(iv) Let $\pi : M^n \to M^m$ be a coordinate projection and let X be a member of $\mathfrak{S}_n$. Then, the projection image $\pi(X)$ belongs to $\mathfrak{S}_m$.

(v) Let σ be a permutation of $\{1, \dots, n\}$. We define the map $\overline{\sigma} : M^n \to M^n$ by $\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We have $\overline{\sigma}(X) \in \mathfrak{S}_n$ if $X \in \mathfrak{S}_n$.

When a structure $\mathcal{M}$ is given, the set M is called the *universe* or the *underlying set* of the structure $\mathcal{M}$. Members in $\mathfrak{S}_n$ are called *definable sets*. Let X and Y be definable sets. A map $f : X \to Y$ is called *definable* if its graph is a definable set.

We sometimes need to consider the family of structures such that some sets other than those given in (i) are definable. When M is a densely linearly ordered set with the order $<$, a structure $\mathcal{M} = (M, \mathfrak{S})$ is called an *expansion* of the dense linear order if the set $\{(x, y) \mid x < y\}$ is definable. When $(M, \cdot)$ is a group, a structure $\mathcal{M}$ with the universe M is called an *expansion* of the group if the set $\{(x, y, z) \in M^3 \mid x \cdot y = z\}$ is definable. We define an expansion of an ordered group, an expansion of an ordered field and so on in the same manner.

An *o-minimal* structure $\mathcal{M} = (M, \mathfrak{S})$ is an expansion of a dense linear order without endpoints such that

(vi) any definable subset of M is a finite union of points and open intervals.

Readers who are interested in o-minimal structures should consult van den Dries's book [3] and Coste's book [2]. The paper [4] is also recommended.

Many structures relaxing the condition (vi) are proposed and investigated. We focus on locally o-minimal structures [21]. A locally o-minimal structure is defined by localizing the condition (vi). A *locally o-minimal structure* is an expansion of a dense linear order without endpoints satisfying the following condition:

(vi)' Let X be a definable subset of M. For any $x \in M$, there exists an open interval I containing the point x such that $X \cap I$ is a finite union of points and open intervals.

We consider an expansion $\mathcal{M} = (M, \mathfrak{S})$ of a dense linear order without endpoints. It is *definably complete* if every definable subset of M has both a supremum and an infimum in $M \cup \{\pm\infty\}$ [16]. A definably complete expansion of an ordered group is divisible and abelian [16, Proposition 2.2]. An o-minimal structure is inevitably definably complete, but a locally o-minimal structure is not necessarily definably complete. Recent studies on local o-minimality often assume definable completeness such as [6, 8, 9, 11].

Here, we give a definition of dimension.

Definition 2.1 (Dimension of a definable set)**.** Let $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$ be an expansion of dense linear order. We assume that M^0 is a singleton with the trivial topology.

- A definable set $X \subset M^n$ is of $\dim(X) \geq m$ if there exists a coordinate projection $\pi : M^n \to M^m$ such that $\pi(X)$ has a non-empty interior.
- The empty set is defined to be of dimension $-\infty$.

Definably complete locally o-minimal structures enjoy tame dimension theory. See [9, 11] for details.

When the structure in consideration is o-minimal, each definable set is partitioned into good-shaped definable sets called *cells*. This fact is very useful in studying o-minimal structures including the study of topology definable in o-minimal structures. However, localized version of cell decomposition is not necessarily available in definably complete locally o-minimal structure. A weaker version of decomposition called decomposition into quasi-special submanifolds is only available as demonstrated in [9]. The first author demonstrated that a definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is uniformly locally o-minimal of the second kind defined below [7].

Definition 2.2. A locally o-minimal structure with the universe M is a *uniformly locally o-minimal structure of the second kind* if, for any positive integer n, any definable set $X \subset M^{n+1}$, $a \in M$ and $b \in M^n$, there exist an open interval I containing the point a and an open box B containing b such that the definable sets $X_y \cap I$ are finite unions of points and open intervals for all $y \in B$, where X_y denotes the fiber $\{x \in M \mid (x, y) \in X\}$.

