Convex combinations associated with the curvature of the space and their natures
曲率に対応して定義される凸結合の幾何的性質

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Abstract

In this paper, we consider another type of convex combinations associated with the curvature, and investigate their natures.

1 Introduction

A convex combination is one of the basic notion for the convex analysis, and its definition is very simple. In a real vector space V, a convex combination of two points x and y with a ratio $\alpha \in [0,1]$, which is usually denoted by $\alpha x+(1-\alpha)y$, is a weighted average of x and y for weights α and $1-\alpha$. The concept of convex combination is defined not only for real vector spaces but also for geodesic spaces. A geodesic space X is a metric space that any two points on X have the shortest path joining these points. In a geodesic space X, a convex combination of two points x and y with a ratio $\alpha \in [0,1]$ is generally defined as a point z satisfying $d(x,z) = (1-\alpha)d(x,y)$ and $d(y,z) = \alpha d(x,y)$. We usually write that point z as $\alpha x \oplus (1-\alpha)y$.

In 2020, we defined a new breed of convex combination $\overset{1}{\oplus}$ and showed the following theorem in the context of fixed point approximation on a complete CAT(1) space:

Theorem 1.1 ([3]). *Let X be an admissible complete* CAT(1) *space such that* $\sup_{s,s'\in X} d(s,s') < \pi/2$. *Let $S,T\colon X \to X$ be strongly quasinonexpansive and Δ-demiclosed mappings such that S and T have a common fixed point. Let $\{\alpha_n\},\{\gamma_n\} \subset\ ]0,1[$ and suppose $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, and $\gamma_n \to \gamma \in\ ]0,1[$. Take $v,w,x_1 \in X$ and generate a iterative sequence $\{x_n\} \subset X$ by $s_n = \alpha_n v \overset{1}{\oplus} (1-\alpha_n)Sx_n$,*

$t_n = \alpha_n w \overset{1}{\oplus} (1-\alpha_n)Tx_n$*, and* $x_{n+1} = \gamma_n s_n \overset{1}{\oplus} (1-\gamma_n)t_n$ *for* $n \in \mathbb{N}$*. Then* $\{x_n\}$ *converges to a common fixed point of* S *and* T*. Moreover, its limit is a maximizer of the function* $g\colon F \to]0,1]$ *defined by* $g(x) = \gamma \cos d(v,x) + (1-\gamma)\cos d(w,x)$ *for* $x \in F$*, where* F *is the set of all common fixed points of* S *and* T*.*

In Theorem 1.1, we need to use a new convex combination $\overset{1}{\oplus}$ instead of the traditional convex combination $\oplus$ for the limit of the sequence $\{x_n\}$ to be the maximizer of the function g. Indeed, if we only use $\oplus$ instead of $\overset{1}{\oplus}$, then we can verify that the limit of $\{x_n\}$ may differ from the maximizer of g. This result suggests that the traditional convex combination is somewhat incompatible with CAT(1) spaces, and that $\overset{1}{\oplus}$ may be better adapted to CAT(1) spaces; note that the function g is well compatible with CAT(1) spaces. Particularly, since the model space of CAT(1) spaces is the unit sphere $\mathbb{S}^2$, it is expected that the new convex combination $\overset{1}{\oplus}$ is adapted to a geodesic space with the constant curvature 1. In this paper, we consider the natures of the new convex combination $\overset{1}{\oplus}$ and investigate its behavior on the unit sphere on Hilbert spaces, and its generalization $\overset{\kappa}{\oplus}$.

2 Preliminaries

Let A be a set and $f\colon A \to \mathbb{R}$. If f has the unique minimizer t_0, then we write t_0 by $\operatorname{argmin}_{t\in A} f(t)$. Similarly, $\operatorname{argmax}_{t\in A} f(t)$ denotes the unique maximizer of f.

Let X be a metric space. For $x, y \in X$, a mapping $\gamma\colon [0,1] \to X$ is called a *geodesic* joining x and y if $\gamma(0) = y$, $\gamma(1) = x$, and $d(\gamma(s), \gamma(t)) = |s-t| d(x,y)$ hold for any $s, t \in [0,1]$. For $D \in]0,\infty]$, X is called a *uniquely* D*-geodesic space* if a geodesic joining x and y exists uniquely for any two points $x, y \in X$ with $d(x,y) < D$. In particular, a uniquely ∞-geodesic space is simply called a *uniquely geodesic space*.

Let X be a uniquely D-geodesic space and let $x, y \in X$ such that $d(x,y) < D$. Then a point $tx \oplus (1-t)y := \gamma(t)$ is called a *convex combination* of x and y, where γ is the unique geodesic joining x and y. The set of all convex combinations of x and y is denoted by $[x,y]$, that is, $[x,y] = \{tx \oplus (1-t)y \mid x, y \in X,\ t \in [0,1]\}$. Then we get $[x,y] = [y,x]$ obviously. We call $[x,y]$ a *geodesic segment* (on X) joining x and y. Furthermore, a subset $C \subset X$ is said to be *convex* if $[x,y] \subset C$ for any $x, y \in C$.

Let M_κ be the complete simply connected 2-dimensional Riemannian manifold with constant sectional curvature $\kappa \in \mathbb{R}$ and a metric ρ. It is equal to $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$, $\mathbb{R}^2$, $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ if $\kappa > 0$, $\kappa = 0$, $\kappa < 0$, respectively, where $\mathbb{S}^2$ is the 2-dimensional unit sphere, and $\mathbb{H}^2$ is the 2-dimensional hyperbolic space. We define $D_\kappa \in]0,\infty]$ by $D_\kappa = \infty$ if $\kappa \le 0$, and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$, which means a diameter of M_κ. M_κ is a uniquely D_κ-geodesic space. In what follows, $[u,v]_{M_\kappa}$ denotes a geodesic segment joining $u, v \in M_\kappa$.

