# Note on the space of algebraic loops on a toric variety

山口耕平　(Kohhei Yamaguchi)
電気通信大学 情報理工学研究科　(University of Electro-Communications)

**Abstract**

The homotopy type of the space of rational curves on a toric variety has been well studied by several authors since the work of Segal [27] appeared (cf. [9], [10], [12], [15], [18], [25]). In this note we shall consider the real analogue of these spaces. In particular, we report about the homotopy type of spaces of algebraic loops on a toric variety. This result is based on the joint works with A. Kozlowski given in [19].

## 1 Introduction

First we shall recall several basic definitions and facts about toric topology.

**Fans and toric varieties.** *A convex rational polyhedral cone* $\sigma$ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\sigma = \mathrm{Cone}(S) = \mathrm{Cone}(\boldsymbol{m}_1, \cdots, \boldsymbol{m}_s) = \Big\{ \sum_{k=1}^{s} \lambda_k \boldsymbol{m}_k : \lambda_k \geq 0 \text{ for any } k \Big\} \tag{1.1}$$

for a finite set $S = \{\boldsymbol{m}_k\}_{k=1}^s \subset \mathbb{Z}^n$.[1] A convex rational polyhedral cone $\sigma$ is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_n\}$, and its dimension $\dim \sigma$ is the dimension of the smallest subspace in $\mathbb{R}^n$ which contains $\sigma$. *A face* $\tau$ of $\sigma$ is a subset $\tau \subset \sigma$ of the form

$$\tau = \sigma \cap \{\boldsymbol{x} \in \mathbb{R}^n : L(\boldsymbol{x}) = 0\} \tag{1.2}$$

for some linear form $L$ on $\mathbb{R}^n$, such that $L(\boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in \sigma$. If $\{k : L(\boldsymbol{m}_k) = 0, 1 \leq k \leq s\} = \{i_1, \cdots, i_t\}$, we easily see that $\tau = \mathrm{Cone}(\boldsymbol{m}_{i_1}, \cdots, \boldsymbol{m}_{i_t})$. Thus, a face $\tau$ of $\sigma$ is also a strongly convex rational polyhedral cone if $\sigma$ is so.

A finite collection $\Sigma$ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called *a fan* in $\mathbb{R}^n$ if every face $\tau$ of $\sigma \in \Sigma$ belongs to $\Sigma$ and the intersection of any two elements of $\Sigma$ is a face of each.

[1] When $S$ is the emptyset $\emptyset$, we set $\mathrm{Cone}(\emptyset) = \{\mathbf{0}_n\}$ and we may also regard it as one of strongly convex rational polyhedral cones in $\mathbb{R}^n$, where we denote by $\mathbf{0}_n$ the zero vector in $\mathbb{R}^n$ defined by $\mathbf{0}_n = (0, \cdots, 0) \in \mathbb{R}^n$.

An $n$ dimensional irreducible normal variety $X$ (over $\mathbb{C}$) is called *a toric variety* if it has a Zariski open subset $\mathbb{T}^n_{\mathbb{C}} = (\mathbb{C}^*)^n$ and the action of $\mathbb{T}^n_{\mathbb{C}}$ on itself extends to an action of $\mathbb{T}^n_{\mathbb{C}}$ on $X$. The most significant property of a toric variety is the fact that it is characterized up to isomorphism entirely by its associated fan $\Sigma$. We denote by $X_\Sigma$ the toric variety associated to a fan $\Sigma$.

Since the fan of $\mathbb{T}^n_{\mathbb{C}}$ is $\{\mathbf{0}_n\}$ and this case is trivial, we always assume that any fan $\Sigma$ in $\mathbb{R}^n$ satisfies the condition $\{\mathbf{0}_n\} \subsetneqq \Sigma$.

