Structure of bicentralizer algebras and inclusions of type III factors

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概 要

We investigate the structure of the relative bicentralizer algebra $\mathrm{B}(N \subset M, \varphi)$ for inclusions of von Neumann algebras with normal expectation where N is a type III_1 subfactor and $\varphi \in N_*$ is a faithful state. We first construct a canonical flow $\beta^\varphi : \mathbf{R}_+^* \curvearrowright \mathrm{B}(N \subset M, \varphi)$ on the relative bicentralizer algebra and we show that the W*-dynamical system $(\mathrm{B}(N \subset M, \varphi), \beta^\varphi)$ is independent of the choice of φ up to a canonical isomorphism. In the case when $N = M$, we deduce new results on the structure of the automorphism group of $\mathrm{B}(M, \varphi)$ and we relate the period of the flow β^φ to the tensorial absorption of Powers factors. For general irreducible inclusions $N \subset M$, we relate the ergodicity of the flow β^φ to the existence of irreducible hyperfinite subfactors in M that sit with normal expectation in N. When the inclusion $N \subset M$ is discrete, we prove a relative bicentralizer theorem and we use it to solve Kadison's problem when N is amenable.

This is a set of notes for my talk at RIMS conference 量子場の数理とその周辺2021.

This talk is based on a joint work with Uffe Haagerup, Cyril Houdayer and Amine Marrakchi [AHHM18].

1. Connes' Classification of hyperfinite factors and the bicentralizer problem

A von Neumann algebra M on a Hilbert space H is called *hyperfinite* if there exists an increasing sequence $M_1 \subset M_2 \subset \cdots \subset M$ of finite-dimensional $*$-subalgebras of M whose union is dense in the strong operator topology (SOT). Hyperfinite factors forms one of the most important class of factors in von Neumann algebra theory. While it has the simplest structure among factors, they appear in quite a few places in application. Already Murray and von Neumann showed that there exists only one hyperfinite factor of type II_1, denoted by R, up to $*$-isomorphism. However, their argument does not apply to prove the uniqueness result for e.g. type II_∞ factors and it was a long standing open problem to prove that there exist only one hyperfintie II_∞ factor, namely the one $R_{0,1}$ constructed as an infinite tensor product of matrix algebras.

Because the hyperfiniteness is hard to check, one needs to find a characterization of hyperfiniteness that does not involve finite-dimensional $*$-subalgebras. Many mathematicians have worked on this problem and several conditions (injectivity, semi-discreteness, Schwarz' property, $\cdots$) that imply hyperfiniteness have been introduced. To cut a long story short, Connes [Co75b] showed that all these properties, especially the injectivity, are equivalent to hyperfiniteness. M is said to be *injective* if there exists a norm one projection $\mathrm{E} \colon \mathbb{B}(H) \to M$. Thanks to Tomiyama's Theorem, E is a condi-

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tional expectation although it is typically non-normal (in fact, E can be chosen to be normal if and only if M is atomic). The equivalence of injectivity and hyperfiniteness immediately leads to the theorem that there exists only one factors of type II_1 and II_∞ up to $*$-isomorphism, and any subfactor of the hyperfinite II_1 factor R is again hyperfinite. On the other hand, in the early 1970's Tomita–Takesaki theory has been invented. This led Connes and Takesaki to their structure theorem for type III factors. By Connes' structure theorem for type III factors, any type III_λ $(0 \leq \lambda < 1)$ factor is of the form $M = N \rtimes_\theta \mathbf{Z}$ where N is of type II_∞ (and is a factor if $\lambda \neq 0$) with θ a centrally ergodic action on N which scales down a semifinite trace τ of N. For the type III_1 case, thanks to Takesaki's duality theorem, there is a continuous decomposition $M = N \rtimes_\theta \mathbf{R}$ where N is a type II_∞ factor and θ is a flow on N scaling the semifinite trace τ. Then the classification of hyperfinite type III factors is reduced to the classification of hyperfinite type II_∞ factors and of the actions of $\mathbf{Z}$ (or $\mathbf{R}$ for the III_1 case) on them. Then by the classification of automorphisms of R and of $R_{0,1}$, Connes [Co72, Co74b, Co75a, Co75b, Co75c, CT76, Ta73] showed that for each $0 < \lambda < 1$, there exists only one hyperfinite III_λ factor, denoted R_λ and together with the work of Krieger [Kr76] on ergodic flows, he showed that the isomorphism classes of hyperfinite III_0 factors are in 1-1 correspondence with the isomorphism classes of properly ergodic flows. There only remained the type III_1 case. In order to settle the III_1 case, he found several strategies to prove the uniqueness. Among them, he discovered the following [Co85]: let M be an injective III_1 factor with separable predual and fix a faithful state $\varphi \in M_*$. Let $T = -\frac{2\pi}{\log \lambda}$. Then $N = M \rtimes_{\sigma_T^\varphi} \mathbf{Z}$ is an injective type III_λ factor, hence $N \cong R_\lambda$ and if we let θ the dual action of σ_T^φ, then $M \cong R_\lambda \rtimes_\theta \mathbb{T}$. Then if one shows the uniqueness of the $\mathbb{T}$ action on R_λ, the uniqueness result for the III_1 factor follows. He then showed that this can be achieved if one can show that $\sigma_T^\varphi \in \overline{\mathrm{Inn}}(M)$ (the approximately inner automorphisms). For an automorphism α of a factor N with separable predual, consider the following conditions:

(i) $\alpha \in \overline{\mathrm{Inn}}(N)$.

(ii) $\alpha \odot \mathrm{id} \in \mathrm{Aut}(N \odot N^{\mathrm{op}})$ extends to an automorphism of the C*-algebra $\mathrm{C}^*_{\lambda \cdot \rho}(N)$ generated by the standard representation of $N \odot N^{\mathrm{op}}$ on $\mathrm{L}^2(N)$ given by

$$(a \otimes b^{\mathrm{op}}) \cdot \xi := aJb^*J\xi, \quad a, b \in N, \quad \xi \in \mathrm{L}^2(N).$$

Here, we fix a standard form $(N, \mathrm{L}^2(N), J, \mathrm{L}^2(N)^+)$. Then always (i)⇒(ii), and when $N = M$ is an injective type III_1 factor, (ii) is satisfied for every α, and (ii)⇒(i) follows if in addition the bicentralizer $\mathrm{B}(M, \varphi)$ is trivial $(= \mathbb{C})$. Here, the *bicentralizer* of M with respect to φ is defined by

$$\mathrm{B}(M, \varphi) = \left\{ x \in M \mid xa_n - a_n x \overset{n \to \infty}{\to} 0 \text{ strongly}, \forall (a_n)_n \in \mathrm{AC}(M, \varphi) \right\}$$

where

$$\mathrm{AC}(M, \varphi) = \left\{ (a_n)_n \in \ell^\infty(\mathbf{N}, M) \mid \lim_{n \to \infty} \|a_n \varphi - \varphi a_n\| = 0 \right\}$$

is the *asymptotic centralizer* of φ. The question of the triviality of the bicentralizer was solved affirmatively by Haagerup in [Ha85] for *amenable* M, thus settling the problem of the classification of amenable factors with separable predual (see [Co75b, Co85]). Connes also asked whether or not the bicentralizer is trivial for general type III_1 factors with separable predual.