For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space. For each $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, we define a *geodesic triangle* with vertices x, y, z by $[x,y] \cup [y,z] \cup [z,x]$, and write it by $\triangle(x,y,z)$. For each $\triangle(x,y,z)$, there exists three points $\overline{x}, \overline{y}, \overline{z} \in M_\kappa$ such that $d(x,y) = \rho(\overline{x}, \overline{y})$, $d(y,z) = \rho(\overline{y}, \overline{z})$, and $d(z,x) =$

$\rho(\overline{z},\overline{x})$. For these points $\overline{x},\overline{y},\overline{z}$, we define a *comparison triangle* $\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ by $[\overline{x},\overline{y}]_{M_\kappa}\cup[\overline{y},\overline{z}]_{M_\kappa}\cup[\overline{z},\overline{x}]_{M_\kappa}$. For any $\triangle(x,y,z)$ and a point $p\in\triangle(x,y,z)$, there exists a point $\overline{p}\in\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ such that the distances from two adjacent vertices are identical. That point $\overline{p}$ is called a *comparison point* of p.

Let $\kappa\in\mathbb{R}$. A uniquely D_κ-geodesic space X is called a CAT(κ) *space* if for any $\triangle := \triangle(x,y,z)$ and its comparison triangle $\overline{\triangle} := \overline{\triangle}(\overline{x},\overline{y},\overline{z})$, and for any two points $p,q\in\triangle$ and these comparison points $\overline{p},\overline{q}\in\overline{\triangle}$, the inequality $d(p,q)\le\rho(\overline{p},\overline{q})$ holds. A CAT(κ) space X is said to be *admissible* if $d(x,y)<D_\kappa/2$ for every $x,y\in X$. If $\kappa\le 0$, then every CAT(κ) space is admissible.

By the definition of CAT(κ) spaces, the unit sphere $\mathbb{S}^2$ embedded in a Euclidean space $\mathbb{R}^3$, a Hilbert space H, the hyperbolic space $\mathbb{H}^2$ are a CAT(1) space, a CAT(0) space, a CAT(-1) space, respectively. For more details, see [1].

3 κ-convex combination

In this section, we introduce the definition of new convex combination which is called the κ-convex combination, and we investigate its nature.

For each $\kappa\in\mathbb{R}$, define $c_\kappa\colon\mathbb{R}\to\mathbb{R}$ by

$$c_\kappa(d)=\begin{cases}\dfrac{1}{-\kappa}(\cosh(\sqrt{-\kappa}\,d)-1) & (\text{if } \kappa<0),\\[2mm] \dfrac{1}{2}d^2 & (\text{if } \kappa=0),\\[2mm] \dfrac{1}{\kappa}(1-\cos(\sqrt{\kappa}\,d)) & (\text{if } \kappa>0)\end{cases}$$

for $d\in\mathbb{R}$. In particular, $c_{-1}(d)=\cosh d-1$ and $c_1(d)=1-\cos d$. Note that c_κ is strictly convex and increasing on $[0,D_\kappa]$ for any $\kappa\in\mathbb{R}$.

The first definition of κ-convex combinations $\overset{\kappa}{\oplus}$ for $\kappa=-1$ and $\kappa=1$ were given by [2] and [3], respectively. Later, properties of the κ-convex combination for general $\kappa\in\mathbb{R}$ was shown in [4].

Let X be a uniquely D_κ-geodesic space. In [2], [3] and [4], the κ-convex combination of x and y is defined under the condition $d(x,y)<D_\kappa/2$. Actually, we can weaken the assumption to $d(x,y)<D_\kappa$ when define the κ-convex combination. In this paper, we use the condition $d(x,y)<D_\kappa$ to define the κ-convex combination.

Theorem 3.1. *Let $\kappa\in\mathbb{R}$ and X a uniquely D_κ-geodesic space. Take $x,y\in X$ with $d(x,y)<D_\kappa$ and $\alpha\in[0,1]$. Define $g_\kappa\colon X\to\mathbb{R}$ by*

$$g_\kappa(z)=\alpha c_\kappa(d(x,z))+(1-\alpha)c_\kappa(d(y,z))$$

for $z\in X$. Then the restriction $g_\kappa|_{[x,y]}$ has the unique minimizer, where $[x,y]$ is the geodesic segment joining x and y.

Proof. If $d(x,y)<D_\kappa/2$, then we obtain the conclusion, see [2], [3] and [4]. Furthermore, if $\kappa\le 0$, then we also have the conclusion, since $D_\kappa=\infty=D_\kappa/2$. Thus

we only show the case where $\kappa > 0$. It is sufficient to prove the case where $\kappa = 1$, henceforth we will assume $\kappa = 1$.