**Definition 1.1.** Let $\Sigma$ be a fan in $\mathbb{R}^n$ such that $\{\mathbf{0}_n\} \subsetneqq \Sigma$ and let

$$\Sigma(1) = \{\rho_1, \cdots, \rho_r\} \tag{1.3}$$

denote the set of all one dimensional cones in $\Sigma$. For each integer $1 \leq k \leq r$, we denote by $\boldsymbol{n}_k \in \mathbb{Z}^n$ *the primitive generator* of $\rho_k$, such that

$$\rho_k \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} \cdot \boldsymbol{n}_k. \tag{1.4}$$

Note that $\rho_k = \mathrm{Cone}(\boldsymbol{n}_k) = \mathbb{R}_{\geq 0} \cdot \boldsymbol{n}_k$ for each $1 \leq k \leq r$. $\square$

**Polyhedral products and homogenous coordinates.** Next, recall the definition of polyhedral products and homogenous coordinates of toric varieties.

**Definition 1.2.** Let $K$ be a simplicial complex on the vertex set $[r] = \{1, 2, \cdots, r\}$,[2] and let $(X, A)$ be a pair of based spaces such that $A \subset X$.

(i) Let $\mathcal{Z}_K(X, A)$ denote *the polyhedral product* of the pair $(X, A)$ with respect to $K$ given by the union

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma, \tag{1.5}$$

where we set $(X, A)^\sigma = \{(x_1, \cdots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}$.

When $(X, A) = (D^2, S^1)$, we write $\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1)$ and it is called *the moment-angle complex* of $K$.

(ii) For a fan $\Sigma$ in $\mathbb{R}^n$, let $\mathcal{K}_\Sigma$ denote *the underlying simplicial complex of* $\Sigma$ defined by

$$\mathcal{K}_\Sigma = \Big\{\{i_1, \cdots, i_s\} \subset [r] : \mathrm{Cone}(\boldsymbol{n}_{i_1}, \boldsymbol{n}_{i_2}, \cdots, \boldsymbol{n}_{i_s}) \in \Sigma\Big\}. \tag{1.6}$$

Note that $\mathcal{K}_\Sigma$ is a simplicial complex on the vertex set $[r]$.

(iii) Let $G_\Sigma \subset \mathbb{T}^r_{\mathbb{C}} = (\mathbb{C}^*)^r$ denote the multiplicative subgroup of $\mathbb{T}^r_{\mathbb{C}}$ defined by

$$G_\Sigma = \{(\mu_1, \cdots, \mu_r) \in \mathbb{T}^r_{\mathbb{C}} : \prod_{k=1}^r (\mu_k)^{\langle \boldsymbol{n}_k, \boldsymbol{m} \rangle} = 1 \text{ for all } \boldsymbol{m} \in \mathbb{Z}^n\}, \tag{1.7}$$

[2] Let $K$ be some set of subsets of $[r]$. Then the set $K$ is called *an abstract simplicial complex* on the vertex set $[r]$ if the following condition holds: if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In this paper by a simplicial complex $K$ we always mean an *an abstract simplicial complex*, and we always assume that a simplicial complex $K$ contains the empty set $\emptyset$.

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$ given by $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \sum_{k=1}^n u_k v_k$ for $\boldsymbol{u} = (u_1, \cdots, u_n)$ and $\boldsymbol{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$.

(iv) Consider the natural $G_\Sigma$-action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)$ given by coordinate-wise multiplication, i.e. $\mu \cdot \boldsymbol{x} = (\mu_1 x_1, \cdots, \mu_r x_r)$ for $(\mu, \boldsymbol{x}) = ((\mu_1, \cdots, \mu_r), (x_1, \cdots, x_r)) \in G_\Sigma \times \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)$. We denote by $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$ the corresponding orbit space and let

$$q_\Sigma : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma \tag{1.8}$$

denote the canonical projection. □

**Lemma 1.3** ([6], [7], [19]). *Suppose that the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^n$ (i.e. $\sum_{k=1}^r \mathbb{R} \cdot \boldsymbol{n}_k = \mathbb{R}^n$).*

(i) *There is a natural isomorphism*

$$X_\Sigma \cong \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma. \tag{1.9}$$

