Semi-classical limits for the Nelson model

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Abstract

We are concerned with the Nelson Hamiltonian $H_\hbar$ with semi-classical parameter $\hbar > 0$. A classical object $\mathcal{H}(q_s, p_s, u_s, \bar{u}_s)$ is defined by the solution $\{q_s, p_s, u_s\}$ to the Hamilton-Jacobi equation associated with the Nelson Hamiltonian. We show the asymptotic behaviour of

$$e^{-i\frac{t}{\hbar}H_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s}$$

as $\hbar \to 0$. Furthermore we introduce Wigner measures μ_0 on the particle-field phase space $X = \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$ appearing in the semi-classical limits of a family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$. I.e.,

$$\lim_{\hbar \to 0} \mathrm{Tr}(\rho_\hbar \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_0(x)$$

for $\xi' \in X$ and $\mathcal{W}(\xi)$ denotes an exponential operator. The Wigner measure μ_t associated with the family of time evolutions of trace class operators $\{\rho_\hbar(t), \hbar \in (0,1)\}$ are given by

$$\lim_{\hbar \to 0} \mathrm{Tr}(\rho_\hbar(t) \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_t(x).$$

We show that $\mu_t(\cdot) = \mu_0 \circ \Phi_t^{-1}(\cdot)$, where Φ_t is the flow for the solution to the Hamilton-Jacobi equation.

1 Hamilton-Jacobi equation for the Nelson model

In the RIMS conference held on December 6-8, 2021 we gave a talk on the title "Newton Maxwell equation through semi-classical analysis". In this article, however, we are concerned with the Nelson model on coherent states for the simplicity and demonstrate a motivation why we are interested in the semiclassical analysis. This results are ultimately developed in [2] for the Pauli-Fierz model in non-relativistic QED [10] and the semi-classical limit is

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investigated through the so-called Wigner measures. The Wigner measure is a probability measure on the total phase space $\mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$. The Wigner measure is applied to the semi-classical analysis in [7] for Schrödinger operators and is extended to an infinite dimensional phase space in [3]. We refer [1, 4, 6, 5] for related investigations.

1.1 Semi-classical limit of Schrödinger operators

Before going to our main results, we introduce a semi-classical limit of Schrödinger operators for the readers convenient. Let us consider $3D$-Schrödinger operator of the form:

$$h_\hbar = \frac{\hbar^2}{2m} D_x^2 + V(x),$$

where $\hbar > 0$ is a semi-classical parameter, $m > 0$ a mass of a particle, $D_x = -i\nabla_x$ and V is an external potential. Let

$$\mathcal{H} = \mathcal{H}(q,p) = \frac{p^2}{2m} + V(q),$$

where $(p,q) \in \mathbb{R}^3 \times \mathbb{R}^3$. The Hamilton-Jacobi equation associated with $h_\hbar$ is

$$\begin{cases} \dot{q}_t & = \dfrac{\delta \mathcal{H}}{\delta p_t} & = \dfrac{p_t}{m}, \\ \\ \dot{p}_t & = -\dfrac{\delta \mathcal{H}}{\delta q_t} & = -\nabla V(q_t). \end{cases} \tag{1.1}$$

Let $(q_t, p_t) \in \mathbb{R}^3 \times \mathbb{R}^3$ be the solution to (1.1). We are interested in the asymptotic behaviour of

$$e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}$$

as $\hbar \to 0$.

Let $\xi_\hbar$ be the $3D$-dilation defined by

$$\xi_\hbar f(x) = \hbar^{3/4} f(\sqrt{\hbar} x),$$

and hence $\xi_\hbar^* f(x) = \hbar^{-3/4} f(x/\sqrt{\hbar})$. Let us define the quadratic operator Q_t^{Sch} by

$$Q_t^{Sch} = \frac{1}{2m} D_x^2 + x \cdot \nabla^2 V(q_t) x.$$

Then it actually follows that

$$\lim_{\hbar \to 0} \| e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s} \varphi - e^{\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} \xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s} \xi_\hbar e^{-\frac{i}{\hbar}(p_0 x - \hbar q_0 D_x)} \varphi \| = 0. \tag{1.2}$$

This can be proven as follows. Let

$$\gamma_t = \xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s} \xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi.$$

Then

$$\|e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi - e^{\frac{i}{\hbar}(p_t x-\hbar q_t D_x)}\xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_0x-\hbar q_0 D_x)}\varphi\| = \|\gamma_t - \gamma_0\| \leq \int_0^t \|\dot\gamma_s\|\mathrm{d}s.$$

We see that

$$\begin{aligned}
\dot\gamma_t =& e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s} \frac{i}{\hbar}\mathcal{H}(q_t,p_t)\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* i\dot{Q} e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar \left\{-\frac{i}{\hbar}(\dot p_t x - \hbar \dot q_t D_x)\right\} e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)} \left\{-\frac{i}{\hbar}h_\hbar\right\} e^{-i\frac{t}{\hbar}h_\hbar}\varphi.
\end{aligned}$$