Let $x, y \in X$, $\alpha \in [0,1]$ and put $D = d(x,y)$. If $D = 0$, then we obtain the desired result obviously. Suppose that $0 < D < \pi$. Then we have

$$g_1(tx \oplus (1-t)y) = 1 - (\alpha \cos((1-t)D) + (1-\alpha)\cos tD)$$

for any $t \in [0,1]$. Define $f\colon [0,1] \to \mathbb{R}$ by $f(t) = \alpha \cos((1-t)D) + (1-\alpha)\cos tD$ for $t \in [0,1]$. Then $f'(t)/D = \alpha \sin((1-t)D) - (1-\alpha)\sin tD$ holds for each $t \in [0,1]$. Let $\tan^{-1}\colon \mathbb{R} \to [0,\pi[\setminus \{\pi/2\}$ be the inverse of the trigonometric tangent function. Then putting

$$t_0 = \frac{1}{D}\tan^{-1}\frac{\alpha \sin D}{1-\alpha+\alpha\cos D},$$

we get $t_0 \in [0,1]$ and $f'(t_0) = 0$. Take $t \in [0,1]$ arbitrarily. If $t < t_0$, then we obtain

$$f'(t)/D = \alpha \sin((1-t)D) - (1-\alpha)\sin tD > \alpha \sin((1-t_0)D) - (1-\alpha)\sin t_0 D = 0.$$

Similarly, if $t > t_0$, then $f'(t)/D < 0$. It concludes t_0 is the unique maximizer of f, and hence $t_0 x \oplus (1-t_0)y = \operatorname{argmin}_{z \in [x,y]} g_1(z)$. $\square$

Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. Let $\alpha \in [0,1]$ and $x, y \in X$ such that $d(x,y) < D_\kappa$. Suppose that $d(x,y) < D_\kappa$. Then the unique minimizer of $g_\kappa|_{[x,y]}$ in Theorem 3.1 is called a *κ-convex combination* of x and y, and we write it by $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$. That is, $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z \in [x,y]} g_\kappa(z)$. Note that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$ can be expressed by using a traditional convex combination $tx \oplus (1-t)y$. In fact, define $t \in [0,1]$ by

$$t = \begin{cases} \dfrac{1}{\sqrt{-\kappa}\, d(x,y)} \tanh^{-1} \dfrac{\alpha \sinh\left(\sqrt{-\kappa}\, d(x,y)\right)}{1-\alpha+\alpha\cosh\left(\sqrt{-\kappa}\, d(x,y)\right)} & (\text{if } \kappa < 0 \text{ and } x \neq y); \\ \alpha & (\text{if } \kappa = 0 \text{ or } x = y); \\ \dfrac{1}{\sqrt{\kappa}\, d(x,y)} \tan^{-1} \dfrac{\alpha \sin\left(\sqrt{\kappa}\, d(x,y)\right)}{1-\alpha+\alpha\cos\left(\sqrt{\kappa}\, d(x,y)\right)} & (\text{if } \kappa > 0 \text{ and } x \neq y). \end{cases}$$

Then we get

$$1-t = \begin{cases} \dfrac{1}{\sqrt{-\kappa}\, d(x,y)} \tanh^{-1} \dfrac{(1-\alpha) \sinh\left(\sqrt{-\kappa}\, d(x,y)\right)}{\alpha+(1-\alpha)\cosh\left(\sqrt{-\kappa}\, d(x,y)\right)} & (\text{if } \kappa < 0 \text{ and } x \neq y); \\ 1-\alpha & (\text{if } \kappa = 0 \text{ or } x = y); \\ \dfrac{1}{\sqrt{\kappa}\, d(x,y)} \tan^{-1} \dfrac{(1-\alpha) \sin\left(\sqrt{\kappa}\, d(x,y)\right)}{\alpha+(1-\alpha)\cos\left(\sqrt{\kappa}\, d(x,y)\right)} & (\text{if } \kappa > 0 \text{ and } x \neq y) \end{cases}$$

and $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$, where $\tanh^{-1}\colon [0,1[\to [0,\infty[$ is the inverse of the hyperbolic tangent function, and $\tan^{-1}\colon \mathbb{R} \to [0,\pi[\setminus \{\pi/2\}$ is the inverse of the trigonometric tangent function.

For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space. Then, the following properties hold for any $\kappa \in \mathbb{R}$, $\alpha \in [0,1]$, and $x, y \in X$ with $d(x,y) < D_\kappa$.

(a) $1x \overset{\kappa}{\oplus} 0y = x$ and $0x \overset{\kappa}{\oplus} 1y = y$.

(b) $\alpha x \overset{\kappa}{\oplus} (1-\alpha)x = x$.

(c) $\frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y = \frac{1}{2}x \oplus \frac{1}{2}y$.

These properties (a), (b) and (c) are obtained directly from the definition of κ-convex combination.

Theorem 3.2. *The* 0*-convex combination* $\overset{0}{\oplus}$ *is identical with the traditional convex combination* $\oplus$.

Proof. For $D \in]0,\infty]$, let X be a uniquely D-geodesic space and take $x, y \in X$ with $d(x,y) < D$. Then we show $\alpha x \overset{0}{\oplus} (1-\alpha)y = \alpha x \oplus (1-\alpha)y$ for any $\alpha \in [0,1]$. Since $\alpha x \overset{0}{\oplus} (1-\alpha)y \in [x,y]$, we get

$$\alpha x \overset{0}{\oplus} (1-\alpha)y = \operatorname*{argmin}_{z \in [x,y]} \left(\alpha d(x,z)^2 + (1-\alpha) d(y,z)^2\right) = \alpha' x \oplus (1-\alpha')y,$$

where

$$\alpha' = \operatorname*{argmin}_{t \in [0,1]} \left(\alpha((1-t)d(x,y))^2 + (1-\alpha)(td(x,y))^2\right) = \alpha.$$

Thus we get the conclusion. □

Lemma 3.3. *Let* $\kappa \in \mathbb{R}$ *and* X *a uniquely* D_κ*-geodesic space. Take* $x, y \in X$ *with* $d(x,y) < D_\kappa$ *and* $\alpha \in [0,1]$. *Define* $g_\kappa \colon X \to \mathbb{R}$ *by* $g_\kappa(z) = \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))$ *for* $z \in X$. *Let* C *be a subset of* X *such that* $d(u,v) < D_\kappa$ *for any* $u, v \in C$ *and* $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y \in C$. *Then* $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z \in C} g_\kappa(z)$.