3 O-minimal case

The purpose of this paper is to introduce Open problem 1.2 and partial results on the problem. As we pointed out in Section 1, we do not have a solution of the problem except one-dimensional case even when the structure is o-minimal. The following is due to Peterzil and Rosel:

Theorem 3.1 ([17, Main theorem]). *Let $\mathcal{M} = (M, \mathfrak{S})$ be an o-minimal expansion of an ordered group. Assume that arbitrary two closed bounded intervals are definably homeomorphic. Let $X \subseteq M^n$ be a definable bounded set with $\dim X = 1$, and let τ be a definable Hausdorff topology on X. Then the following are equivalent:*

(1) (X, τ) is definably homeomorphic to a definable subset of M^k for some k, with its affine topology.

(2) There is a finite set $G \subseteq X$ such that every τ-open subset of $X \setminus G$ is open with respect to the affine topology on $X \setminus G$.

(3) Every definable subset of X has finitely many definably connected components, with respect to τ.

(4) τ is regular and X has finitely many definably connected components with respect to τ.

The assumption that arbitrary two closed bounded intervals are definably homeomorphic is not found in the original paper [17], but this assumption is used in the proof implicitly. The assumption that X is bounded could be omitted when there exists a definable bijection between a bounded interval and an unbounded interval. The structure not satisfying the above condition is investigated in [5]. It is called a *semi-bounded* o-minimal structure. Theorem 3.1 is not true if we drop the assumption that X is bounded as in the example in [17, Section 4.3]. In the non-bounded case, the authors have demonstrated the following proposition:

Proposition 3.2. *Let $\mathcal{M} = (M, \mathfrak{S})$ be a semi-bounded o-minimal expansion of an ordered group. Assume that arbitrary two closed bounded intervals are definably homeomorphic. Let $X \subseteq M^n$ be a definable set with $\dim X = 1$, and let τ be a definable Hausdorff topology on X. Then the following are equivalent:*

(1) (X, τ) is definably homeomorphic to a definable subset of M^k for some k, with its affine topology.

(2) There is a finite set $G \subseteq X$ such that the restriction of τ to $X \setminus G$ coincides with the affine topology on $X \setminus G$.

Let X be a set definable in an o-minimal structure and of dimension one. If we take a large closed box C, the set $X \setminus C$ is of very simple form. We demonstrated Proposition 3.2 using Theorem 3.1 and this fact.

4 Locally o-minimal case

Our main contribution is the solution of Open problem 1.2 when the structure is definably complete uniformly locally o-minimal of the second kind and the definable set is of dimension one. As we assumed that arbitrary two closed bounded intervals are definably homeomorphic in Theorem 3.1 and Proposition 3.2, we need a weak assumption on the existence of definable homeomorphism. This assumption is too technical, so we prefer a sufficient condition explained below.

Definition 4.1. An expansion $\mathcal{M} = (M, \mathfrak{S})$ of a densely linearly ordered abelian group has *definable bounded multiplication compatible to* $+$ if there exist an element $1 \in M$ and a map $\cdot : M \times M \to M$ such that

(1) the tuple $(M, <, 0, 1, +, \cdot)$ is an ordered field;
(2) for any bounded open interval, the restriction $\cdot|_{I \times I}$ of the product $\cdot$ to $I \times I$ is definable in $\mathcal{M}$.

We simply say that $\mathcal{M}$ has *definable bounded multiplication* when the addition in consideration is clear from the context. The condition (2) does not imply that the product $\cdot : M \times M \to M$ itself is definable. The ordered field $(M, <, 0, 1, +, \cdot)$ is an ordered real closed field by [11, Proposition 3.3].

We are now ready to introduce our main result.

Theorem 4.2. *Consider a definably complete uniformly locally o-minimal expansion $\mathcal{M} = (M, \mathfrak{S})$ of the second kind of an ordered group having definable bounded multiplication. Let X be a definable bounded subset of M^n of dimension one. Let τ be a definable topology on X which is Hausdorff and regular.*

The following are equivalent:

(1) The definable topological space (X, τ) is definably homeomorphic to a definable subset of M^k with its affine topology for some k.
(2) There is a definable τ-closed and τ-discrete subset G of X at most of dimension zero satisfying the following conditions:
  (i) The restriction of τ to $X \setminus G$ coincides with the affine topology on $X \setminus G$;
  (ii) There exists a positive integer K such that, for any $x \in G$ and a definable τ-open neighborhood U of x, we can find a definable τ-open neighborhood V of x contained in U such that $V \setminus \{x\}$ has at most K τ^{af}-definably connected components.