Nowadays, the bicentralizer problem is still of premium importance. Indeed by [Ha85, Theorem 3.1], for any type III_1 factor M with separable predual, M has trivial bicentralizer if and only if there exists a faithful state $\varphi \in M_*$ with an irreducible centralizer, meaning that $(M_\varphi)' \cap M = \mathbf{C}1$. Then by [Ha85, Theorem 3.1] and [Po81, Theorem 3.2], M has trivial bicentralizer if and only if there exists a maximal abelian subalgebra $A \subset M$ that is the range of a normal conditional expectation (see [Ta71, Question] where the problem of finding such maximal abelian subalgebras is mentioned). For these reasons, Connes' bicentralizer problem appears naturally when one tries to use Popa's deformation/rigidity theory in the type III context (see for instance [HI15, Theorem C]). The bicentralizer problem is known to have a positive solution for particular classes of nonamenable type III_1 factors: factors with a Cartan subalgebra; Shlyakhtenko's free Araki–Woods factors ([Ho08]); (semi-)solid factors ([HI15]); free product factors ([HU15]). However, the bicentralizer problem is still wide open for arbitrary type III_1 factors.

2. Connes' isomorphism $\beta_{\psi,\varphi}$ and the bicentralizer flow β^φ

2.1. The relative bicentralizer flow $\beta^\varphi \curvearrowright \mathrm{B}(N \subset M, \varphi)$

In his attempt to solve the bicentralizer problem, Connes observed that for any type III_1 factor M, the bicentralizer $\mathrm{B}(M, \varphi)$ does not depend on the choice of the state φ up to a canonical isomorphism. In around 2012–2013, Haagerup found out that the idea of Connes' isomorphism (denoted by $\beta_{\psi,\varphi}$ below) can be enhanced to construct a canonical flow (u-continuous action) $\beta^\varphi \colon \mathbf{R}^*_+ \curvearrowright \mathrm{B}(M, \varphi)$ with interesting properties. This flow was independently discovered by Marrakchi and this was the starting point of our joint research.

Let $N \subset M$ be any inclusion of σ-finite von Neumann algebras *with expectation*, meaning that there exists a faithful normal conditional expectation $\mathrm{E}_N : M \to N$. Following [Ma03, Definition 4.1], we define the *relative bicentralizer* $\mathrm{B}(N \subset M, \varphi)$ of the inclusion $N \subset M$ with respect to the faithful state $\varphi \in N_*$ by

$$\mathrm{B}(N \subset M, \varphi) = \left\{ x \in M \mid xa_n - a_n x \overset{n\to\infty}{\to} 0 \text{ strongly}, \forall (a_n)_n \in \mathrm{AC}(N, \varphi) \right\}.$$

Observe that we always have $N' \cap M \subset \mathrm{B}(N \subset M, \varphi) \subset (N_\varphi)' \cap M$. When $N = M$, we simply have $\mathrm{B}(N \subset M, \varphi) = \mathrm{B}(M, \varphi)$.

Our first main result deals with the construction of the canonical flow on the relative bicentralizer $\mathrm{B}(N \subset M, \varphi)$.

Theorem A. Let $N \subset M$ be any inclusion of σ-finite von Neumann algebras with expectation. Assume that N is a type III_1 factor. Then the following assertions hold:

(i) For every pair of faithful states $\varphi, \psi \in N_*$, there exists a canonical isomorphism

$$\beta_{\psi,\varphi} : \mathrm{B}(N \subset M, \varphi) \to \mathrm{B}(N \subset M, \psi)$$

characterized by the following property: for any uniformly bounded sequence $(a_n)_{n\in\mathbf{N}}$ in N and any $x \in \mathrm{B}(N \subset M, \varphi)$, we have

$$\|a_n\varphi - \psi a_n\| \overset{n\to\infty}{\to} 0 \quad \Rightarrow \quad a_n x - \beta_{\psi,\varphi}(x)a_n \overset{n\to\infty}{\to} 0 \ \ast\text{-strongly}.$$

(ii) There exists a canonical flow

$$\beta^{\varphi} : \mathbf{R}^*_+ \curvearrowright \mathrm{B}(N \subset M, \varphi)$$

characterized by the following property: for any uniformly bounded sequence $(a_n)_{n\in\mathbf{N}}$ in N, any $x \in \mathrm{B}(N \subset M, \varphi)$ and any $\lambda > 0$, we have

$$\|a_n\varphi - \lambda\varphi a_n\| \overset{n\to\infty}{\to} 0 \quad \Rightarrow \quad a_n x - \beta^{\varphi}_{\lambda}(x)a_n \overset{n\to\infty}{\to} 0 \ \ast\text{-strongly}.$$

(iii) We have $\beta_{\varphi_3,\varphi_2} \circ \beta_{\varphi_2,\varphi_1} = \beta_{\varphi_3,\varphi_1}$ for every faithful state $\varphi_i \in N_*$, $i \in \{1,2,3\}$, and $\beta^{\psi}_{\lambda} \circ \beta_{\psi,\varphi} = \beta_{\psi,\varphi} \circ \beta^{\varphi}_{\lambda}$ for every pair of faithful states $\psi, \varphi \in N_*$ and every $\lambda > 0$.

(iv) For every pair of faithful states $\psi, \varphi \in N_*$ and every $\lambda > 0$, we have

$$\mathrm{E}^{\psi}_{N'\cap M} \circ \beta_{\psi,\varphi} = \mathrm{E}^{\varphi}_{N'\cap M} = \mathrm{E}^{\varphi}_{N'\cap M} \circ \beta^{\varphi}_{\lambda}$$

where $\mathrm{E}^{\varphi}_{N'\cap M} : M \to N' \cap M$ is the unique normal conditional expectation such that $\mathrm{E}^{\varphi}_{N'\cap M}(x) = \varphi(x)1$ for all $x \in N$.

The proof of Theorem A uses ultraproduct von Neumann algebras [Oc85, AH12] and relies on Connes–Størmer transitivity theorem [CS76] and the fact that any $\lambda > 0$ is an approximate eigenvalue for the faithful state $\varphi \in N_*$ (see Lemma 8).

The meaning of the compatibility relations given in item (iii) is that the W*-dynamical system $(\mathrm{B}(N \subset M, \varphi), \beta^{\varphi})$ does not depend on the choice of $\varphi \in N_*$ up to the canonical isomorphism $\beta_{\psi,\varphi}$. Thus, $(\mathrm{B}(N \subset M, \varphi), \beta^{\varphi})$ is an invariant of the inclusion $N \subset M$. We call it the *relative bicentralizer flow* of the inclusion $N \subset M$. When $N = M$, we simply call it the *bicentralizer flow* of M. In this talk, we study this invariant and we relate it to some structural properties of the inclusion $N \subset M$. In these notes, we also give proofs to some basic facts needed to the construction of the relative bicentralizer flow but are omitted in [AHHM18].

Let $N \subset M$ be any irreducible inclusion of factors with separable predual and with expectation. In [Po81], Popa proved that if N is *semifinite*, then there exists a hyperfinite subfactor with expectation $P \subset N$ such that $P' \cap M = \mathbf{C}1$. We extend this theorem to the case when N is a type III_{λ} factor $(0 < \lambda < 1)$ in Theorem **??**. In the case when N is a type III_1 factor, we relate this question to the ergodicity of the relative bicentralizer flow.

Theorem B. Let $N \subset M$ be any inclusion of von Neumann algebras with separable predual and with expectation. Assume that N is a type III_1 factor. Let $\varphi \in N_*$ be any faithful state. The following assertions are equivalent:

(i) $\mathrm{B}(N \subset M, \varphi)^{\beta^\varphi} = N' \cap M$.

(ii) There exists a hyperfinite subfactor with expectation $P \subset N$ such that $P' \cap M = N' \cap M$.

We can always choose $P = R_\infty$ to be the hyperfinite type III_1 factor.