We compute $\xi_\hbar e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)}\left\{-\frac{i}{\hbar}h_\hbar\right\}$. By a shift operator $e^{\frac{i}{\hbar}(px-q\hbar D_x)}$,

$$h_\hbar \to \frac{(\hbar D_x + p_t)^2}{2m} + V(x+q_t),$$

and by a scaling $\xi_\hbar$,

$$\to \frac{(\sqrt{\hbar}D_x + p_t)^2}{2m} + V(\sqrt{\hbar}x + q_t).$$

Then the right-hand side above is

$$\begin{aligned}
\xi_\hbar e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)}h_\hbar &= \frac{(\sqrt{\hbar}D_x + p_t)^2}{2m} + V(\sqrt{\hbar}x + q_t) \\
&= \mathcal{H}(q_t,p_t) + \sqrt{\hbar}\left(\frac{p_t D_x}{m} + \nabla V(q_t)x\right) + \hbar Q_t^{Sch} + O(\hbar^{3/2}). \qquad (1.3)
\end{aligned}$$

Furthermore

$$\xi_\hbar\left\{-\frac{i}{\hbar}(\dot p_t x - \hbar\dot q_t D_x)\right\} = -\frac{i}{\sqrt{\hbar}}(\dot p_t x - \dot q_t D_x)\xi_\hbar.$$

Hence

$$\begin{aligned}
\dot\gamma_t =& \xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\left\{\frac{i}{\hbar}\mathcal{H}(q_t,p_t) + iQ_t^{Sch} - \frac{i}{\sqrt{\hbar}}(\dot p_t x - \dot q_t D_x) - \frac{i}{\hbar}(1.3)\right\}\xi_\hbar \\
&\times e^{-\frac{i}{\hbar}(p_t x-\hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi.
\end{aligned}$$

By (1.1) we have

$$\frac{i}{\hbar}\mathcal{H}(q_t,p_t)+iQ_t^{Sch}-\frac{i}{\sqrt{\hbar}}(\dot{p}_t x-\dot{q}_t D_x)-\frac{i}{\hbar}(1.3)=O(\sqrt{\hbar}).$$

Then (1.2) follows. In the semi-classical region we can see that

$$e^{-i\frac{t}{\hbar}h_\hbar}e^{+\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s)\mathrm{d}s}\sim e^{\frac{i}{\hbar}(p_t x-\hbar q_t D_x)}\xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_0 x-\hbar q_0 D_x)}.$$

Here we emphasize that Q_s^{Sch} is independent of $\hbar$. We extend this kind of arguments to the Nelson model in quantum field theory in what follows.

1.2 Nelson model

Let $a^\dagger(f)$ and $a(f)$ be the annihilation operator and the creation operator, respectively on the boson Fock space over $L^2(\mathbb{R}^3)$:

$$\mathcal{F}=\bigoplus_{n=0}^\infty[\otimes_s^n L^2(\mathbb{R}^3)].$$

The adjoint relation is $a(f)^*=a^\dagger(\bar{f})$ and the CCR is given by $[a(f),a^\dagger(g)]=(\bar{f},g)\mathbb{1}$ and $[a^\sharp(f),a^\sharp(g)]=0$, where (f,g) denotes a scalar product on $L^2(\mathbb{R}^3)$ and it is linear in g and anti-linear in f. Formally we write $a^\sharp(f)=\int a^\sharp(k)f(k)\mathrm{dk}$. The field operator is given by

$$\phi(f)=\frac{1}{\sqrt{2}}(a^\dagger(f)+a(\bar{f}))$$

and its momentum conjugate by

$$\Pi(f)=\frac{i}{\sqrt{2}}(a^\dagger(f)-a(\bar{f})).$$

Thus $[\phi(f),\Pi(g)]=i\,\mathrm{Re}(f,g)$, $[\phi(f),\phi(g)]=i\,\mathrm{Im}(f,g)$ and $[\Pi(f),\Pi(g)]=i\,\mathrm{Im}(f,g)$ hold true. Let $H_\mathrm{f}=d\Gamma(\omega)$ be the second quantization of the multiplication by $\omega(k)=|k|$. Here $|k|$ denotes the energy of a massless boson with momentum $k\in\mathbb{R}^3$.