Proof. Put $v = \alpha x \overset{\kappa}{\oplus} (1-\alpha)\, y = \operatorname{argmin}_{z \in [x,y]} g_\kappa(z) \in C$. If $x = y$, then we obtain $v = x = \operatorname{argmin}_{z \in C} c_\kappa(d(x,z)) = \operatorname{argmin}_{z \in C} g_\kappa(z)$, which is the conclusion. Suppose that $x \neq y$ and take $w \in C \setminus \{v\}$ arbitrarily. Put $t = d(y,w)/(d(x,w) + d(y,w))$ and $v' = tx \oplus (1-t)\, y$. Then $d(x,v') : d(y,v') = d(x,w) : d(y,w)$. Moreover, we obtain $g_\kappa(v) \le g_\kappa(v')$, notably we get $g_\kappa(v) < g_\kappa(v')$ if $v \neq v'$.

Suppose that $v = v'$. Then we get $w \neq v'$ and hence $w \notin [x,y]$. Thus we have $d(x,v') + d(y,v') = d(x,y) < d(x,w) + d(y,w)$. It implies that $d(x,v') < d(x,w)$ and $d(y,v') < d(y,w)$. Therefore we get $g_\kappa(v') < g_\kappa(w)$ and it follows that $g_\kappa(v) < g_\kappa(w)$.

Next we assume $v \neq v'$. Then we have $d(x,v') \le d(x,w)$ and $d(y,v') \le d(y,w)$, and hence $g_\kappa(v') \le g_\kappa(w)$. It implies $g_\kappa(v) < g_\kappa(w)$ and thus we get the conclusion. □

Corollary 3.4. *Let* $\kappa \in \mathbb{R}$ *and* X *a uniquely geodesic space such that* $d(u,v) < D_\kappa$ *for any* $u, v \in X$. *Take* $x, y \in X$, $\alpha \in [0,1]$ *and define* $g_\kappa \colon X \to \mathbb{R}$ *by* $g_\kappa(z) = \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))$ *for* $z \in X$. *Then* $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z \in X} g_\kappa(z)$.

4 1-convex combination

The κ-convex combination is not defined only in geodesic manifolds with a curvature κ. For instance, we can define κ-convex combinations on an Euclidean space $\mathbb{R}^n$ for any $\kappa \in \mathbb{R}$. However, not all of κ-convex combinations have good properties on $\mathbb{R}^n$. In fact, it is obvious that the most useful κ-convex combination on $\mathbb{R}^n$ is the 0-convex combination. We consider that the κ-convex combination defined on a geodesic manifold with a curvature exactly κ should play a beneficial role, that is implied by previous studies [2, 3, 4].

In this section, we investigate properties of the 1-convex combination on geodesic spaces. Additionally, we confirm that the 1-convex combination has good behavior on the unit sphere in an Hilbert space, especially the 2-dimensional unit sphere $\mathbb{S}^2$.

4.1 1-convex combination on geodesic spaces

For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space. Then the 1-convex combination of $x, y \in X$ is defined by

$$\begin{aligned} \alpha x \overset{1}{\oplus} (1-\alpha) y &= \operatorname*{argmin}_{z \in X} \left(\alpha c_1(d(x,z)) + (1-\alpha) c_1(d(y,z)) \right) \\ &= \operatorname*{argmax}_{z \in [x,y]} \left(\alpha \cos d(x,z) + (1-\alpha) \cos d(y,z) \right) \end{aligned}$$

for each $\alpha \in [0,1]$, where $d(x,y) < D$.

Lemma 4.1. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space. Let $x, y \in X$ such that $0 < d(x,y) < D$, and put $d_0 = d(x,y)$. Then for any $\alpha \in [0,1]$,*

$$\begin{aligned} &\alpha x \overset{1}{\oplus} (1-\alpha) y \\ &\qquad = \left(\frac{1}{d_0} \tan^{-1} \frac{\alpha \sin d_0}{1 - \alpha + \alpha \cos d_0} \right) x \oplus \left(\frac{1}{d_0} \tan^{-1} \frac{(1-\alpha) \sin d_0}{\alpha + (1-\alpha) \cos d_0} \right) y. \end{aligned}$$

Proof. The proof of Theorem 3.1 exactly implies the conclusion. □

Let X be a CAT(1) space, and take $\triangle(x,y,z) \subset X$ and $\alpha \in [0,1]$ arbitrarily. Then

$$\cos d(\alpha x \oplus (1-\alpha) y, z) \sin D \geq \sin(\alpha D) \cos d(x,z) + \sin((1-\alpha)D) \cos d(y,z) \tag{i}$$

holds, where $D = d(x,y)$. This inequality is often called the parallelogram law on CAT(1) spaces. In an admissible subspace S of the unit sphere $\mathbb{S}^2$, the inequality (i) holds as the equation. On the other hand, for any $\triangle(x,y,z) \subset X$ and $\alpha \in [0,1]$,

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha) y, z) \geq \frac{\alpha \cos d(x,z) + (1-\alpha) \cos d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha) \cos D + (1-\alpha)^2}} \tag{ii}$$

holds. Incidentally, we know that two inequalities are equivalent, which can be proved from Lemma 4.1, see [3]. Therefore, in S, the inequality (ii) also holds as the equation.