- We can moreover choose $P = R_\lambda$ to be the hyperfinite type III_λ factor $(0 < \lambda < 1)$ if and only if $\mathrm{B}(N \subset M, \varphi)^{\beta^\varphi_\lambda} = N' \cap M$.

- We can moreover choose $P = R$ to be the hyperfinite type II_1 factor if and only if $\mathrm{B}(N \subset M, \varphi) = N' \cap M$.

The proof of Theorem B generalizes the methods developed by Popa in [Po81, Theorem 3.2] and Haagerup in [Ha85, Theorem 3.1].

Following [Co72, Co74a], a σ-finite von Neumann algebra Q is *almost periodic* if Q possesses an *almost periodic* state, that is, a faithful normal state for which the corresponding modular operator is diagonalizable. By [Co72, Co74a], any σ-finite type III_λ factor with $0 \leq \lambda < 1$ is almost periodic. When $N \subset M$ is an irreducible inclusion of factors with separable predual and with expectation, a sufficient condition for the relative bicentralizer flow $\beta^\varphi : \mathbf{R}^*_+ \curvearrowright \mathrm{B}(N \subset M, \varphi)$ to be ergodic is the existence of an almost periodic subfactor with expectation $Q \subset N$ such that $Q' \cap M = \mathbf{C}1$. Using Theorem B, we derive the following application which is new even in the case when $N = M$.

Application 1. Let $N \subset M$ be any irreducible inclusion of factors with separable predual and with expectation. Assume that N is a type III_1 factor and that there exists an almost periodic subfactor with expectation $Q \subset N$ such that $Q' \cap M = \mathbf{C}1$.

Then there exists a hyperfinite subfactor with expectation $P \subset N$ such that $P' \cap M = \mathbf{C}1$.

We point out that it is unclear whether we can choose P as a subfactor of Q. We can do so if Q possesses an almost periodic faithful state $\varphi \in Q_*$ such that its centralizer Q_φ is a type II_1 factor. However, when Q is a type III_0 factor, no such almost periodic state exists on Q and so we really need to exploit the ergodicity of the relative bicentralizer flow to construct the AFD subfactor $P \subset N$.

A sufficient condition for an inclusion of factors $N \subset M$ to be irreducible is the existence of an abelian von Neumann subalgebra $A \subset N$ that is maximal abelian in M. One of Kadison's well-known problems in [Ka67] asks whether the converse is true as well. We will say that an irreducible inclusion of factors with expectation $N \subset M$ satisfies *Kadison's property* if there exists an abelian subalgebra with expectation $A \subset N$ that is maximal abelian in M.

Popa proved in [Po81, Theorem 3.2] that any irreducible inclusion $N \subset M$ with separable predual and with expectation such that N is semifinite satisfies Kadison's property. Combining Theorem B with [Po81, Theorem 3.2], we obtain the following characterization:

Corollary C. Let $N \subset M$ be any irreducible inclusion of factors with separable predual and with expectation. Assume that N is a type III_1 factor. Then the following assertions are equivalent:

(i) $\mathrm{B}(N \subset M, \varphi) = \mathbf{C}1$ for some (or any) faithful state $\varphi \in N_*$.

(ii) The inclusion $N \subset M$ satisfies Kadison's property.

In the case when $N \subset M$ has finite index, Corollary C follows from [Po95, Theorem 4.2]. In order to find new examples of inclusions $N \subset M$ that satisfy Kadison's property, we will prove a relative bicentralizer theorem for *discrete* inclusions.

Finally, let us point out that in the case $M = N$, a breakthrough result has been obtained by Marrakchi:

Theorem 1 (Marrakchi [Ma20])**.** *Let M be a type III_1 factor with separable predual and φ a faithful normal state on M. Then the bicentralizer flow β^φ is ergodic. Furthermore, M has trivial bicentralizer if $M \cong M \overline{\otimes} R_\lambda$ holds for some $0 < \lambda < 1$.*

2.2. Ocneanu ultrapower M^ω and the Groh–Raynaud ultrapower M^ω_{GR}

Let M be a von Neumann algebra with a faithful state $\varphi \in M_*$. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Let $\mathcal{I}_\omega$ be the C*-algebra of all bounded sequences $(x_n)_n$ in M which tends to 0 $*$-strongly along ω: $\lim_{n\to\omega} \|x_n\|_\varphi^\sharp = 0$. Let $\mathcal{M}^\omega = \{x \in \ell^\infty(M) \mid x\mathcal{I}_\omega + \mathcal{I}_\omega x \subset \mathcal{I}_\omega\}$ be the normalizer of $\mathcal{I}_\omega$ in the C*-algebra $\ell^\infty(M)$ of all bounded sequences in M. The quotient C*-algebra $M^\omega := \mathcal{M}^\omega/\mathcal{I}_\omega$ is a W*-algebra (the *Ocneanu ultrapower* [Oc85]) equipped with the faithful normal state φ^ω given by the formula

$$\varphi^\omega((x_n)^\omega) = \lim_{n\to\omega} \varphi(x_n), \quad (x_n)^\omega \in M^\omega.$$

Here, $(x_n)^\omega$ is the element in M^ω represented by a sequence $(x_n)_n \in \mathcal{M}^\omega$. We will be using the following results repeatedly.

Theorem 2 ([AH12])**.** *Let M be a von Neumann algebra with a faithful state $\varphi \in M_*$. Then*

(i) $\sigma_t^{\varphi^\omega}((x_n)^\omega) = (\sigma_t^\varphi(x_n))^\omega$ *holds for every $t \in \mathbf{R}$ and $(x_n)^\omega \in M^\omega$. In particular, $(M_\varphi)^\omega \subset M^\omega_{\varphi^\omega}$ holds.*

(ii) *If M is a type III_1 factor, then M^ω is a type III_1 factor with strictly homogeneous state space, meaning tha any two faithful normal states on M^ω are unitarily equivalent.*

(iii) *If M is a type III_1 factor, then $M^\omega_{\varphi^\omega}$ is a type II_1 factor and is independent of φ up to a conjugation by a unitary in M^ω.*

Let $A = (M, \|\cdot\|_\infty)^\omega$ be the ultrapower Banach space of M with respect to ω. Then A is naturally a C*-algebra but it is not a von Neumann algebra in general. Let A^{**} be the bidual of A which is a von Neumann algebra. Let $(M_*)^\omega$ be the ultraproduct Banach space of M_*. Then $(M_*)^\omega$ can be naturally identified with a closed subspace of A^* via the embedding

$$(\varphi_n)^\omega \mapsto \Big((x_n)^\omega \mapsto \lim_{n\to\omega} \varphi_n(x_n)\Big)$$

The orthogonal of $(M_*)^\omega$ in A^{**} defined by

$$\mathfrak{J} = \{x \in A^{**} \mid \forall \psi \in (M_*)^\omega,\ \psi(x) = 0\}$$

is a weak* closed ideal in the von Neumann algebra A^{**} which means that the quotient $M^\omega_{GR} = A^{**}/\mathfrak{J}$ is a von Neumann algebra. It is called the *Groh–Raynaud ultrapower* [Ra02] of M with respect to ω. By construction, the predual of M^ω_{GR} is exactly $(M_*)^\omega$ and M^ω_{GR} contains the ultrapower Banach space $A = (M, \|\cdot\|_\infty)^\omega$ as a dense C*-subalgebra. The $*$-homomorphism $M \to M^\omega_{GR} : x \mapsto x^\omega$ is not normal in general and so M is not a von Neumann subalgebra of M^ω_{GR}. The von Neumann algebra M^ω_{GR} is very large (not separable and not even σ-finite in general). The main interest in this ultraproduct comes from the fact that, as explained in [AH12], there is a natural identification $\mathrm{L}^2(M^\omega_{GR}) = \mathrm{L}^2(M)^\omega$. We have $\varphi^\omega \in (M^\omega_{GR})^+_*$ but φ^ω is not faithful in general. Let e be the support of φ^ω in M^ω_{GR}. The projection e does not depend on the choice of φ and the corner $e(M^\omega_{GR})e$ is isomorphic to the Ocneanu ultrapower M^ω [AH12]. By the identification $e(M^\omega_{\mathrm{GR}})e = M^\omega$, we have $(M^\omega)_* = e(M_*)^\omega e$ and $\mathrm{L}^2(M^\omega) = e(\mathrm{L}^2(M)^\omega)e$.