The Nelson Hamiltonian [9, 8] is defined as s self-adjoint operator on the product Hilbert space

$$\mathcal{H}=L^2(\mathbb{R}^3)\otimes\mathcal{F}$$

and is given by

$$H=(\frac{1}{2m}D_x^2+V)\otimes\mathbb{1}+\mathbb{1}\otimes H_\mathrm{f}+H_\mathrm{I},$$

and the interaction by

$$H_{\mathrm{I}} = H_{\mathrm{I}}(x)\phi(e^{-ikx}\hat{\varphi}/\sqrt{\omega}) = \frac{1}{\sqrt{2}}\int\left\{\frac{e^{-ikq}\hat{\varphi}(k)}{\sqrt{\omega(k)}}a^{\dagger}(k) + \frac{e^{ikq}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}a(k)\right\}\mathrm{d}k.$$

Here $\hat{\varphi}$ is a cutoff function. We assume that $\omega\sqrt{\omega}\hat{\varphi}$, $\sqrt{\omega}\hat{\varphi}$, $\hat{\varphi}/\sqrt{\omega}$, $\hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$. Throughout we suppose that $V \in C^2(\mathbb{R}^3)$ and bounded. Then H is self-adjoint on $D(D_x^2) \cap D(H_{\mathrm{f}})$ and bounded from below. We introduce the semi-classical parameter $\hbar > 0$ by

$$H_{\hbar} = (\frac{\hbar^2}{2m}D_x^2 + V) \otimes \mathbb{1} + \sqrt{\hbar}H_{\mathrm{I}} + \hbar\mathbb{1} \otimes H_{\mathrm{f}}.$$

Let $(q, p, u) \in \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$. The classical Nelson Hamiltonian is given by

$$\mathcal{H}(p, q, u, \bar{u}) = \frac{p^2}{2m} + V(q) + \int_{\mathbb{R}^3}\omega(k)|u(k)|^2\mathrm{d}k + U(q, u).$$

Here

$$U(q, u) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{\frac{e^{-ikq}\hat{\varphi}(k)}{\sqrt{\omega(k)}}\bar{u}(k) + \frac{e^{ikq}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}u(k)\right\}\mathrm{d}k.$$

The time evolution of $(p, q, u) \in \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$ are governed by the Hamilton-Jacobi equation:

$$(N)\begin{cases} \dot{q}_t &= \dfrac{\delta\mathcal{H}}{\delta p_t} &= \dfrac{p_t}{m}, \\ \dot{p}_t &= -\dfrac{\delta\mathcal{H}}{\delta q_t} &= -\nabla V(q_t) - \nabla U(q_t, u_t), \\ i\dot{u}_t(k) &= \dfrac{\delta\mathcal{H}}{\delta\bar{u}_t} &= \omega(k)u_t(k) + \dfrac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}}. \end{cases}$$

Here

$$\nabla U(q_t, u_t) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{-ik\frac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}}\bar{u}_t(k) + ik\frac{e^{ikq_t}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}u_t(k)\right\}\mathrm{d}k.$$

Note that $\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and then the right-hand side above is finite.

2 Coherent states and Weyl commutation relations

Now we define coherent states for the field and the particle. In general, when $[A, B]$ is c-number, then formally

$$e^A e^B = e^{\frac{1}{2}[A,B]}e^{A+B}$$

holds true. Let $W(f) = e^{i\Pi(f)}$. Then Weyl commutation relation holds:

$$W(f)W(g) = e^{-\frac{i}{2}\operatorname{Im}(f,g)}W(f+g).$$

Since $W(ig) =^{-\Phi(g)}$, we can see that

$$W(f)W(ig) = e^{-\frac{i}{2}\operatorname{Re}(f,g)}W(f+ig).$$

Let $z = q + ip \in \mathbb{R}^3 + i\mathbb{R}^3$. Define $T(z) = e^{i(px - q\hbar D_x)}$. Note that

$$[px - q\hbar D_x, p'x - q'\hbar D_x] = i\hbar(qp' - pq') = i\hbar \operatorname{Im}\bar{z} \cdot z'.$$

Hence

$$T(z)T(z') = e^{-\frac{i}{2}\hbar \operatorname{Im}\bar{z}\cdot z'}T(z+z')$$

and

$$T(z)T(iz') = e^{-\frac{i}{2}\hbar \operatorname{Re}\bar{z}\cdot z'}T(z+iz').$$

The coherent state smeared by u is defined by

$$W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega,$$

where $\Omega \in \mathcal{F}$ is the Fock vacuum. Note that

$$W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) = e^{-\frac{1}{\sqrt{\hbar}}(a^\dagger(u) - a(\bar{u}))}.$$

Let $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ be a point in the phase space and

$$\psi_\hbar(x) = (\pi\hbar)^{-3/4}e^{-|x|^2/(2\hbar)}.$$

Thus $\|\psi_\hbar\| = 1$. The coherent state for the particle part is given by

$$\psi^\hbar_{q,p}(x) = T^\hbar_{q,p}\psi_\hbar,$$

where $T^\hbar_{q,p} = T\left(\frac{z}{\hbar}\right)$ for $z = q + ip$, i.e.,

$$T^\hbar_{q,p} = \exp\left(\frac{i}{\hbar}(px - \hbar q D_x)\right).$$

Note that $\psi^\hbar_{q,p}$ is normalized in $L^2(\mathbb{R}^3)$ for each $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3$. We see that

$$T^\hbar_{q,p} = e^{-\frac{i}{2}\frac{1}{\hbar}pq}e^{\frac{i}{\hbar}px}e^{-iqD_x} = e^{\frac{i}{2}\frac{1}{\hbar}pq}e^{-iqD_x}e^{\frac{i}{\hbar}px}.$$