Lemma 4.2. *Let $d \in \,]0, \pi/2[$ and define $f\colon]0,1[\to \mathbb{R}$ by $f(t) = (\sin td)/t$ for $t \in \,]0,1[$. Then f is strictly decreasing.*

Lemma 4.3. *Let $d \in \,]0, \pi/2[$, $\alpha \in \,]0,1[$ and put*

$$\sigma = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d} \in \,]0,1[.$$

Then the following hold:

- *If $\alpha < 1/2$, then $\alpha > \sigma$;*
- *if $\alpha = 1/2$, then $\alpha = \sigma$;*
- *if $\alpha > 1/2$, then $\alpha < \sigma$.*

Proof. The case where $\alpha = 1/2$ is obviously true. It is enough to prove only the case where $\alpha < 1/2$ by the symmetric property.

Suppose that $\alpha < 1/2$, and define a strictly concave function $g\colon [0,1] \to \mathbb{R}$ by $g(t) = \alpha \cos((1-t)d) + (1-\alpha)\cos td$ for $t \in [0,1]$. Then σ is a unique maximizer of g. In addition, we obtain

$$\begin{aligned} g'(\alpha) &= \alpha d \sin((1-\alpha)d) - (1-\alpha) d \sin \alpha d \\ &= \alpha(1-\alpha)d \cdot \left(\frac{\sin((1-\alpha)d)}{1-\alpha} - \frac{\sin \alpha d}{\alpha} \right) < 0 \end{aligned}$$

from Lemma 4.2. It implies $\alpha > \sigma$ and thus we get the conclusion. □

Corollary 4.4. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x,y) < R$. Let $\alpha \in \,]0,1[$. Then $\alpha x \overset{1}{\oplus} (1-\alpha) y = \alpha x \oplus (1-\alpha) y$ holds if and only if $\alpha = 1/2$.*

Proof. Lemma 4.3 implies the conclusion. □

Corollary 4.5. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x,y) < R$. Let $\alpha \in \,]0,1[\setminus \{1/2\}$. Then a point $u_1 = \alpha x \overset{1}{\oplus} (1-\alpha) y$ is farther from the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ than $u_0 = \alpha x \oplus (1-\alpha) y$.*

Proof. Put $\sigma x \oplus (1-\sigma) y \coloneqq u_1$. If $\alpha < 1/2$, then we have $1/2 > \alpha > \sigma$ by Lemma 4.3. Otherwise, we get $1/2 < \alpha < \sigma$. Therefore u_1 is farther from the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ than u_0 in both cases. □

Lemma 4.6. *Let $d \in \,]0, \pi/2[$, and define a function $f\colon [0,1] \to [0,1]$ by*

$$f(\alpha) = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d}$$

for $\alpha \in [0,1]$. Then f is continuous, strictly increasing, and bijective.

Proof. By basic calculations, we get $f'(\alpha) > 0$ for any $\alpha \in [0,1]$. Since $f(0) = 0$ and $f(1) = 1$, we get the conclusion. □

Corollary 4.7. *For $D \in]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x, y) < D$. Then $[x, y] = \{tx \overset{1}{\oplus} (1-t)y \mid t \in [0,1]\}$.*

Proof. Define a function $f \colon [0,1] \to [0,1]$ by

$$f(\alpha) = \frac{1}{D} \tan^{-1} \frac{\alpha \sin D}{1 - \alpha + \alpha \cos D}$$

for $\alpha \in [0,1]$. Then we have $\{tx \overset{1}{\oplus} (1-t)y \mid t \in [0,1]\} = \{f(t)x \oplus (1-f(t))y \mid t \in [0,1]\}$ by Lemma 4.1, thus we get the conclusion by bijectivity of f. □

Corollary 4.8. *For $D \in]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x, y) < D$. Put $d_0 = d(x, y)$. Then for any $\sigma \in [0,1]$,*

$$\sigma x \oplus (1-\sigma)y = \frac{\sin(\sigma d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)} x \overset{1}{\oplus} \frac{\sin((1-\sigma)d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)} y.$$

Proof. Take $\sigma \in [0,1]$. Then there exists $\alpha \in [0,1]$ such that $\alpha x \overset{1}{\oplus} (1-\alpha)y = \sigma x \oplus (1-\sigma)y$ by Corollary 4.7. Thus, using Lemma 4.1, we obtain

$$\sigma = \frac{1}{d_0} \tan^{-1} \frac{\alpha \sin d_0}{1 - \alpha + \alpha \cos d_0},$$

which is equivalent to

$$\alpha = \frac{\sin(\sigma d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)}.$$

Consequently we obtain the conclusion. □

Lemma 4.9. *For $a, b, c, d \in \mathbb{R}$,*

$$\begin{aligned}\sin((a+b)(c-d))\sin((a-b)(c+d)) - \sin((a+b)(c+d))\sin((a-b)(c-d))\\ = -\sin 2ac \sin 2bd + \sin 2ad \sin 2bc.\end{aligned}$$

Lemma 4.10. *Let $k \in]0,1[$ and define $f \colon]0, \pi[\to \mathbb{R}$ by $f(x) = (\sin kx)/\sin x$ for $x \in]0, \pi[$. Then f is strictly increasing.*