2.3. Iterated Ultrapower

Let I, J be directed sets, and let $\mathcal{U}, \mathcal{V}$ be cofinal ultrafilters on I and J, respectively. Then the *product ultrafilter*, denoted $\mathcal{U} \otimes \mathcal{V}$ is a filter on $I \times J$ (with the partial ordering $(i, j) \le (i', j')$ if $i \le i'$ and $j \le j'$) given by

$$\mathcal{U} \otimes \mathcal{V} = \{A \subset I \times J; \{i \in I; \{j \in J; (i, j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

The next lemma is well-known in the theory of ultrafilters and can be checked by a straightforward computations.

Lemma 3. *$\mathcal{U} \otimes \mathcal{V}$ is a cofinal ultrafilter on $I \times J$. Moreover, if $(x_{i,j})_{(i,j)\in I\times J}$ is a doubly indexed sequence in a compact Hausdorff space X, then*

$$\lim_{(i,j)\to\mathcal{U}\otimes\mathcal{V}} x_{i,j} = \lim_{i\to\mathcal{U}} \lim_{j\to\mathcal{V}} x_{i,j}.$$

Proof. The result is well-known, but for the convenience of the reader we include its proof. For $X \subset I \times J$ and $i \in I$, we write $X_i = \{j \in J;\ (i, j) \in X\}$. First, we show that $\mathcal{U} \otimes \mathcal{V}$ is a filter on $I \times J$. It is clear that $\emptyset \notin \mathcal{U} \otimes \mathcal{V}$. Let $A, B \subset I \times J$ be such that $A \subset B$ and $A \in \mathcal{U} \otimes \mathcal{V}$. Then for each $i \in I$, $A_i \subset B_i (\subset J)$ and $\{i \in I; A_i \in \mathcal{V}\} \in \mathcal{U}$.

This shows that $\{i \in I;\ B_i \in \mathcal{V}\} \in \mathcal{U}$, whence $B \in \mathcal{U} \otimes \mathcal{V}$. Next, let $A, B \in \mathcal{U} \otimes \mathcal{V}$. Then for each $i \in I$, we have $A_i \cap B_i = (A \cap B)_i$, whence

$$\{i \in I; A_i \in \mathcal{V}\} \cap \{i \in I;\ B_i \in \mathcal{V}\} \subset \{i \in I;\ (A \cap B)_i \in \mathcal{V}\},$$

which implies that $\{i \in I;\ (A \cap B)_i \in \mathcal{V}\} \in \mathcal{U}$. Therefore $A \cap B \in \mathcal{U} \otimes \mathcal{V}$. This shows that $\mathcal{U} \otimes \mathcal{V}$ is a filter on $I \times J$.

Next, let $(i_0, j_0) \in I \times J$ and let $S := \{(i,j) \in I \times J;\ i \geq i_0,\ j \geq j_0\}$. For each $i \in I$, $S_i = \begin{cases} \{j \in J;\ j \geq j_0\} & (i \geq i_0) \\ \emptyset & \text{(otherwise)} \end{cases}$, and if $i \geq i_0$, then $\{j \in J;\ j \geq j_0\} \in \mathcal{V}$ because $\mathcal{V}$ is cofinal. Thus $\{i \in I;\ S_i \in \mathcal{V}\} = \{i \in I;\ i \geq i_0\} \in \mathcal{U}$ because $\mathcal{U}$ is cofinal. Therefore $\mathcal{U} \otimes \mathcal{V}$ is cofinal. Finally, let $A \subset I \times J$ be such that $A \notin \mathcal{U} \otimes \mathcal{V}$. Then because $\mathcal{U}, \mathcal{V}$ are ultrafilters, we have

$$\begin{aligned} \{i \in I;\ A_i \in \mathcal{V}\} \notin \mathcal{U} &\Leftrightarrow \{i \in I;\ A_i \notin \mathcal{V}\} \in \mathcal{U} \\ &\Leftrightarrow \{i \in I;\ (J \setminus A_i) = (I \times J \setminus A)_i \in \mathcal{V}\} \in \mathcal{U}, \end{aligned}$$

and the last condition is equivalent to $I \times J \setminus A \in \mathcal{U} \otimes \mathcal{V}$. Therefore $\mathcal{U} \otimes \mathcal{V}$ is a cofinal ultrafilter on $I \times J$. This finishes the proof of the first assertion. We show the second assertion. Set $x := \lim_{(i,j) \to \mathcal{U} \otimes \mathcal{V}} x_{i,j}$ and $x_i := \lim_{j \to \mathcal{V}} x_{i,j}$ $(i \in I)$. Let W be an open neighborhood of x in X. Since a compact Hausdorff space is regular, there exists an open neighborhood W_1 of x such that $x \in W_1 \subset \overline{W_1} \subset W$. Then $\{(i,j) \in I \times J;\ x_{i,j} \in W_1\} \in \mathcal{U} \otimes \mathcal{V}$, whence $I_0 := \{i \in I;\ \{j \in J;\ x_{i,j} \in W_1\} \in \mathcal{V}\} \in \mathcal{U}$ holds. Let $i \in I_0$. Then $B := \{j \in J;\ x_{i,j} \in W_1\} \in \mathcal{V}$. If V is any open neighborhood of x_i, then $B' := \{j \in J;\ x_{i,j} \in V\} \in \mathcal{V}$, whence $B \cap B' \in \mathcal{V}$ holds. In particular, we can take $j \in B \cap B'$. Then $x_{i,j} \in V \cap W_1 \neq \emptyset$. Since V is arbitrary, this shows that $x_i \in \overline{W_1} \subset W$. Therefore $\mathcal{U} \ni I_0 \subset \{i \in I;\ x_i \in W\}$, which shows that $\{i \in I;\ x_i \in W\} \in \mathcal{U}$. Since W is arbitrary, we have $\lim_{i \to \mathcal{U}} x_i = x$. □

As a consequence of Lemma 3, we see that for any Banach space E, the natural isomorphism

$$\ell^\infty(I \times J, E) \ni (x_{i,j})_{(i,j) \in I \times J} \mapsto ((x_{i,j})_{j \in J})_{i \in I} \in \ell^\infty(I, \ell^\infty(J, E))$$

induces an isomorphism of the ultrapowers

$$E^{\mathcal{U} \otimes \mathcal{V}} \ni (x_{i,j})^{\mathcal{U} \otimes \mathcal{V}} \mapsto ((x_{i,j})^{\mathcal{V}})^{\mathcal{U}} \in (E^{\mathcal{V}})^{\mathcal{U}}.$$

If we apply this to $E = M_*$ where M is a von Neumann algebra, we obtain the following proposition which extends [CP12, Proposition 2.1] on iterated ultrapowers of II_1 factors to arbitrary σ-finite von Neumann algebras. We leave the details to the reader.