Let (q_t, p_t, u_t) be the solution to (N). Define

$$\Phi_t^\hbar = T_{q_t,p_t,u_t}^\hbar (\psi_\hbar \otimes \Omega), \quad t \geq 0.$$

Here

$$T_{q,p,u}^\hbar = T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right), \quad z = q + ip,$$

is unitary. The unitary operator $T_{q_t,p_t,u_t}^\hbar$ is the shift operator such that

$$\begin{aligned}
&T_{q_t,p_t,u_t}^{\hbar *} x T_{q_t,p_t,u_t}^\hbar = x + q_t, \\
&T_{q_t,p_t,u_t}^{\hbar *} \hbar D_x T_{q_t,p_t,u_t}^\hbar = \hbar D_x + p_t, \\
&T_{q_t,p_t,u_t}^{\hbar *} \sqrt{\hbar} a(k) T_{q_t,p_t,u_t}^\hbar = \sqrt{\hbar} a(k) + u_t(k), \\
&T_{q_t,p_t,u_t}^{\hbar *} \sqrt{\hbar} a^\dagger(k) T_{q_t,p_t,u_t}^\hbar = \sqrt{\hbar} a^\dagger(k) + \bar{u}_t(k).
\end{aligned}$$

From these relations we can see that

$$\begin{aligned}
&(x + i\hbar D_x)\Phi_t^\hbar = (q_t + ip_t)\Phi_t^\hbar, \\
&\sqrt{\hbar} a(k)\Phi_t^\hbar = u_t(k)\Phi_t^\hbar, \\
&\sqrt{\hbar} a^\dagger(k)\Phi_t^\hbar = \bar{u}_t(k)\Phi_t^\hbar.
\end{aligned}$$

The classical objects appear as the eigenvalues.

3 Semi-classical limits

In this section we shall prove that

$$\lim_{\hbar \to 0} \| e^{-i\frac{t}{\hbar} H_\hbar} \Phi_\hbar - e^{-\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s} T_{q_t,p_t,u_t}^\hbar e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s} T_{q_0,p_0,u_0}^{\hbar *} \Phi_\hbar \| = 0. \tag{3.1}$$

Here $\int_0^t Q_{\hbar,s}\mathrm{d}s$ is a quadratic operator derived from $H_\hbar$. The strategy to see (3.1) is due to the fact

$$T_{q_t,p_t,u_t}^{\hbar *} H_\hbar T_{q_t,p_t,u_t}^\hbar = \mathcal{H}(q_t, p_t, u_t, \bar{u}_t) + Q_{\hbar,t} + \text{reminder} + O(\sqrt{\hbar}).$$

This corresponds to (1.3) for Schrödinger operators. See (3.3). The quadratic term is given by

$$Q_{\hbar,t} = \frac{\hbar^2}{2m} D_x^2 + \frac{1}{2} x \cdot \left(\nabla^2 V(q_t) + \nabla^2 U(q_t, u_t)\right) x + \sqrt{\hbar} \nabla H_{\mathrm{I}}(q_t) x + \hbar H_{\mathrm{f}}.$$

Here $\nabla H_{\rm I}(q_s) = \phi(-ike^{-ikq_s}\hat{\varphi}/\sqrt{\omega})$, $\nabla^2 V(q_t) = (\nabla_\alpha\nabla_\beta V(q_t))_{1\le\alpha,\beta\le 3}$ and $\nabla^2 U(q_t,u_t) = (\nabla_\alpha\nabla_\beta U(q_t,u_t))_{1\le\alpha,\beta\le 3}$ with

$$\nabla_\alpha\nabla_\beta U(q_t,u_t) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{-k_\alpha k_\beta\frac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}}\bar{u}_t(k) - k_\alpha k_\beta\frac{e^{ikq_t}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}u_t(k)\right\}\mathrm{d}k.$$

The main theorem is as follows.

Theorem 3.1 *Let $(q,p,u)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$. Suppose that $(q_t,p_t,u_t)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$ is the solution to (N) with initial condition $(q_0,p_0,u_0)=(q,p,u)$. Then*

$$\|e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s}\Phi_\hbar - T^\hbar_{q_t,p_t,u_t}e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}T^{\hbar*}_{q,p,u}\Phi_\hbar\| \le C\sqrt{\hbar}. \tag{3.2}$$