(ii) *If $f : \mathbb{C}\mathrm{P}^m \to X_\Sigma$ is a holomorphic map, there exists an $r$-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers satisfying the condition $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}$ and homogenous polynomials $f_i \in \mathbb{C}[z_0, \cdots, z_m]$ of degree $d_i$ $(i = 1, 2, \cdots, r)$ such that polynomials $\{f_i\}_{i\in\sigma}$ have no common root except $\mathbf{0} \in \mathbb{C}^{m+1}$ for each $\sigma \in I(\mathcal{K}_\Sigma)$ and that the diagram*

$$\begin{array}{ccc} \mathbb{C}^{m+1} \setminus \{\mathbf{0}\} & \xrightarrow{(f_1, \cdots, f_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\ \gamma_m \downarrow & & q_\Sigma \downarrow \\ \mathbb{C}\mathrm{P}^m & \xrightarrow{f} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma \end{array} \tag{1.10}$$

*is commutative, where $\gamma_m : \mathbb{C}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{C}\mathrm{P}^m$ denotes the canonical Hopf fibering and the map $q_\Sigma$ is a canonical projection induced from the identification (1.9). In this case, we call this holomorphic map $f$ as a holomorphic map of degree $D = (d_1, \cdots, d_r)$ and we represent it as*

$$f = [f_1, \cdots, f_r]. \tag{1.11}$$

*Moreover, if $g_i \in \mathbb{C}[z_0, \cdots, z_m]$ is a homogenous polynomial of degree $d_i$ $(1 \leq i \leq r)$ such that $f = [f_1, \cdots, f_r] = [g_1, \cdots, g_r]$, there exists some element $(\mu_1, \cdots, \mu_r) \in G_\Sigma$ such that $f_i = \mu_i \cdot g_i$ for each $1 \leq i \leq r$. Thus, such $r$-tuple $(f_1, \cdots, f_r)$ of homogenous polynomials representing the holomorphic map $f$ is uniquely determined up to $G_\Sigma$-action.*

(iii) *Let $h_k \in \mathbb{C}[z_0, \cdots, z_m]$ be a homogenous polynomial of the degree $d_k$ for each $1 \leq k \leq r$ such that the polynomials $\{h_k\}_{k\in\sigma}$ have no common real root except $\mathbf{0}_{m+1} \in \mathbb{R}^{m+1}$ for each $\sigma \in I(\mathcal{K}_\Sigma)$. Then there is a unique map $h : \mathbb{R}\mathrm{P}^m \to X_\Sigma$ such that the following diagram*

$$\begin{array}{ccc} \mathbb{R}^{m+1} \setminus \{\mathbf{0}\} & \xrightarrow{(h_1, \cdots, h_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\ \gamma_{m,\mathbb{R}} \downarrow & & q_\Sigma \downarrow \\ \mathbb{R}\mathrm{P}^m & \xrightarrow{h} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma \end{array} \tag{1.12}$$

*is commutative if and only if* $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_n$*, where* $\gamma_{m,\mathbb{R}} : \mathbb{R}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{RP}^m$ *denotes the canonical double covering.* □

**Remark 1.4.** We call the map $h$ determined by an $r$-tuple $(h_1, \cdots, h_r)$ of homogenous polynomials given in (iii) of Lemma 1.3 as *an algebraic map* and we write $h = [h_1, \cdots, h_r]$.

Note that two different such $r$-tuples of polynomials can determine the same maps. In fact, if we multiply all polynomials in such an $r$-tuple by the same polynomial which does not have any real roots except $\mathbf{0}_m$, we obtain the same algebraic map. For example, suppose that $(h_1, \cdots, h_r)$ is the $r$-tuple of homogenous polynomials in $\mathbb{C}[z_0, \cdots, z_m]$ of degree $d_1, \cdots, d_r$ satisfying the same condition as before. If $(a_1, \cdots, a_r) \in \mathbb{N}^r$ is the $r$-tuple of positive integers and it satisfies the condition $\sum_{k=1}^{r} a_k \boldsymbol{n}_k = \mathbf{0}_n$, we can easily see that $h = [h_1, \cdots, h_r] = [(g_1)^{a_1} h_1, \cdots, (g_1)^{a_r} h_r] = [(g_2)^{a_1} h_1, \cdots, (g_2)^{a_r} h_r]$ for $g_1 = \sum_{k=0}^{m} z_k^2$ and $g_2 = (z_0 + z_1)^2 + \sum_{k=2}^{m} z_k^2$. □

**Assumptions.** Let $\Sigma$ be a fan in $\mathbb{R}^n$ satisfying the condition (1.3) as in Definition 1.1. From now on, we assume that the following two conditions hold.