See [10] for the complete proof. We only give a sketch of the strategy of the proof here. The implication (1) $\Rightarrow$ (2) is easier to be proven than the converse implication. The strategy of the proof of (2) $\Rightarrow$ (1) is as follows: The set of 'bad points' is defined as the union of G and the frontier of X. It is discrete and closed under the affine topology. We demonstrate that there are only finitely many curves such that X is the union of the subfamily of a shifted curves at any point of bad points using the local definable cell decomposition theorem [10, Lemma 2.3]. We explicitly construct a definable homeomorphism announced in (1) with the aid of this fact.

参考文献

[1] C. C. Chang and H. J. Keisler, Model theory 3rd eds., Studies in logic and the foundation of mathematics, vol. 73, North-Holland Press, Amsterdam, 1990.

[2] M. Coste, An introduction to o-minimal geometry, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Mathematica, Instituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[3] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, Vol. 248. Cambridge University Press, Cambridge, 1998.

[4] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.

[5] M. J. Edmundo, *Structure theorems for o-minimal expansions of groups*, Ann. Pure Appl. Logic, **102** (2000), 159-181.

[6] A. Fornasiero, Locally o-minimal structures and structures with locally o-minimal open core, Ann. Pure Appl. Logic, 164 (2013), 211-229.

[7] M. Fujita, *Uniformly locally o-minimal structures and locally o-minimal structures admitting local definable cell decomposition*, Ann. Pure Appl. Logic, **171** (2020), 102756.

[8] M. Fujita, *Dimension inequality for a definably complete uniformly locally o-minimal structure of the second kind*, J. Symbolic Logic, **85** (2020), 1654-1663.

[9] M. Fujita, *Locally o-minimal structures with tame topological properties*, J. Symbolic Logic, to appear.

[10] M. Fujita and T. Kawakami, *Affiness of topological space definable in a definably complete uniformly locally o-minimal structure of the second kind*, preprint (2021).

[11] M. Fujita, T. Kawakami and W. Komine, *Tameness of definably complete locally o-minimal structures and definable bounded multiplication*, preprint (2021).

[12] P. A. Guerrero, M. E. M. Thomas and E. Walsberg, *Directed sets and topological spaces definable in o-minimal structures*, J. London Math. Soc. (2), **104** (2021), 989-1010.

[13] W. Hodge, A shorter model theory, Cambridge university press, Cambridge,

1997.

[14] T. Kawakami, K. Takeuchi, H. Tanaka and A. Tsuboi, *Locally o-minimal structures*, J. Math. Soc. Japan, **64** (2012), 783-797.

[15] D. Marker, Model theory: An introduction, Graduate texts in mathematics, vol. 217, Springer-Verlag, New York, 2002.

[16] C. Miller, *Expansions of dense linear orders with the intermediate value property*, J. Symbolic Logic, **66** (2001), 1783-1790.

[17] Y. Peterzil and A. Rosel, *Definable one-dimensional topologies in o-minimal structures*, Archiv Math. Logic, **59** (2020), 103-125.

[18] B. Poizat, A course in model theory: An introduction to contemporary mathematical logic, Universitext, Springer-Verlag, New York, 2000.

[19] M. Shiota, *Semialgebraic metric spaces and resolution of singularities of definable sets*, arXiv:1701.03858 (2017).

[20] K. Tent and M. Ziegler, A course in model theory, Lecture notes in logic, Cambridge university press, Cambridge, 2012.

[21] C. Toffalori and K. Vozoris, *Notes on local o-minimality*, Math. Log. Q., **55** (2009), 617-632.

[22] G. Vallette, *On metric types that are definable in an o-minimal structure*, J. Symbolic Logic, **73** (2008), 439-447.

[23] E. Walsberg, *On the topology of metric spaces definable in o-minimal expansions of fields*, arXiv:1510.07291 (2015).