Proof: Let $\xi_\hbar$ be the dilation defined by $\xi_\hbar f(x) = \hbar^{3/4}f(\sqrt{\hbar}x)$, and hence $\xi^*_\hbar f(x) = \hbar^{-3/4}f(x/\sqrt{\hbar})$. In particular we have

$$\xi^*_\hbar\psi_1(x) = \psi_\hbar(x).$$

To show (3.2), the Cook method is applied. Let

$$\nu_t = \xi_\hbar e^{\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}T^{\hbar*}_{q_t,p_t,u_t}e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s}\xi^*_\hbar\Phi_1,$$

where $\Phi_1 = \psi_1\otimes\Omega$. Since $\nu_0 = \xi_\hbar T^*_{q,p,u}\xi^*_\hbar$, we have $\nu_t - \nu_0 = \int_0^t\dot{\nu}_s\mathrm{d}s$ and then the left-hand side of (3.2) can be written as $\|\nu_t-\nu_0\|$. We note that

$$\nu_t-\nu_0 = \int_0^t\xi_\hbar e^{\frac{i}{\hbar}\int_0^s Q_{\hbar,r}\mathrm{d}r}\left(\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)+\frac{i}{\hbar}Q_{\hbar,s}+C_s-\frac{i}{\hbar}H_\hbar(s)\right)T^{\hbar*}_{q_s,p_s,u_s}e^{-i\frac{s}{\hbar}H_\hbar}\xi^*_\hbar\Phi_1\mathrm{d}s,$$

where

$$\frac{d}{ds}T^{\hbar*}_{q_s,p_s,u_s} = C_sT^{\hbar*}_{q_s,p_s,u_s}.$$

Since $T^\hbar_{q_t,p_t,u_t}$ acts as the shift by $x\to x+q_t$, $\hbar D_x\to\hbar D_x+p_t$ and $\sqrt{\hbar}a(k)\to\sqrt{\hbar}a(k)+u(k)$, we used the intertwining property:

$$T^{\hbar*}_{q_t,p_t,u_t}H_\hbar = H_\hbar(t)T^{\hbar*}_{q_t,p_t,u_t},$$

where

$$\begin{aligned}
H_\hbar(t) =& \frac{(\hbar D_x+p_t)^2}{2m} + V(x+q_t) + \int_{\mathbb{R}^3}\omega(k)(\sqrt{\hbar}a^\dagger(k)+\bar{u}_t(k))(\sqrt{\hbar}a(k)+u_t(k))\mathrm{d}k \\
&+ \int_{\mathbb{R}^3}\left\{\frac{e^{-ik(x+q_t)}}{\sqrt{\omega(k)}}\hat{\varphi}(k)(\sqrt{\hbar}a^\dagger(k)+\bar{u}_t(k)) + \frac{e^{+ik(x+q_t)}}{\sqrt{\omega(k)}}\bar{\hat{\varphi}}(k)(\sqrt{\hbar}a(k)+u_t(k))\right\}\mathrm{d}k \\
=& \frac{(\hbar D_x+p_t)^2}{2m} + V(x+q_t) + U(x+q_t,u_t) + \hbar H_{\rm f} + \int_{\mathbb{R}^3}\omega(k)|u_t(k)|^2\mathrm{d}k \\
&+ \sqrt{\hbar}\sqrt{2}\phi(\omega u_t) + \sqrt{\hbar}H_{\rm I}(x+q_t).
\end{aligned} \tag{3.3}$$

We shall estimate the term $\xi_\hbar\left(\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)+\frac{i}{\hbar}Q_{\hbar,s}+C_s-\frac{i}{\hbar}H_\hbar(s)\right)\xi_\hbar^*$. Since

$$\xi_\hbar\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\xi_\hbar^*=\mathcal{H}(q_s,p_s,u_s,\bar{u}_s),$$

we investigate

$$\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)+\frac{i}{\hbar}\xi_\hbar Q_{\hbar,s}\xi_\hbar^*+\xi_\hbar\left(C_s-\frac{i}{\hbar}H_\hbar(s)\right)\xi_\hbar^*.$$

We can directly compute C_t as

$$C_t=-\frac{i}{\hbar}\left(\dot{p}_t x-\hbar\dot{q}_t D_x\right)-\frac{1}{\sqrt{\hbar}}\left\{a^\dagger(\dot{u}_t)-a(\overline{\dot{u}_t})\right\}.$$

Note that $\xi_\hbar x\xi_\hbar^*=x\sqrt{\hbar}$ and $\xi_\hbar D_x\xi_\hbar^*=D_x/\sqrt{\hbar}$. Then

$$\xi_\hbar C_t\xi_\hbar^*=-\frac{i}{\sqrt{\hbar}}\left\{\dot{p}_t x-\dot{q}_t D_x-(a^\dagger(i\dot{u}_t)+a(\overline{i\dot{u}_t}))\right\}.$$

Next we compute $\xi_\hbar H_\hbar(t)\xi_\hbar^*$. By (3.3) we have

$$\begin{aligned}\xi_\hbar H_\hbar(t)\xi_\hbar^*=&\frac{(\sqrt{\hbar}D_x+p_t)^2}{2m}+V(\sqrt{\hbar}x+q_t)+U(\sqrt{\hbar}x+q_t,u_t)\\&+\sqrt{\hbar}H_{\mathrm{I}}(\hbar x+q_t)+\sqrt{\hbar}\sqrt{2}\phi(\omega u_t)+\int_{\mathbb{R}^3}\omega(k)|u_t(k)|^2\mathrm{d}k+\hbar H_{\mathrm{f}}.\end{aligned}$$