(1.9.1) There is an $r$-tuple $D_* = (d_1^*, \cdots, d_r^*) \in \mathbb{N}^r$ of positive integers such that $\sum_{k=1}^{r} d_k^* \boldsymbol{n}_k = \mathbf{0}_n$.

(1.9.2) The set $\{\boldsymbol{n}_k\}_{k=1}^{r}$ of primitive generators spans $\mathbb{Z}^n$ over $\mathbb{Z}$.

**Remark 1.5.** Note that $X_\Sigma$ is a compact iff $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$. Note also that $X_\Sigma$ is simply connected if and only if $\sum_{k=1}^{r} \mathbb{Z} \cdot \boldsymbol{n}_k = \mathbb{Z}^n$. Hence, the condition (1.9.2) always holds if $X_\Sigma$ is compact or simply connected. On the other hand, if the condition (1.9.2) holds, one can easily see that the set $\{\boldsymbol{n}_k\}_{k=1}^{r}$ spans $\mathbb{R}^n$ over $\mathbb{R}$, and there is an isomorphism (1.9) for the space $X_\Sigma$. Moreover, we know that the condition (1.9.1) holds if $X_\Sigma$ is compact and non-singular [7, Theorem 3.1]. □

**Remark 1.6.** Let $\Sigma$ denote the fan in $\mathbb{R}^2$ given by $\Sigma = \{\{\mathbf{0}_2\}, \mathrm{Cone}(\boldsymbol{e}_1), \mathrm{Cone}(\boldsymbol{e}_2)\}$ for the standard basis $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$. Then the toric variety $X_\Sigma$ of $\Sigma$ is $\mathbb{C}^2$ which has trivial homogenous coordinates. It is clearly a (simply connected) smooth toric variety, and the condition (1.9.1) also holds. However, in this case, $\sum_{k=1}^{2} d_k \boldsymbol{n}_k = \mathbf{0}_2$ iff $(d_1, d_2) = (0, 0)$. Hence, it follows from Lemma 1.3 that there are no algebraic maps $\mathbb{RP}^m \to X_\Sigma = \mathbb{C}^2$ other than the constant maps. Assuming the condition (1.9.1) guarantees the existence of non-trivial algebraic maps $\mathbb{RP}^m \to X_\Sigma$. Of course, it would be sufficient to assume that $D = (d_1, \ldots, d_r) \neq (0, \ldots 0)$ but if $d_i = 0$ for some $i$, then the number $d(D, \Sigma)$ (defined in (2.2)) is not a positive integer and our assertion (Theorem 2.2 below) is vacuous. For this reason, we will assume the condition $d_k^* \geq 1$ for each $1 \leq k \leq r$ in (1.9.1). □

Let $X_\Sigma$ be a non-singular toric variety and make the identification

$$X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma. \tag{1.13}$$

Let $z_0, \cdots, z_m$ be variables. Now we consider the space of all tuples of polynomials which define based algebraic maps.

**Definition 1.7.** (i) For each $d, m \in \mathbb{N}$, let $\mathcal{H}_m^d(\mathbb{C})$ denote the space of all homogenous polynomials $f(z_0, \cdots, z_m) \in \mathbb{C}[z_0, \cdots, z_m]$ of degree $d$.

(ii) For each $r$-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, let $\mathrm{Pol}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote the space of $r$-tuples $f = (f_1(z_0, \cdots, z_m), \cdots, f_r(z_0, \cdots, z_m)) \in \mathcal{H}_m^{d_1}(\mathbb{C}) \times \cdots \times \mathcal{H}_m^{d_r}(\mathbb{C})$ of homogenous polynomials satisfying the following two conditions:

(1.14.1) $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_r(\boldsymbol{x})) \in U(\mathcal{K}_\Sigma)$ for any point $\boldsymbol{x} = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}_{m+1}\}$.