Proposition 4. *Let M be any σ-finite von Neumann algebra. There exists a natural isomorphism of the Groh–Raynaud ultrapowers*

$$\pi_{\mathrm{GR}} : M_{\mathrm{GR}}^{\mathcal{U} \otimes \mathcal{V}} \to (M_{\mathrm{GR}}^{\mathcal{V}})_{\mathrm{GR}}^{\mathcal{U}}$$

characterized by

$$\pi_{\rm GR}((x_{i,j})^{\mathcal{U}\otimes\mathcal{V}}) = ((x_{i,j})^{\mathcal{V}})^{\mathcal{U}} \text{ for all } (x_{i,j})_{(i,j)\in I\times J} \in \ell^\infty(I\times J, M).$$

Its predual map is the isomorphism

$$(\pi_{\rm GR})_* : (M_*)^{\mathcal{U}\otimes\mathcal{V}} \ni (\varphi_{i,j})^{\mathcal{U}\otimes\mathcal{V}} \mapsto ((\varphi_{i,j})^{\mathcal{V}})^{\mathcal{U}} \in ((M_*)^{\mathcal{V}})^{\mathcal{U}}.$$

In particular, $\pi_{\rm GR}$ restricts to an isomorphism between the Ocneanu corners

$$\pi : M^{\mathcal{U}\otimes\mathcal{V}} \to (M^{\mathcal{V}})^{\mathcal{U}}.$$

Recall that if $N \subset M$ is a von Neumann subalgebra with faithful normal conditional expectation $\mathrm{E}_N^M : M \to N$, then we have a natural embedding $N^{\mathcal{U}} \subset M^{\mathcal{U}}$ with faithful normal conditional expectation $\mathrm{E}_{N^{\mathcal{U}}}^{M^{\mathcal{U}}} = \left(\mathrm{E}_N^M\right)^{\mathcal{U}} : M^{\mathcal{U}} \to N^{\mathcal{U}}$.

The conditional expectations satisfy

$$\mathrm{E}_M^{M^{\mathcal{U}}} \circ \mathrm{E}_{N^{\mathcal{U}}}^{M^{\mathcal{U}}} = \mathrm{E}_{N^{\mathcal{U}}}^{M^{\mathcal{U}}} \circ \mathrm{E}_M^{M^{\mathcal{U}}} = \mathrm{E}_N^{N^{\mathcal{U}}} \circ \mathrm{E}_{N^{\mathcal{U}}}^{M^{\mathcal{U}}} = \mathrm{E}_N^M \circ \mathrm{E}_M^{M^{\mathcal{U}}}.$$

By applying this to the inclusion $M \subset M^{\mathcal{V}}$ with the canonical faithful normal conditional expectation $\mathrm{E}_M^{M^{\mathcal{V}}} : M^{\mathcal{V}} \to M$, we obtain the following result.

Proposition 5. *Let M be any σ-finite von Neumann algebra. Then we have a commuting square*

$$\begin{array}{ccccc} M & \subset & M^{\mathcal{V}} & & \\ \cap & & \cap & & \\ M^{\mathcal{U}} & \subset & (M^{\mathcal{V}})^{\mathcal{U}} & = M^{\mathcal{U}\otimes\mathcal{V}} \end{array}$$

where the canonical faithful normal conditional expectations satisfy

$$\mathrm{E}_M^{M^{\mathcal{U}\otimes\mathcal{V}}} = \mathrm{E}_{M^{\mathcal{V}}}^{(M^{\mathcal{V}})^{\mathcal{U}}} \circ \mathrm{E}_{M^{\mathcal{U}}}^{(M^{\mathcal{V}})^{\mathcal{U}}} = \mathrm{E}_{M^{\mathcal{U}}}^{(M^{\mathcal{V}})^{\mathcal{U}}} \circ \mathrm{E}_{M^{\mathcal{V}}}^{(M^{\mathcal{V}})^{\mathcal{U}}} = \mathrm{E}_M^{M^{\mathcal{U}}} \circ \mathrm{E}_{M^{\mathcal{U}}}^{(M^{\mathcal{V}})^{\mathcal{U}}} = \mathrm{E}_M^{M^{\mathcal{V}}} \circ \mathrm{E}_{M^{\mathcal{V}}}^{(M^{\mathcal{V}})^{\mathcal{U}}}.$$

In particular, we have $M^{\mathcal{U}} \cap M^{\mathcal{V}} = M$.

The following Lemma will be repeatedly used.

Lemma 6. *Let X be a Hausdorff space and let $(x_n)_{n=1}^\infty$ be a sequence in X. If there exists $x \in X$ such that $\lim_{n\to\omega} x_n = x$ for every $\omega \in \beta\mathbf{N}\setminus\mathbf{N}$, then $\lim_{n\to\infty} x_n = x$.*

Proof. Let $\omega_0 := \{A \subset \mathbf{N};\ \mathbf{N}\setminus A \text{ is finite}\}, \omega' := \{A \subset \mathbf{N};\ \forall\omega \in \beta\mathbf{N}\setminus\mathbf{N}\ [A \in \omega]\}$. We show that $\omega' = \omega_0$. Clearly $\omega_0 \subset \omega'$ holds. Assume by contradiction that there exists $A \in \omega'\setminus\omega_0$. Then $A^c = \mathbf{N}\setminus A$ is infinite by $A \notin \omega_0$. In particular, $S := \omega_0 \cup \{A^c\}$ is a family of subsets of $\mathbf{N}$ with finite intersection property. Therefore there exists a ultrafilter ω_1 extending S, which is free by $\omega_0 \subset \omega_1$. But $A \notin \omega_1$, contradicting $A \in \omega'$. Thus $\omega' = \omega_0$. Then if U is an open neighborhood of x in X, then for each $\omega \in \beta\mathbf{N}\setminus\mathbf{N}$, (ii) implies that $A := \{n \in \mathbf{N}; x_n \in U\}$ belongs to ω. Thus $A \in \omega' = \omega_0$. Thus there exists $N \in \mathbf{N}$ such that $x_n \in U$ $(n \geq N)$. Since U is arbitrary, $\lim_{n\to\infty} x_n = x$ holds. □

2.4. Construction of the β^φ

We sketch the construction of the relative bicentralizer flow. The relative bicentralizer $\mathrm{B}(N \subset M, \varphi)$ has the following ultraproduct interpretation.

Proposition 7. *Let $N \subset M$ be any inclusion of σ-finite von Neumann algebras with expectation. Let $\varphi \in N_*$ be any faithful state. For any nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$, we have*

$$\mathrm{B}(N \subset M, \varphi) = (N^\omega_{\varphi^\omega})' \cap M.$$

Proof. Let $x \in M$. If $x \notin \mathrm{B}(N \subset M, \varphi)$, then there exists $(a_n)_n \in \mathrm{AC}(N, \varphi)$ such that $\|[x, a_n]\|_\varphi$ does not tend to 0. By passing to a subsequence, we may assume that $\|[x, a_n]\|_\varphi \geq c > 0$ for each $n \in \mathbb{N}$. Since $\|a_n\varphi - \varphi a_n\| \overset{n\to\infty}{\to} 0$, $(a_n)_n \in \mathcal{M}^\omega(N)$ and $(a_n)^\omega \in N^\omega_{\varphi^\omega}$ holds. This shows that $x \notin (N^\omega_{\varphi^\omega})' \cap M$. Therefore $(N^\omega_{\varphi^\omega})' \cap M \subset \mathrm{B}(N \subset M, \varphi)$ holds. Conversely, if $x \notin (N^\omega_{\varphi^\omega})' \cap M$, then there exists $(a_n)^\omega \in N^\omega_{\varphi^\omega}$ for which $\lim_{n\to\omega} \|[x, a_n]\|_\varphi = c > 0$ holds. By [AH12, Lemma 4.35], $\lim_{n\to\omega} \|a_n\varphi - \varphi a_n\| = 0$. For each $k \in \mathbb{N}$, we have

$$I_k := \left\{ n \in \mathbb{N}; \|[x, a_n]\|_\varphi \geq \frac{c}{2},\ \|[a_n, \varphi]\| \leq \frac{1}{k} \right\} \in \omega.$$