Theorem 4.11. *Let $\alpha \in]0,1[$, and define a function $f \colon]0, \pi/2[\to]0,1[$ by*

$$f(d) = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d}$$

for $d \in]0, \pi/2[$. Then the following hold:

- $\lim_{d \to 0} f(d) = \alpha$;
- *if $\alpha < 1/2$, then f is strictly decreasing;*
- *if $\alpha > 1/2$, then f is strictly increasing.*

Proof. The equation $\lim_{d\to 0} f(d) = \alpha$ can be verified easily, thus we prove the other properties. It suffices to show the case where $\alpha < 1/2$. Let $\alpha \in]0, 1/2[$, $d_1, d_2 \in]0, \pi/2[$ and suppose $d_1 < d_2$. Put $\sigma_1 = f(d_1)$ and $\sigma_2 = f(d_2)$. Then we obtain $\sigma_1 < 1/2$ and $\sigma_2 < 1/2$ by Lemma 4.3. Moreover, using the equation $\sigma_2 = f(d_2)$, we get

$$\alpha = \frac{\sin(\sigma_2 d_2)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}. \tag{iii}$$

Define a strictly concave function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha\cos((1-t)d_1) + (1-\alpha)\cos td_1$$

for $t \in [0,1]$. Then σ_1 is a unique maximizer of g. By the formula (iii), we obtain

$$g(t) = \frac{\sin(\sigma_2 d_2)\cos((1-t)d_1) + \sin((1-\sigma_2)d_2)\cos td_1}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in [0,1]$ and hence

$$g'(t) = \frac{d_1(\sin(\sigma_2 d_2)\sin((1-t)d_1) - \sin((1-\sigma_2)d_2)\sin td_1)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in [0,1]$. Put

$$C = \frac{d_1}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}.$$

Then we get $C > 0$ and

$$\frac{1}{C}g'(\sigma_2) = \sin(\sigma_2 d_2)\sin((1-\sigma_2)d_1) - \sin((1-\sigma_2)d_2)\sin(\sigma_2 d_1).$$

Put $p = (d_1 + d_2)/2$, $q = (d_2 - d_1)/2$, and $k = 1 - 2\sigma_2$. Then using Lemma 4.9, we have

$$\begin{aligned}\frac{1}{C}g'(\sigma_2) &= \sin\left((p+q)\left(\frac{1}{2} - \frac{1}{2}k\right)\right)\sin\left((p-q)\left(\frac{1}{2} + \frac{1}{2}k\right)\right)\\ &\quad - \sin\left((p+q)\left(\frac{1}{2} + \frac{1}{2}k\right)\right)\sin\left((p-q)\left(\frac{1}{2} - \frac{1}{2}k\right)\right)\\ &= -\sin kp\sin q + \sin kq\sin p\\ &= \sin p\sin q\left(\frac{\sin kq}{\sin q} - \frac{\sin kp}{\sin p}\right).\end{aligned}$$

Since $0 < q < p < \pi/2$ and $0 < k < 1$, we get $g'(\sigma_2) > 0$ from Lemma 4.10. Therefore we obtain $\sigma_1 > \sigma_2$ and it implies $f(d_1) > f(d_2)$. □

Theorem 4.11 implies that the greater the distance between two points x and y, the further the point $\alpha x \overset{1}{\oplus} (1-\alpha)y$ is from the midpoint of x and y as a ratio than the point $\alpha x \oplus (1-\alpha)y$.

4.2 1-convex combination on unit spheres

Next, we observe the nature of the 1-convex combination on a unit sphere of a Hilbert space to know a relation between $\oplus$ and $\overset{1}{\oplus}$. Hereafter, we consider S_H the unit sphere embedded in a Hilbert space H, that is, $S_H = \{x \in H \mid \|x\| = 1\}$. Suppose that a metric $d\colon S_H \to [0,\pi]$ is defined by $d(x,y) = \cos^{-1}\langle x,y\rangle$ for each $x,y \in S_H$, where $\cos^{-1}\colon [-1,1] \to [0,\pi]$ is the inverse of the trigonometric cosine function. Then S_H is a complete CAT(1) space. If $H = \mathbb{R}^3$, then S_H becomes a model of the unit sphere $\mathbb{S}^2$, which has a constant curvature 1.

In what follows, $[x,y]$ denotes a geodesic segment on S_H joining $x,y \in S_H$, and $[x,y]_H$ denotes a geodesic segment on H joining $x,y \in H$. Furthermore, we write 0_H for the origin of H.

Theorem 4.12. *Let $x,y \in S_H$ such that $0 < d(x,y) < \pi$. Then a convex combination $tx \oplus (1-t)y \in S_H$ is expressed by*

$$tx \oplus (1-t)y = \frac{\sin(td(x,y))}{\sin d(x,y)}x + \frac{\sin((1-t)d(x,y))}{\sin d(x,y)}y$$

for any $t \in [0,1]$.

Theorem 4.13. *Let $x,y \in S_H$ such that $d(x,y) < \pi$. Then a 1-convex combination $tx \overset{1}{\oplus} (1-t)y \in S_H$ is expressed by*

$$tx \overset{1}{\oplus} (1-t)y = \frac{tx + (1-t)y}{\|tx + (1-t)y\|}$$

for any $t \in [0,1]$.