(1.14.2) $f(\boldsymbol{e}_1) = (f_1(\boldsymbol{e}_1), \cdots, f_r(\boldsymbol{e}_1)) = (1, 1, \cdots, 1)$, where $\boldsymbol{e}_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$. □

**Definition 1.8.** We always assume the identification $X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma$, and denote by $[y_1, \cdots, y_r]$ the point in $X_\Sigma$ represented by $(y_1, \cdots, y_r) \in U(\mathcal{K}_\Sigma)$. Moreover, we choose the two points $[1 : 0 : \cdots : 0] \in \mathbb{R}\mathrm{P}^m$ and $* = [1, \cdots, 1] \in X_\Sigma$ as the base-points of $\mathbb{R}\mathrm{P}^m$ and $X_\Sigma$ respectively.

Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an $r$-tuple of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_n$. Then by using Lemma 1.3, for each $r$-tuple

$$f = (f_1(z_0, \cdots, z_m), \cdots, f_r(z_0, \cdots, z_m)) \in \mathrm{Pol}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$$

one can define based algebraic map

$$[f] = [f_1, \cdots, f_r] : (\mathbb{R}\mathrm{P}^m, [\boldsymbol{e}_1]) \to (X_\Sigma, *) \quad \text{by} \tag{1.14}$$

$$[f]([\boldsymbol{x}]) = [f_1(\boldsymbol{x}), \cdots, f_r(\boldsymbol{x})] \tag{1.15}$$

for $[\boldsymbol{x}] = [x_0 : \cdots : x_m] \in \mathbb{R}\mathrm{P}^m$, where $\boldsymbol{x} = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}_{m+1}\}$. Hence, we denote by $\mathrm{Map}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ the path-component of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ which contains all algebraic maps of degree $D$, and we obtain the natural map

$$i_{D,m} : \mathrm{Pol}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma) \to \mathrm{Map}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma) \tag{1.16}$$

given by

$$i_{D,m}(f) = [f] = [f_1, \cdots, f_r] \tag{1.17}$$

for $f = (f_1(z_0, \cdots, z_m), \cdots, f_r(z_0, \cdots, z_m)) \in \mathrm{Pol}_D^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$,. □

When $m = 1$, we make the identification $\mathbb{R}\mathrm{P}^1 = S^1 = \mathbb{R} \cup \infty$ and choose the points $\infty$ as the base-point of $\mathbb{R}\mathrm{P}^1$. Then, by setting $z = \frac{z_0}{z_1}$, we can view a homogenous polynomial $f(z_0, z_1) \in \mathbb{C}[z_0, z_1]$ of degree $d$ as a monic polynomial $f_k(z) \in \mathbb{C}[z]$ of degree $d$. Thus, when $m = 1$, one can redefine the space $\mathrm{Pol}_D^*(S^1, X_\Sigma)$ as follows.

**Definition 1.9.** (i) Let $\mathrm{P}^d$ denote the space of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d \in \mathbb{C}[z]$ of degree $d$, and let

$$\mathrm{P}^D = \mathrm{P}^{d_1} \times \mathrm{P}^{d_2} \times \cdots \times \mathrm{P}^{d_r}. \tag{1.18}$$

Note that there is a homeomorphism $\phi : \mathrm{P}^d \cong \mathbb{C}^d$ given by $\phi(z^d + \sum_{k=1}^{d} a_k z^{d-k}) = (a_1, \cdots, a_d) \in \mathbb{C}^d$.

(ii) For any $r$-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, let $\mathrm{Pol}^*_D(S^1, X_\Sigma)$ denote the space of all $r$-tuples $(f_1(z), \cdots, f_r(z)) \in \mathrm{P}^D$ of monic polynomials satisfying the following condition (†):

(†) The polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common *real* root for any $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$, i.e. $(f_{i_1}(\alpha), \cdots, f_{i_s}(\alpha)) \neq \mathbf{0}_s$ for any $\alpha \in \mathbb{R}$.

When the condition $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_n$ holds, by identifying $X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$ and $\mathbb{R}\mathrm{P}^1 = S^1 = \mathbb{R} \cup \infty$, one can define a natural map

$$i_D = i_{D,1} : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \mathrm{Map}^*(S^1, X_\Sigma) = \Omega X_\Sigma \quad \text{by} \tag{1.19}$$

$$i_D(f_1(z), \cdots, f_r(z))(\alpha) = \begin{cases} [f_1(\alpha), \cdots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [1, 1, \cdots, 1] & \text{if } \alpha = \infty \end{cases} \tag{1.20}$$

for $(f_1(z), \cdots, f_r(z)) \in \mathrm{Pol}^*_D(S^1, X_\Sigma)$ and $\alpha \in S^1 = \mathbb{R} \cup \infty$, where we choose the points $\infty$ and $[1, 1, \cdots, 1]$ as the base-points of $S^1$ and $X_\Sigma$.