By

$$\begin{aligned}&V(\sqrt{\hbar}x+q_t)=V(q_t)+\sqrt{\hbar}\nabla V(q_t)x+\frac{1}{2}\hbar x\cdot\nabla^2V(q_t)x+O(\hbar^{3/2}),\\&U(\sqrt{\hbar}x+q_t,u_t)=U(q_t,u_t)+\sqrt{\hbar}\nabla U(q_t,u_t)x+\frac{1}{2}\hbar x\cdot\nabla^2U(q_t,u_t)x+O(\hbar^{3/2}),\\&\sqrt{\hbar}H_{\mathrm{I}}(\sqrt{\hbar}x+q_t)=\sqrt{\hbar}H_{\mathrm{I}}(q_t)+\hbar\nabla H_{\mathrm{I}}(q_t)x+O(\hbar^{3/2}),\end{aligned}$$

we see that

$$\begin{aligned}&\xi_\hbar\left(C_t-\frac{i}{\hbar}H_\hbar(t)\right)\xi_\hbar^*\\&=-i\left\{\frac{1}{2m}D_x^2+\frac{1}{2}x\cdot\nabla^2V(q_t)x+\frac{1}{2}x\cdot\nabla U(q_t,u_t)x+\nabla H_{\mathrm{I}}(q_t)x+H_{\mathrm{f}}\right\}\\&-\frac{i}{\sqrt{\hbar}}\left\{\frac{1}{m}pD_x+\nabla V(q_t)x+\nabla U(q_t,u_t)x+\sqrt{2}\phi(\sqrt{\omega}u_t)+H_{\mathrm{I}}(q_t)\right.\\&\qquad\qquad\left.+\dot{p}_t x-\dot{q}_t D_x-(a^\dagger(i\dot{u}_t)+a(\overline{i\dot{u}_t}))\right\}\\&-\frac{i}{\hbar}\left\{\frac{1}{2m}p_t^2+V(q_t)+\int\omega(k)|u_t(k)|^2\mathrm{d}k+U(q_t,u_t)\right\}+O(\sqrt{\hbar}).\end{aligned}$$

The second term of the right-hand side above is identically zero by equation (N). Hence

$$\xi_\hbar \left(C_t - \frac{i}{\hbar} H_\hbar(t) \right) \xi_\hbar^* = -\frac{i}{\hbar} \xi_\hbar Q_{\hbar,t} \xi_\hbar^* - \frac{i}{\hbar} \mathcal{H}(q_t, p_t, u_t, \bar{u}_t) + O(\sqrt{\hbar}). \tag{3.4}$$

It follows that

$$\begin{aligned} &\|\nu_t - \nu_0\| \\ &\leq \int_0^t \left\| \left(\frac{i}{\hbar} \mathcal{H}(q_s, p_s, u_s, \bar{u}_s) + \frac{i}{\hbar} \xi_\hbar Q_{\hbar,s} \xi_\hbar^* + \xi_\hbar (C_s - \frac{i}{\hbar} H_\hbar(s)) \xi_\hbar^* \right) \xi_\hbar T^*_{q_s,p_s,u_s} e^{-i\frac{s}{\hbar} H_\hbar} \xi_\hbar^* \Phi_1 \right\| \mathrm{d}s \\ &\leq tC\sqrt{\hbar} \|\Phi_1\| \end{aligned}$$

with some constant $C > 0$ by (3.4). Then the theorem follows. ■

4 Wigner measures

In this section we introduce Wigner measures on the phase space $\mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$ appearing in the semi-classical limits of a family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$. This has been studied in e.g., [7, 3].

4.1 Examples

We recall that

$$T^\hbar_{q_t,p_t,u_t} = T\left(\frac{z_t}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u_t}{\sqrt{\hbar}}\right),$$

where $z_t = q_t + ip_t \in \mathbb{R}^3 + i\mathbb{R}^3$ and $u_t \in L^2(\mathbb{R}^3)$ are the solution to (N). In the previous section we consider the asymptotic behavior of $T^\hbar_{q_t,p_t,u_t}$ as $\hbar \to 0$ in the sense of Theorem 3.1. Note that $\|T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar\| = 1$ but $(\Phi_\hbar, T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar) \to 0$ as $\hbar \to 0$.

In this section the following strategy is taken to analyze the asymptotic behavior of coherent vector $T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar$ as $\hbar \to 0$. For each $z = q + ip \in \mathbb{R}^3 + i\mathbb{R}^3$ and $u \in L^2(\mathbb{R}^3)$, we define the trace class operator $\mathcal{C}_\hbar(z,u)$ by

$$\mathcal{C}_\hbar = \mathcal{C}_\hbar(z,u) = |T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar\rangle\langle T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar|.$$

This is a one-rank operator. Let $z' = q' + ip' \in \mathbb{R}^3 + i\mathbb{R}^3$ nd $u' \in L^2(\mathbb{R}^3)$. We prepare the operator

$$\mathcal{W} = \mathcal{W}(z', u') = T(2\pi i z') \otimes W(\sqrt{2}\pi i\sqrt{\hbar}u') = e^{2\pi i(q'x + p'\hbar D_x)} \otimes e^{-\sqrt{2}\pi i\sqrt{\hbar}\phi(u')}.$$

We consider the asymptotic behaviour of the trace $\mathrm{Tr}(\mathcal{C}_\hbar \mathcal{W})$.