Since ω is free, we may choose inductively $n_1 < n_2 < \cdots$ such that $n_k \in I_k$ $(k \in \mathbb{N})$. Now $(a_{n_k})_k \in \mathrm{AC}(N, \varphi)$, and $\|[x, a_{n_k}]\| \geq \frac{c}{2}$ $(k \in \mathbb{N})$, so that $x \notin \mathrm{B}(N \subset M, \varphi)$. This shows that $\mathrm{B}(N \subset M, \varphi) \subset (N^\omega_{\varphi^\omega})' \cap M$. $\square$

Lemma 8. *Let M be any nontrivial factor with strictly homogeneous state space. Let $\varphi \in M_*$ be any faithful state. Then M_φ is a type II_1 factor and for any $\lambda > 0$, we can find a finite family $v_1, \dots, v_n$ of partial isometries in M such that $v_k\varphi = \lambda\varphi v_k$ for all $k = 1, \dots, n$ and $\sum_{k=1}^n v_k v_k^* = 1$. If $\lambda \leq 1$, then we can take $n = 1$.*

Proof. By [AH12, Proposition 4.24], M_φ is a type II_1 factor and by the proof of [AH12, Proposition 4.22], we know that if $p, q \in M_\varphi$ are two nonzero projections, then we can find $v \in M$ such that $v^*v = p$, $vv^* = q$ and $v\varphi = \frac{\varphi(p)}{\varphi(q)}\varphi v$. Indeed, set $\psi_q = \frac{1}{\varphi(q)} q\varphi$ and $\psi_p = \frac{1}{\varphi(p)} p\varphi$. Then by [AH12, Proposition 4.22], there exists a partial isometry $v \in M$ such that $v^*v = \mathrm{supp}(\psi_p) = p$, $vv^* = \mathrm{supp}(\psi_q) = q$ and $v\psi_p v^* = \psi_q$. Then by $p, q \in M_\varphi$, it follows that

$$\begin{aligned} v\varphi = vv^*v\varphi &= \varphi(p)v\psi_p = \varphi(p)(v\psi_p v^*)v \\ &= \varphi(p)\psi_q v = \frac{\varphi(p)}{\varphi(q)} q\varphi v = \frac{\varphi(p)}{\varphi(q)} \varphi(v\,\cdot\,q) \\ &= \frac{\varphi(p)}{\varphi(q)} \varphi(qv\,\cdot\,) = \frac{\varphi(p)}{\varphi(q)} \varphi v. \end{aligned}$$

If $\lambda \leq 1$, we can take $q = 1$ and $p \in M_\varphi$ such that $\varphi(p) = \lambda$ and we obtain a partial isometry $v \in M$ such that $v\varphi = \lambda\varphi v$ and $vv^* = 1$. If $\lambda > 1$, choose $n \geq 1$ such that $\lambda \leq n$. Then we can find a finite partition of unity $q_1, \dots, q_n$ in M_φ (hence $\varphi(q_k) = \frac{1}{n}$, $1 \leq k \leq n$) and some projections $p_1, \dots, p_n \in M_\varphi$ (not necessarily orthogonal) such that $\varphi(p_k) = \lambda\varphi(q_k)(\leq 1)$. Then by the first part, we can find a family $v_k \in M$ of partial isometries such that $v_k^* v_k = p_k$, $v_k v_k^* = q_k$ and $v_k \varphi = \frac{\varphi(p_k)}{\varphi(q_k)} \varphi v_k = \lambda \varphi v_k$ as we wanted. □

Proof of Theorem A. (i) Let $\omega_1, \omega_2 \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilters. Let $u \in \mathcal{U}(N^{\omega_1})$ (resp. $v \in \mathcal{U}(N^{\omega_2})$) such that $u\varphi^{\omega_1}u^* = \psi^{\omega_1}$ (resp. $v\varphi^{\omega_2}v^* = \psi^{\omega_2}$). Then, inside $M^{\omega_2 \otimes \omega_1}$, we have $v^*u \in N^{\omega_2 \otimes \omega_1}_{\varphi^{\omega_2 \otimes \omega_1}}$. For every $x \in \mathrm{B}(N \subset M, \varphi)$, we have $v^*ux = xv^*u$ which means that $uxu^* = vxv^*$. Since $uxu^* \in M^{\omega_1}$ and $vxv^* \in M^{\omega_2}$, Proposition 5 shows that $uxu^* = vxv^*$ is an element of M. Thus, we have shown that for every $x \in \mathrm{B}(N \subset M, \varphi)$, there exists an element $\beta_{\psi,\varphi}(x) \in M$ given by $\beta_{\psi,\varphi}(x) = uxu^*$ where $u \in N^\omega$ is any unitary such that $u\varphi^\omega u^* = \psi^\omega$ and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ is any nonprincipal ultrafilter. In particular, if w is a unitary in $N^\omega_{\psi^\omega}$, we can replace u by wu, so that we have $\beta_{\psi,\varphi}(x) = wuxu^*w^* = w\beta_{\psi,\varphi}(x)w^*$. This shows that $\beta_{\psi,\varphi}(x) \in \mathrm{B}(N \subset M, \psi)$. Now, if $(a_n)_{n\in\mathbf{N}}$ is a uniformly bounded sequence in N such that $\|a_n\varphi - \psi a_n\| \to 0$, then it defines an element $a = (a_n)^\omega \in M^\omega$ such that $a\varphi^\omega = \psi^\omega a$ and so $u^*a \in N^\omega_{\varphi^\omega}$. This shows that $u^*ax = xu^*a$, that is, $ax = uxu^*a = \beta_{\psi,\varphi}(x)a$. Since the nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ is arbitrary, by Lemma 6, we conclude that $a_nx - \beta_{\psi,\varphi}(x)a_n \to 0$ $*$-strongly as $n \to \infty$. It is straightforward to check that $\beta_{\psi,\varphi}$ is a $*$-homomorphism and that $\beta_{\varphi_3,\varphi_2} \circ \beta_{\varphi_2,\varphi_1} = \beta_{\varphi_3,\varphi_1}$ for every faithful state $\varphi_i \in N_*$, $i \in \{1,2,3\}$. This shows in particular that $\beta_{\psi,\varphi} : \mathrm{B}(N \subset M, \varphi) \to \mathrm{B}(N \subset M, \psi)$ is an isomorphism with inverse $\beta_{\varphi,\psi}$. Let $\mathrm{E}_N : M \to N$ be any faithful normal conditional expectation and use it to extend φ and ψ to faithful normal states on M. Then we clearly have $\psi \circ \beta_{\psi,\varphi} = \varphi$. Since $N' \cap M$ is clearly fixed by $\beta_{\psi,\varphi}$, this implies that

$$\mathrm{E}^{\psi}_{N'\cap M} \circ \beta_{\psi,\varphi} = \mathrm{E}^{\varphi}_{N'\cap M}.$$