Note that $\mathrm{Pol}^*_D(S^1, X_\Sigma)$ is simply connected and that the map $\Omega q_\Sigma : \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to \Omega X_\Sigma$ is a universal covering. Thus, when $\sum_{k=1}^{r} d_k \boldsymbol{n}_k = \mathbf{0}_n$, the map $i_D$ lifts to the space $\Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \simeq \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}$ and there is a map

$$j_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma} \tag{1.21}$$

such that

$$\Omega q_\Sigma \circ j_D = i_D. \tag{1.22}$$

**Remark 1.10.** Even if $\sum_{k=1}^{r} d_k \boldsymbol{n}_k \neq \mathbf{0}_n$ we can define the two maps

$$i_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega X_\Sigma, \quad j_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}$$

by using stabilization maps. The detail is given in [19]. □

Now we need to define the numbers $r_{\min}(\Sigma)$ and $d(D, \Sigma)$.

**Definition 1.11.** Let $\Sigma$ be a fan in $\mathbb{R}^n$ as in Definition 1.1.

(i) We say that a set $S = \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\}$ is *primitive in* $\Sigma$ if $\mathrm{Cone}(S) \notin \Sigma$ but $\mathrm{Cone}(T) \in \Sigma$ for any proper subset $T \subsetneqq S$.

(ii) For $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ define integers $r_{\min}(\Sigma)$ and $d(D, \Sigma; m)$ by

$$\begin{cases} r_{\min}(\Sigma) & = \min\{s \in \mathbb{N} : \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\} \text{ is primitive in } \Sigma\}, \\ d(D, \Sigma; m) & = (2r_{\min}(\Sigma) - m - 1)d_{\min} - 2, \quad \text{where } d_{min} = \min\{d_1, \cdots, d_r\}. \end{cases} \tag{1.23}$$

**Definition 1.12.** Recall that a map $g : V \to W$ is called *a homology* (*resp. homotopy*) *equivalence through dimension* $N$ if the induced homomorphism $g_* : H_k(V;\mathbb{Z}) \to H_k(W;\mathbb{Z})$ (resp. $g_* : \pi_k(V) \to \pi_k(W)$) is an isomorphism for all $k \leq N$. $\square$

Now recall the following result.

**Theorem 1.13** ([13])**.** *Let $m \geq 2$ be a positive integer, $X_\Sigma$ be a compact smooth toric variety and $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an $r$-tuple of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_n$. Then the natural map $i_{D,m} : \mathrm{Pol}^*_D(\mathbb{RP}^m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{RP}^m, X_\Sigma)$ is a homology equivalence through dimension $d(D,\Sigma;m)$.* $\square$

Note that the above result does not hold for the case $m = 1$. For example, this can be seen in [11] for the case $X_\Sigma = \mathbb{CP}^n$. In fact, the main purpose of this paper is to investigate the result corresponding to this theorem for the case $m = 1$.

## 2 Main results

**Previous results.** First, recall the following result concerning to the homotopy type of space of rational curves one a toric variety.

**Theorem 2.1** ([18])**.** *Let $X_\Sigma$ be a simply connected non-singular toric variety associated to the fan $\Sigma$ such that the condition (1.9.1) is satisfied. Then if $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_n$, the inclusion map*

$$i_{D,hol} : \mathrm{Hol}^*_D(S^2, X_\Sigma) \xrightarrow{\subset} \Omega^2_D X_\Sigma$$

*is a homotopy equivalence through dimension $d_*(D,\Sigma)$ if $r_{\min}(\Sigma) \geq 3$ and a homology equivalence through dimension $d_*(D,\Sigma) = d_{min} - 2$ if $r_{\min}(\Sigma) = 2$.*

*Here, $\Omega^2_D X_\Sigma$ (resp. $\mathrm{Hol}^*_D(S^2, X_\Sigma)$) denotes the space of based continuous (resp. based holomorphic) maps from $S^2$ to $X_\Sigma$ of degree $D$, and $d_*(D,\Sigma)$ is the number given by*

$$(2.1) \qquad d_*(D,\Sigma) = (2r_{\min}(\Sigma) - 3)d_{min} - 2, \quad \text{where } d_{min} = \min\{d_1, \cdots, d_r\}. \quad \square$$

**The main results of this note.** The main result of this paper is to consider the real analogue of the above result and this is stated as follows.