Lemma 4.1 *Let $z = q + ip, z' = q' + ip' \in \mathbb{R}^3 + i\mathbb{R}^3$ and $u, u' \in L^2(\mathbb{R}^3)$. Then it follows that*

$$\lim_{\hbar\to 0} \mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u')) = e^{2\pi i \,\mathrm{Re}((u,u')+\bar{z}\cdot z')}.$$

Proof: The formulae $W(f)^* = W(-f)$ and $T(z)^* = T(-z)$, and

$$(W(f)\Omega, W(ig)W(f)\Omega) = (\Omega, W(ig)\Omega)e^{i\,\mathrm{Re}(f,g)}$$

and

$$(T(z)\psi, T(iz')T(z)\psi) = (\psi, T(iz')\psi)e^{i\,\mathrm{Re}\bar{z}\cdot z'}$$

are useful. We see that $\mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u'))$ can be decomposed into two factors:

$$\begin{aligned}&\mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u'))\\ &= (T\left(\frac{z}{\hbar}\right)\psi_\hbar, T(2\pi i z')T\left(\frac{z}{\hbar}\right)\psi_\hbar)\cdot(W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega).\end{aligned}$$

Then the field part turns out to be

$$(W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega) = (\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')\Omega)e^{2\pi i\,\mathrm{Re}(u,u')}$$

and the particle part

$$(T\left(\frac{z}{\hbar}\right)\psi_\hbar, T(2\pi i z')T\left(\frac{z}{\hbar}\right)\psi_\hbar) = (\psi_\hbar, T(2\pi i z')\psi_\hbar)e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'}.$$

We also see that

$$\lim_{\hbar\to 0}(\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')\Omega)e^{2\pi i\,\mathrm{Re}(u,u')} = e^{2\pi i\,\mathrm{Re}(u,u')}.$$

Since $\psi_\hbar^2 \to \delta(x)$ and $T(2\pi i z') \to e^{2\pi i q' x}$ as $\hbar \to 0$, we can see that

$$\lim_{\hbar\to 0}(\psi_\hbar, T(2\pi i z')\psi_\hbar)e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'} = e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'}.$$

Then the lemma is proven. ■

4.2 Wigner measures

Let $X = \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$. Set

$$(\xi, \xi')_X = qq' + pp' + i(qp' - pq') + (u, u')$$

for $\xi = (q,p,u) \in X$ and $\xi' = (q',p',u') \in X$. We define $\mathcal{W}(\xi') = \mathcal{W}(z',u') = \mathcal{W}(q',p',u')$ and $\mathcal{C}_\hbar(\xi) = \mathcal{C}_\hbar(z,u) = \mathcal{C}_\hbar(q,p,u)$. Then the statements of Lemma 4.1 can be rewritten as

$$\lim_{\hbar\to 0} \mathrm{Tr}(\mathcal{C}_\hbar(\xi)\mathcal{W}(\xi')) = e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X}.$$

Furthermore

$$e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X} = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_\xi(x),$$

where $\mu_\xi(x)$ is the Dirac measure $\delta_\xi(x)$ on the phase space X with mass at $x = \xi$. This is called the Wigner measure associated with $\{\mathcal{C}_\hbar(\xi), \hbar \in (0,1)\}$. In [2] we consider Wigner measures μ_0 associated with a general family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$ on the total Hilbert space $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. I.e.,

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar\mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_0(x).$$

The existence and the uniqueness of the measure μ_0 associated with $\{\rho_\hbar, \hbar \in (0,1)\}$ are established in [2] but for the Pauli-Fierz model which is rather complicated than the Nelson model.

We can show that any Borel probability measure μ on X is a Wigner measure. We define the family of trace class operators by

$$\rho_\hbar = \int_X \mathcal{C}_\hbar(\xi) d\mu(\xi), \quad \hbar \in (0,1).$$

Proposition 4.2 [2, Lemma 4.3] *The Wigner measure of* $\{\rho_\hbar, \hbar \in (0,1)\}$ *is* μ.