Proof. By the definition of 1-convex combination, we have

$$\begin{aligned} tx \overset{1}{\oplus} (1-t)y &= \operatorname*{argmax}_{z \in S_H} \left(t\cos d(x,z) + (1-t)\cos d(y,z)\right) \\ &= \operatorname*{argmax}_{z \in S_H} \langle tx + (1-t)y, z\rangle . \end{aligned}$$

Put $p = tx + (1-t)y$ and $w = p/\|p\|$. Then for any $z \in S_H$, we obtain

$$\langle tx + (1-t)y, w\rangle - \langle tx + (1-t)y, z\rangle = \|p\| - \langle p, z\rangle = \|p\|\|z\| - \langle p, z\rangle \geq 0.$$

Thus we get $tx \overset{1}{\oplus} (1-t)y = w$, which is the desired result. □

Corollary 4.14. *Take $x,y \in S_H$ with $d(x,y) < \pi$. For $\alpha \in [0,1]$, take $u = \alpha x + (1-\alpha)y \in H$ and put $v = \alpha x \overset{1}{\oplus} (1-\alpha)y \in [x,y]$. Then three points u, v, and 0_H are on a straight line.*

Proof. Since $v = u/\|u\|$, we get the conclusion. $\square$

Theorem 4.13 implies that $\alpha x \overset{1}{\oplus} (1-\alpha)y \in S_H$ is a projection of $\alpha x + (1-\alpha)y \in H$ into the unit sphere S_H.

Lemma 4.15. *Take $x, y \in S_H$ with $d(x, y) < \pi$. Let $k, l \in \,]0, 1]$ and put $x' = kx$, $y' = ly$. Then the geodesic segment $[x, y] \subset S_H$ is expressed by*

$$[x, y] = \left\{ \frac{tx' + (1-t)y'}{\|tx' + (1-t)y'\|} \;\middle|\; t \in [0, 1] \right\} = \left\{ \frac{p}{\|p\|} \;\middle|\; p \in [x', y']_H \right\}.$$

Proof. Take $u \in [x, y]$ arbitrarily. Then there exists $t \in [0, 1]$ such that $u = tx \overset{1}{\oplus} (1-t)y$ by Corollary 4.7. Thus, putting $t' = tl/(tl + (1-t)k)$, we get

$$u = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} = \frac{t'x' + (1-t')y'}{\|t'x' + (1-t')y'\|}.$$

On the other hand, take $s \in [0, 1]$ and put $u' = (sx' + (1-s)y')/\|sx' + (1-s)y'\|$. Then putting $s' = sk/(sk + (1-s)l)$, we obtain

$$u' = \frac{sx' + (1-s)y'}{\|sx' + (1-s)y'\|} = \frac{s'x + (1-s')y}{\|s'x + (1-s')y\|} = s'x \overset{1}{\oplus} (1-s')y \in [x, y],$$

which implies the conclusion. $\square$

Lemma 4.15 yields the following two corollaries.

Corollary 4.16. *Take $x, y \in S_H$ arbitrarily. Let $k, l \in \,]0, 1]$ and put $x' = kx$, $y' = ly$. Then $v/\|v\| \in [x, y]$ holds for any $v \in [x', y']_H$.*

Corollary 4.17. *Take $x, y \in S_H$ arbitrarily. Let $k, l \in \,]0, 1]$ and put $x' = kx$, $y' = ly$. Then for any $u \in [x, y]$, there exists $v \in [x', y']_H$ such that $u = v/\|v\|$.*

Fact 4.18 (Ceva's theorem in plane geometry)**.** *Let V be a real vector space and $x, y, z \in V$. For $\alpha, \beta, \gamma \in \,]0, 1[$, take $p = (1-\alpha)x + \alpha y$, $q = (1-\beta)y + \beta z$ and $r = (1-\gamma)z + \gamma x$. Put $[u, v]_V = \{tu + (1-t)v \mid t \in [0, 1]\}$ for each $u, v \in V$. Suppose that $[x, y]_V \cap [y, z]_V \cap [z, x]_V = \varnothing$. Then $[x, q]_V \cap [y, r]_V \cap [z, p]_V \neq \varnothing$ if and only if*

$$\frac{\alpha}{1-\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\gamma}{1-\gamma} = 1.$$

Using the 1-convex combination and the fact above, we get the following theorem which can be said to be Ceva's theorem on the unit sphere.

Theorem 4.19. *Let S be a nonempty convex subspace of S_H such that $d(u, v) < \pi$ for any $u, v \in S$, and $\triangle(x, y, z)$ a geodesic triangle on S such that $[x, y] \cap [y, z] \cap [z, x] = \varnothing$. For $\alpha, \beta, \gamma \in \,]0, 1[$, take $p = (1-\alpha)x \overset{1}{\oplus} \alpha y$, $q = (1-\beta)y \overset{1}{\oplus} \beta z$ and $r = (1-\gamma)z \overset{1}{\oplus} \gamma x$. Then $[x, q] \cap [y, r] \cap [z, p] \neq \varnothing$ if and only if*

$$\frac{\alpha}{1-\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\gamma}{1-\gamma} = 1.$$

Proof. Let $\triangle_H(x, y, z) = [x, y]_H \cup [y, z]_H \cup [z, x]_H$ be a geodesic triangle on H. Put $\overline{p} = (1-\alpha)x + \alpha y$, $\overline{q} = (1-\beta)y + \beta z$, and $\overline{r} = (1-\gamma)z + \gamma x$. Then we have $p = \overline{p}/\|\overline{p}\|$, $q = \overline{q}/\|\overline{q}\|$, $r = \overline{r}/\|\overline{r}\|$, and $\overline{p}, \overline{q}, \overline{r} \in \triangle_H(x, y, z)$. By Fact 4.18, we obtain $[x, \overline{q}]_H \cap [y, \overline{r}]_H \cap [z, \overline{p}]_H \neq \varnothing$ holds if and only if $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$. Furthermore, Corollaries 4.16 and 4.17 imply that $[x, \overline{q}]_H \cap [y, \overline{r}]_H \cap [z, \overline{p}]_H \neq \varnothing$ if and only if $[x, q] \cap [y, r] \cap [z, p] \neq \varnothing$. □