**Theorem 2.2** ([19])**.** *Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ be an $r$-tuple of positive integers and let $X_\Sigma$ be a simply connected non-singular toric variety such that the condition (1.9.1) holds. Then there is map*

$$j_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}$$

*which is a homotopy equivalence through dimension $d(D,\Sigma)$, where the number $d(D,\Sigma)$ is given by*

$$(2.2) \qquad d(D,\Sigma) = d(D,\Sigma;1) = (2r_{\min}(\Sigma) - 2)d_{\min} - 2. \qquad \square$$

**Corollary 2.3** ([19]). *Under the same assumption as in Theorem 2.2, there is the map* $i_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega X_\Sigma$ *induces an isomorphism*

$$(i_D)_* : \pi_k(\mathrm{Pol}^*_D(S^1, X_\Sigma)) \xrightarrow{\cong} \pi_k(\Omega X_\Sigma) \cong \pi_{k+1}(X_\Sigma)$$

*for any* $2 \leq k \leq d(D, \Sigma)$. □

**Corollary 2.4** ([19]). *Let* $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ *be an* $r$*-tuple of positive integers satisfying the condition* $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_n$, *and let* $X_\Sigma$ *be a simply connected compact non-singular toric variety. Let* $\Sigma(1)$ *denote the set of all one dimensional cones in* $\Sigma$, *and* $\Sigma_1$ *any fan in* $\mathbb{R}^n$ *such that* $\Sigma(1) \subset \Sigma_1 \subsetneqq \Sigma$.

(i) *Then* $X_{\Sigma_1}$ *is a non-singular open toric subvariety of* $X_\Sigma$ *and there is the map*

$$j_D : \mathrm{Pol}^*_D(S^1, X_{\Sigma_1}) \to \Omega \mathcal{Z}_{\Sigma_1}$$

*which is a homotopy equivalence through dimension* $d(D, \Sigma_1)$.

(ii) *Moreover, there is the map* $i_D : \mathrm{Pol}^*_D(S^1, X_{\Sigma_1}) \to \Omega X_{\Sigma_1}$ *which induces the isomorphism*

$$(i_D)_* : \pi_k(\mathrm{Pol}^*_D(S^1, X_{\Sigma_1})) \xrightarrow{\cong} \pi_k(\Omega X_{\Sigma_1}) \cong \pi_{k+1}(X_{\Sigma_1})$$

*for any* $2 \leq k \leq d(D, \Sigma_1)$. □

**Examples.** Finally consider the example of the main results. Since the case $X_\Sigma = \mathbb{C}\mathrm{P}^n$ was already well known, we consider the case that $X_\Sigma$ is the Hirzerbruch surface $H(k)$.

**Definition 2.5.** For an integer $k \in \mathbb{Z}$, let $H(k)$ be *the Hirzerbruch surface* defined by

$$H(k) = \big\{([x_0 : x_1 : x_2], [y_1 : y_2]) \in \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1 : x_1 y_1^k = x_2 y_2^k\big\} \subset \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1.$$

Since there are isomorphisms $H(-k) \cong H(k)$ for $k \neq 0$ and $H(0) \cong \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, without loss of generality we can assume that $k \geq 1$. Let $\Sigma_k$ denote the fan in $\mathbb{R}^2$ given by

$$\Sigma_k = \big\{\mathrm{Cone}(\mathbf{n}_i, \mathbf{n}_{i+1})\ (1 \leq i \leq 3), \mathrm{Cone}(\mathbf{n}_4, \mathbf{n}_1), \mathrm{Cone}(\boldsymbol{n}_j)\ (1 \leq j \leq 4),\ \{\mathbf{0}\}\big\},$$

where we set $\boldsymbol{n}_1 = (1, 0),\ \boldsymbol{n}_2 = (0, 1),\ \boldsymbol{n}_3 = (-1, k),\ \boldsymbol{n}_4 = (0, -1)$.