(ii) Let $\omega_1, \omega_2 \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilters and $\lambda > 0$. By Lemma 8, there exists a family $v_1, \dots, v_n$ of partial isometries in N^{ω_1} such that $v_k\varphi^{\omega_1} = \lambda\varphi^{\omega_1}v_k$ for all $k \in \{1, \dots, n\}$ and $\sum_{k=1}^n v_kv_k^* = 1$. Similarly, let $w_1, \dots, w_m$ a family of partial isometries in N^{ω_2} such that $w_l\varphi^{\omega_2} = \lambda\varphi^{\omega_2}w_l$ for all $l \in \{1, \dots, m\}$ and $\sum_{l=1}^m w_lw_l^* = 1$. Then inside $M^{\omega_2 \otimes \omega_1}$, we have $v_k^*w_l \in N^{\omega_2 \otimes \omega_1}_{\varphi^{\omega_2 \otimes \omega_2}}$ for all $k \in \{1, \dots, n\}$ and all $l \in \{1, \dots, m\}$. Then for all $x \in \mathrm{B}(N \subset M, \varphi)$, we have

$$xv_k^*w_l = v_k^*w_lx$$

and so

$$v_kxv_k^*(w_lw_l^*) = (v_kv_k^*)w_lxw_l^*.$$

By summing over k and l, we obtain

$$\sum_{k=1}^n v_kxv_k^* = \sum_{k=1}^m w_kxw_k^*. \tag{1}$$

But the left hand side of (1) lies in M^{ω_1} and the right hand side of (1) lies in M^{ω_2}. Then they are both in M by Proposition 5 and the element $\beta^\varphi_\lambda(x) = \sum_{k=1}^n v_kxv_k^* \in M$ is independent of the choice of the nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and the family $v_1, \dots, v_n \in N^\omega$ as above. In particular, if u is a unitary in $M^\omega_{\varphi^\omega}$, then we can replace v_k by uv_k for all $k \in \{1, \dots, n\}$ and we obtain $\beta^\varphi_\lambda(x) = u\beta^\varphi_\lambda(x)u^*$. This shows that $\beta^\varphi_\lambda(x) \in \mathrm{B}(N \subset M, \varphi)$. Let $(a_n)_{n\in\mathbf{N}}$ be a uniformly bounded sequence in N such that $\lim_n \|a_n\varphi - \lambda\varphi a_n\| = 0$. Then it defines an element $a = (a_n)^\omega \in N^\omega$ such that $a\varphi^\omega = \lambda\varphi^\omega a$. Then we have $v_k^* a \in N^\omega_{\varphi^\omega}$ for all $k \in \{1, \dots, n\}$. Thus, for all $x \in \mathrm{B}(N \subset M, \varphi)$, we have $v_k^* a x = x v_k^* a$ and so

$$ax = \sum_{k=1}^n v_k v_k^* a x = \sum_{k=1}^n v_k x v_k^* a = \beta^\varphi_\lambda(x) a.$$

Since the nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ is arbitrary, by Lemma 6, we conclude that $a_n x - \beta^\varphi_\lambda(x) a_n \to 0$ $*$-strongly as $n \to \infty$.

It is straightforward to check that β^φ_λ is a unital $*$-homomorphism for all $\lambda > 0$ and that $\beta^\varphi_\lambda \circ \beta^\varphi_\mu = \beta^\varphi_{\lambda\mu}$ for all $\lambda, \mu > 0$. This shows that $\beta^\varphi : \lambda \mapsto \beta^\varphi_\lambda$ is a one-parameter group of automorphisms of $\mathrm{B}(N \subset M, \varphi)$. Also, one checks easily that $\beta^\psi_\lambda \circ \beta_{\psi,\varphi} = \beta_{\psi,\varphi} \circ \beta^\varphi_\lambda$ for all faithful normal states $\varphi, \psi \in N_*$. Extend φ to a state on M by using any faithful normal conditional expectation from M to N. Then β^φ is φ-preserving. Indeed, for all $\lambda > 0$, we have

$$\varphi(\beta^\varphi_\lambda(x)) = \sum_{k=1}^n \varphi^\omega(v_k x v_k^*) = \sum_{k=1}^n \lambda^{-1}\varphi^\omega(x v_k^* v_k) = \sum_{k=1}^n \lambda^{-1}\varphi(x)\varphi^\omega(v_k^* v_k)$$

because x commutes with the factor N^{φ^ω}. Since $\varphi^\omega(v_k^* v_k) = \lambda\varphi^\omega(v_k v_k^*)$, we obtain

$$\varphi(\beta^\varphi_\lambda(x)) = \sum_{k=1}^n \varphi(x)\varphi^\omega(v_k v_k^*) = \varphi(x).$$

Thus β^φ_λ is φ-preserving and since β^φ clearly fixes $N' \cap M$, we obtain

$$\mathrm{E}^\varphi_{N'\cap M} \circ \beta^\varphi_\lambda = \mathrm{E}^\varphi_{N'\cap M}.$$

At this point, we have proved all items (i), (ii), (iii) and (iv). It only remains to check that β^φ is indeed a flow in the sense that it is continuous with respect to the u-topology on $\mathrm{B}(N \subset M, \varphi)$. Take a sequence $\lambda_n \in \mathbf{R}^*_+$ such that $\lambda_n \to 1$ and $\lambda_n \leq 1$. We have to show that $\beta^\varphi_{\lambda_n} \to \mathrm{id}_{\mathrm{B}(N\subset M,\varphi)}$ with respect to the u-topology. Since β^φ is φ-preserving, it is enough to show that $\beta^\varphi_{\lambda_n}(x) \to x$ strongly for all $x \in \mathrm{B}(N \subset M, \varphi)$. Let $\omega_1 \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter and pick, for every $n \in \mathbf{N}$, a co-isometry $v_n \in N^{\omega_1}$ such that $v_n\varphi^{\omega_1} = \lambda_n\varphi^{\omega_1} v_n$ (possible because $\lambda_n \leq 1$). Let $\omega_2 \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any other nonprincipal ultrafilter. Since $\lambda_n \to 1$, then $v = (v_n)^{\omega_2}$ defines a co-isometry of $N^{\omega_2 \otimes \omega_1}$ with $v\varphi^{\omega_2\otimes\omega_1} = \varphi^{\omega_2\otimes\omega_1} v$. Since $x \in \mathrm{B}(N \subset M, \varphi)$, we get $x = vxv^* = (v_n x v_n^*)^{\omega_2} = (\beta^\varphi_{\lambda_n}(x))^{\omega_2}$. Since the nonprincipal ultrafilter $\omega_2 \in \beta(\mathbf{N}) \setminus \mathbf{N}$ is arbitrary, by Lemma 6, we conclude that $\beta^\varphi_{\lambda_n}(x) \to x$ strongly as $n \to \infty$. □