5 Balanced 1-convex combination

In a Hilbert space H, let $x_1, x_2, \ldots, x_m \in H$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ such that $\sum_{i=1}^m \alpha_i = 1$. Then

$$\sum_{i=1}^m \alpha_i x_i = \operatorname*{argmin}_{z \in H} \sum_{i=1}^m \alpha_i \|x_i - z\|^2$$

holds. Based on this fact, we generalize the 1-convex combination to be defined for a finite number of points. Let S be a nonempty convex subspace of S_H such that $d(u, v) < \pi$ for any $u, v \in S$. For $x_1, x_2, \ldots, x_m \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^m \alpha_i = 1$, we define $B(\{x_1, \ldots, x_m\}, \{\alpha_1, \ldots, \alpha_m\}) \in S$ by

$$B(\{x_1, \ldots, x_m\}, \{\alpha_1, \ldots, \alpha_m\}) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z).$$

We often write this point simply as $B(\{x_i\}, \{\alpha_i\})$. We call the point $B(\{x_i\}, \{\alpha_i\})$ a *balanced* 1*-convex combination* of $x_1, x_2, \ldots, x_m$ on S. The 1-convex combination is the case where $m = 2$ for the balanced 1-convex combination.

Theorem 5.1. *Let S be a nonempty convex subspace of S_H such that $d(u, v) < \pi$ for any $u, v \in S$, and take $x_1, x_2, \ldots, x_m \in S$ arbitrarily. Then a balanced* 1*-convex combination $B(\{x_i\}, \{\alpha_i\}) \in S$ is well-defined, and it is expressed by*

$$B(\{x_i\}, \{\alpha_i\}) = \sum_{i=1}^m \alpha_i x_i \Bigg/ \left\| \sum_{i=1}^m \alpha_i x_i \right\|$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ such that $\sum_{i=1}^m \alpha_i = 1$.

Proof. By the definition of $B(\{x_i\}, \{\alpha_i\})$, we have

$$B(\{x_i\}, \{\alpha_i\}) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z) = \operatorname*{argmax}_{z \in S} \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle.$$

Put $p = \sum_{i=1}^m \alpha_i x_i$ and $w = p/\|p\| \in S$. Then for any $z \in S \setminus \{p\}$, we obtain

$$\left\langle \sum_{i=1}^m \alpha_i x_i, w \right\rangle - \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle = \|p\| - \langle p, z \rangle = \|p\|\|z\| - \langle p, z \rangle > 0$$

and hence we get the conclusion. □

Theorem 5.1 is a generalization of Theorem 4.13.

Theorem 5.2. *Let S be a nonempty convex subspace of S_H such that $d(u,v) < \pi$ for any $u, v \in S$, and let $\triangle(x,y,z)$ be a geodesic triangle on S. Take $\alpha_1, \alpha_2, \alpha_3 \in \,]0,1[$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and let $u = B(\{x,y,z\}, \{\alpha_1, \alpha_2, \alpha_3\})$. Put $\beta = \alpha_2/(\alpha_2 + \alpha_3)$ and let $w = \beta y \overset{1}{\oplus} (1-\beta)z$. Then $u \in [x,w]$.*

Proof. Put $p = \beta y + (1-\beta)z$ and $q = \alpha_1 x + \alpha_2 y + \alpha_3 z$. Then, from Theorem 4.13 and Theorem 5.1, we obtain $w = p/\|p\|$ and $u = q/\|q\|$. Since $1 - \alpha_1 = \alpha_2 + \alpha_3$, we also have $q = \alpha_1 x + (1-\alpha_1)p$. Thus, putting $\gamma = \alpha_1/(\alpha_1 + (1-\alpha_1)\|p\|)$, we get $q = (\alpha_1 + (1-\alpha_1)\|p\|)(\gamma x + (1-\gamma)w)$. It implies

$$u = \frac{q}{\|q\|} = \frac{\gamma x + (1-\gamma)w}{\|\gamma x + (1-\gamma)w\|} = \gamma x \overset{1}{\oplus} (1-\gamma)w \in [x,w]$$

from Corollary 4.7. □

We consider that Theorem 5.2 is a crucial result that shows the suitability of the 1-convex combination on the unit sphere. Indeed, if we only use the traditional convex combination $\oplus$ on a unit sphere, then we do not obtain simple results like Theorem 5.2.

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References

[1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non–Positive Curvature*, vol. 319 of *Grundlehren der. Mathematischen Wissenschaften*, Springer, Verlag, Berlin, Germany, 1999.

[2] Y. Kimura and K. Sasaki, *A Halpern type iteration with multiple anchor points in complete geodesic spaces with negative curvature*, Fixed Point Theory **21** (2020), 631–646.

[3] Y. Kimura and K. Sasaki, *A Halpern's iterative scheme with multiple anchor points in complete geodesic spaces with curvature bounded above*, Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis & International Conference on Optimization: Techniques and Applications -I- (Hakodate, Japan, 2019), 313–329.

[4] Y. Kimura and S. Sudo, *New type parallelogram laws in Banach spaces and geodesic spaces with curvature bounded above*, Arab. J. Math. (2022).