It is easy to see that $\Sigma_k$ is the fan of $H(k)$ and that $H(k)$ is a compact non-singular toric variety. Note that $\Sigma_k(1) = \{\mathrm{Cone}(\boldsymbol{n}_i) : 1 \leq i \leq 4\}$. Since $\{\boldsymbol{n}_1, \boldsymbol{n}_3\}$ and $\{\boldsymbol{n}_2, \boldsymbol{n}_4\}$ are only primitive in $\Sigma_k$, $r_{\min}(\Sigma_k) = 2$.

Moreover, for $D = (d_1, d_2, d_3, d_4) \in \mathbb{N}^4$ the equality $\sum_{k=1}^4 d_k \boldsymbol{n}_k = \mathbf{0}_2$ holds iff $(d_3, d_4) = (d_1, kd_1 + d_2)$. Thus, if $\sum_{k=1}^4 d_k \boldsymbol{n}_k = \mathbf{0}_2$, we have $d_{\min} = \min\{d_1, d_2, d_3, d_4\} = \min\{d_1, d_2\}$. □

**Example 2.6.** *Let* $D = (d_1, d_2, d_3, d_4) \in \mathbb{N}^4$, $k \in \mathbb{N}$, *and* $\Sigma$ *be a fan in* $\mathbb{R}^2$ *such that* $\Sigma_k(1) = \{\mathrm{Cone}(\boldsymbol{n}_i) : 1 \leq i \leq 4\} \subset \Sigma \subset \Sigma_k$ *as in Definition 2.5.*

(i) $X_\Sigma$ *is a non-singular open toric subvariety of* $H(k)$ *if* $\Sigma \subsetneqq \Sigma_k$.

(ii) *If* $\sum_{k=1}^{4} d_k \boldsymbol{n}_k = \mathbf{0}_2$*, the equality* $(d_3, d_4) = (d_1, kd_1 + d_2)$ *holds and the map* $j_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}$ *is a homotopy equivalence through dimension* $2\min\{d_1, d_2\} - 2$*. Moreover, the map* $i_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega X_\Sigma$ *induces an isomorphism*

$$(i_D)_* : \pi_k(\mathrm{Pol}^*_D(S^1, X_\Sigma)) \xrightarrow{\cong} \pi_k(\Omega X_\Sigma) \cong \pi_{k+1}(X_\Sigma)$$

*for any* $2 \leq k \leq 2\min\{d_1, d_2\} - 2$*.*

(iii) *If* $\sum_{k=1}^{4} d_k \boldsymbol{n}_k \neq \mathbf{0}_2$*, there is a map* $j_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega\mathcal{Z}_{\mathcal{K}_\Sigma}$ *which is a homotopy equivalence through dimension* $2\min\{d_1, d_2, d_3, d_4\} - 2$*, and there is a map* $i_D : \mathrm{Pol}^*_D(S^1, X_\Sigma) \to \Omega X_\Sigma$ *which induces an isomorphism*

$$(i_D)_* : \pi_k(\mathrm{Pol}^*_D(S^1, X_\Sigma)) \xrightarrow{\cong} \pi_k(\Omega X_\Sigma) \cong \pi_{k+1}(X_\Sigma)$$

*for any* $2 \leq k \leq 2\min\{d_1, d_2, d_3, d_4\} - 2$*.* □

**Remark 2.7.** As we considered as above, the space $\mathrm{Pol}^*_D(S^1, X_\Sigma)$ can be regarded as one of real analogues of the space $\mathrm{Hol}^*_D(S^2, X_\Sigma)$. In our previous paper [17], we investigate the homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{C})$ of resultants of bounded multiplicity. We can also consider the real analogues of it, and we shall investigate the homotopy types of them in the subsequent papers ([20], [21]). □

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Department of Mathematics, University of Electro-Communications
1-5-1 Chufugaoka, Chofu, Tokyo 182-8585, Japan
E-mail: kohhei@im.uec.ac.jp