Example 9. Although it is very likely that $\mathrm{B}(N, \varphi) = \mathbb{C}$ for every III_1 factor N with separable predual, the relative bicentralizer $\mathrm{B}(N \subset M, \varphi)$ need not be trivial even when the inclusion is irreducible. Let N be any type III_1 factor with separable predual and with trivial bicentralizer (e.g. $N = R_\infty$). Choose a faithful state $\varphi \in N_*$. Fix $\mu \in (0, 1)$, put $T = \frac{2\pi}{-\log(\mu)}$, define $M = N \rtimes_{\sigma_T^\varphi} \mathbf{Z}$ and canonically extend φ to M. Then M is a type III_μ factor by [Co85, Lemma 1] and the inclusion $N \subset M$ is irreducible and with expectation. We show that $\mathrm{B}(N \subset M, \varphi) = \mathrm{L}(\mathbf{Z}) \cong \mathrm{L}^\infty(\mathbf{R}_+^*/\mu^{\mathbf{Z}})$. Let $\mathrm{E}_N\colon M \to N$ be the canonical conditional expectation and $x \in \mathrm{B}(N \subset M, \varphi)$. Then we have the Fourier series expansion $x = \sum_{n\in\mathbf{Z}} x_n u^n$ in the Hilbert space topology where u is the unitary implementing the σ_T^φ in N, and $x_n = \mathrm{E}_N(x(u^n)^*) \in N$ $(n \in \mathbf{Z})$. Let $a \in N_{\varphi^\omega}^\omega$. Then by Proposition 7, $ax = xa$ in M^ω. Moreover, $\sigma_T^{\varphi^\omega}(a) = a$ by Theorem 2. Thus

$$\begin{aligned} ax_n &= a\mathrm{E}_{N^\omega}(x(u^n)^*) = E_{N^\omega}(ax(u^n)^*) = \mathrm{E}_{N^\omega}(xa(u^n)^*) \\ &= \mathrm{E}_{N^\omega}(x(u^n)^*\sigma_{nT}^{\varphi^\omega}(a)) = \mathrm{E}_{N^\omega}(x(u^n)^*a) \\ &= x_n a. \end{aligned}$$

Thus $x_n \in N_{\varphi^\omega}^\omega \cap N' = \mathrm{B}(N, \varphi) = \mathbb{C}$ by the hypothesis on N. This shows that $x \in \mathrm{W}^*(\{u\}) \cong \mathrm{L}(\mathbf{Z})$ and therefore $\mathrm{B}(N \subset M, \varphi) \subset \mathrm{L}(\mathbf{Z})$. Conversely, it is clear that $u \in (N_\varphi)' \cap M \subset (N_{\varphi^\omega}^\omega)' \cap M = \mathrm{B}(N \subset M, \varphi)$ holds, whence $\mathrm{L}(\mathbf{Z}) \subset \mathrm{B}(N \subset M, \varphi)$. Next, we identify the flow β^φ. Let $\lambda > 0$, and let $v_1, \dots, v_k \in N^\omega$ be partial isometries such that $v_k\varphi^\omega = \lambda\varphi^\omega v_k$ $(1 \le k \le n)$ and $\sum_{k=1}^\infty v_k v_k^* = 1$. Then for each $m \in \mathbf{Z}$,

$$\begin{aligned} \beta_\lambda^\varphi(u^m) &= \sum_{k=1}^n v_k u^m v_k^* = \sum_{k=1}^n v_k \sigma_{mT}^{\varphi^\omega}(v_k^*) u^m \\ &= \sum_{k=1}^n v_k \lambda^{imT} v_k^* u^m = \lambda^{imT} u^m. \end{aligned}$$

We identify $\widehat{\mathbf{Z}} = \mathbf{R}_+^*/\mu^{\mathbf{Z}}$ with $[\mu, 1]$ with the Haar measure $dm(t) = dt/t$. Let $\mathscr{F}\colon \ell^2(\mathbf{Z}) \to \mathrm{L}^2([\mu, 1], m)$ be the Fourier transform given by $\delta_\ell \mapsto e_\ell$, where $\{\delta_\ell\}_{\ell\in\mathbf{Z}}$ is the canonical orthonormal basis for $\ell^2(\mathbf{Z})$ and $e_\ell(t) = \frac{1}{\sqrt{-\log\mu}} t^{i\ell}$. Then for $\xi = \sum_{\ell\in\mathbf{Z}} a_\ell e_\ell \in \mathrm{L}^2([\mu, 1], m)$, we have $[\mathscr{F} u \mathscr{F}^{-1} \cdot \xi](t) = t^i f(t),\ t \in [\mu, 1]$. Thus $\mathscr{F} u \mathscr{F}^{-1} = \sqrt{-\log\mu}\, e_1$. Let $\tilde{\beta}_\lambda^\varphi = \mathscr{F} \beta_\lambda^\varphi \mathscr{F}^{-1}$ acting on $\mathrm{L}^\infty([\mu, 1], m)$. Let $f \in \mathrm{L}^\infty([\mu, 1], m)$. Expand $f = \sum_{\ell\in\mathbf{Z}} a_\ell \mathscr{F} u^\ell \mathscr{F}^{-1}$ in $\mathrm{L}^2([\mu, 1], m)$. Then for $t \in [\mu, 1]$,

$$\begin{aligned} \tilde{\beta}_\lambda^\varphi(f)(t) &= \mathscr{F} \beta_\lambda^\varphi \left(\sum_{\ell\in\mathbf{Z}} a_\ell u^\ell \right) \mathscr{F}^{-1}(t) = \mathscr{F} \left(\sum_{\ell\in\mathbf{Z}} a_\ell \lambda^{i\ell T} u^\ell \right) \mathscr{F}^{-1}(t) \\ &= \sum_{\ell\in\mathbf{Z}} a_\ell \lambda^{i\ell T} t^{i\ell} = \sum_{\ell\in\mathbf{Z}} a_\ell \left(\lambda^{-\frac{2\pi}{\log\mu}} t \right)^{i\ell} \\ &= f(\lambda^{-\frac{2\pi}{\log\mu}} t). \end{aligned}$$

Thus, with the identification $\mathrm{L}(\mathbf{Z}) = \mathrm{L}^\infty([\mu, 1])$, the relative bicentralizer flow $\beta^\varphi : \mathbf{R}_+^* \curvearrowright \mathrm{B}(N \subset M, \varphi)$ is the multiplication action $\mathbf{R}_+^* \curvearrowright \mathbf{R}_+^*/\mu^{\mathbf{Z}} = [\mu, 1]$ given by

$$\beta_\lambda^\varphi(t) = \lambda^{\frac{2\pi}{\log\mu}} t, \quad t \in \mathbf{R}_+^*/\mu^{\mathbf{Z}}$$

Let $\psi \in (R_\infty)_*$ be a faithful state, and let $\mathcal{M} := (M, \varphi) * (R_\infty, \psi)$. Then by [Ue10, Corollary 3.2] (applied to the case $A = N \subset M_1 = M$ in the cited result), we get that $N' \cap \mathcal{M} = N' \cap M = \mathbb{C}$. Thus $N \subset \mathcal{M}$ is still an irreducible inclusion with expectation.

By [Ue10, Corollary 3.2] again (work in $\mathcal{M} \subset (M^\omega, \varphi^\omega) * (R_\infty^\omega, \psi^\omega)$ and apply the result to $A = N_{\varphi^\omega}^\omega \subset M_{\varphi^\omega}^\omega \subset M_1 = M^\omega$),

$$\begin{aligned}\mathrm{B}(N \subset \mathcal{M}, \varphi) &= (N_{\varphi^\omega}^\omega)' \cap \mathcal{M} = (N_{\varphi^\omega}^\omega)' \cap (\underbrace{M^\omega \cap \mathcal{M}}_{=M}) \\ &= (N_{\varphi^\omega}^\omega)' \cap M = \mathrm{B}(N \subset M, \varphi) = \mathrm{L}(\mathbb{Z}).\end{aligned}$$

Since $\mathrm{B}(\mathcal{M}, \varphi) = \mathbb{C}$ by [HU15], we obtain an irreducible inclusion $N \subset \mathcal{M}$ of III_1 factors, both have trivial bicentralizers such that its relative bicentralizer flow is the tranlsation action $\mathbf{R}_+^* \curvearrowright \mathbf{R}_+^*/\mu^{\mathbf{Z}} = [\mu, 1]$ given above.

Acknowledgment

The author would like to thank the organizers for giving him the opportunity to give a talk. The author was supported by JSPS Kakenhi 16K17608 during the work [AHHM18].

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