Proof: It is straightforward to see that

$$\mathrm{Tr}[\rho_\hbar \mathcal{W}(\xi')] = \int_X \mathrm{Tr}(\mathcal{C}_\hbar(\xi)\mathcal{W}(\xi'))d\mu(\xi) \to \int_X e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X} d\mu(\xi).$$

Then the proposition follows. ■

4.3 Time evolution of Wigner measures and flows

The time evolution of the Wigner measure is given by

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar(t)\mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_t(x),$$

where

$$\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar} \rho_\hbar e^{i\frac{t}{\hbar}H_\hbar}.$$

Here we give an example. Fix $\xi = (z, u) = (q, p, u) \in X$. Let

$$\mathcal{C}_\hbar(\xi)(t) = e^{-i\frac{t}{\hbar}H_\hbar}\mathcal{C}_\hbar(\xi)e^{i\frac{t}{\hbar}H_\hbar}.$$

We set

$$T_\xi = T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right),$$
$$T_{\xi_t} = T\left(\frac{z_t}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u_t}{\sqrt{\hbar}}\right).$$

Here $\xi_t = (z_t, u_t) = (q_t, p_t, u_t) \in X$ is the solution to (N) with the initial condition $\xi_0 = \xi = (z, u) = (q, p, u) \in X$. Since $\mathcal{C}_\hbar(\xi) = |T_\xi\Phi_\hbar\rangle\langle T_\xi\Phi_\hbar|$, we have

$$\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = (T_\xi\Phi_\hbar, e^{i\frac{t}{\hbar}H_\hbar}\mathcal{W}(\xi')e^{-i\frac{t}{\hbar}H_\hbar}T_\xi\Phi_\hbar), \tag{4.1}$$
$$\mathrm{Tr}(\mathcal{C}_\hbar(\xi_t)\mathcal{W}(\xi')) = (T_{\xi_t}\Phi_\hbar, \mathcal{W}(\xi')T_{\xi_t}\Phi_\hbar). \tag{4.2}$$

By Theorem 3.1, we can see that

$$e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s} \sim T_{\xi_t}e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}T_\xi^* \tag{4.3}$$

in a semi-classical region. Let us define

$$\hat{H}_\hbar = H_\hbar - \mathcal{H}(q_s, p_s, u_s, \bar{u}_s),$$
$$Q_t = \frac{1}{2m}D_x^2 + \frac{1}{2}x \cdot (\nabla^2 V(q_t) + \nabla^2 U(q_t, u_t))x + \phi\left(-ike^{-ikq_t}\frac{\hat{\varphi}}{\sqrt{\omega}}\right)x + H_{\mathrm{f}}.$$

Thus Q_t is quadratic and independent of $\hbar$. Note that

$$\xi_\hbar e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}\xi_\hbar^* = e^{-i\int_0^t Q_s\mathrm{d}s}$$

and in particular $e^{-i\int_0^t Q_s\mathrm{d}s}$ is independent of $\hbar$. By (4.3) we have a corollary.

Corollary 4.3 *It follows that*

$$e^{-i\int_0^t \frac{1}{\hbar}\hat{H}_\hbar\mathrm{d}s} \sim T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s\mathrm{d}s}\xi_\hbar T_{\xi_0}^*, \quad \hbar \to 0. \tag{4.4}$$

Here $A \sim B$ *means that* $\lim_{\hbar\to 0}\|A\Phi - B\Phi\| = 0$.

Hence

$$\begin{aligned}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) &\sim (T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{ds}}\xi_\hbar\Phi_\hbar, \mathcal{W}(\xi')T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{ds}}\xi_\hbar\Phi_\hbar) \\ &= (\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{ds}}\Phi_1, T_{\xi_t}^*\mathcal{W}(\xi')T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{ds}}\Phi_1) \\ &= (e^{-i\int_0^t Q_s \mathrm{ds}}\Phi_1, \xi_\hbar\mathcal{W}(\xi')\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{ds}}\Phi_1)e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)}.\end{aligned}$$

Furthermore

$$\xi_\hbar\mathcal{W}(\xi')\xi_\hbar^* = e^{2\pi i\sqrt{\hbar}(q'x+p'D_x)} \otimes e^{-\sqrt{2}\pi i\sqrt{\hbar}\phi(u')} \to \mathbb{1}$$

as $\hbar \to 0$. Then

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \|\Phi_1\|^2 e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)} = e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)}. \tag{4.5}$$

(4.5) has been rigorously proven and ultimately generalized in [2, Theorem 1.4].

A relationship between μ_0 and μ_t is given through solutions to (N). Let $\Phi_t : X \to X$ be such that $\xi_t = \Phi_t(\xi)$ is the solution to (N) with the initial condition $\xi_0 = \xi$.

Theorem 4.4 [2, Theorem 1.4] *It follows that* $\mu_t(\cdot) = \mu_0 \circ \Phi_t^{-1}(\cdot)$.

By this we can see that

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \int_X e^{2\pi i\,\mathrm{Re}(x,\xi')_X} d\mu_\xi \circ \Phi_t^{-1}(x) \tag{4.6}$$

and hence

$$\int_X e^{2\pi i\,\mathrm{Re}(x,\xi')_X} d\mu_\xi \circ \Phi_t^{-1}(x) = e^{2\pi i\,\mathrm{Re}(\xi_t,\xi')_X}.$$

Then (4.5) follows. As a corollary we can see that

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi_t)\mathcal{W}(\xi')).$$

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