A criterion of compact sets in $\mathrm{L^p}$-spaces and its application

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1 Introduction

In functional analysis, it is important to recognize compact subsets of function spaces, and criteria like the Ascoli-Arzelà theorem play central roles. In this article, we shall characterize subsets of an $\mathrm{L^p}$-space on a metric measure space to be compact, and as its application, we will investigate the topological structure of the subspace consisting of lipschitz functions with a bounded support. This is a résumé of the paper [10]. Assume that $X = (X, d, \mu)$ is a Borel-regular Borel metric measure space such that for every $x \in X$ and every $r > 0$, $0 < \mu(B(x, r)) < \infty$, where d is a metric, μ is a measure and $B(x, r)$ is the closed ball centered at x with radius r. Recall that a measure space is *Borel* if its Borel sets are measurable, and that a Borel measure space is *Borel-regular* if any subset is contained in some Borel set with the same measure. For $1 \leq p < \infty$, let $\mathrm{L^p}(X) = (\mathrm{L^p}(X), \|\cdot\|_p)$ be the $\mathrm{L^p}$-space on X. It is known that $\mathrm{L^p}(X)$ is a Banach space. Giving a criterion of compactness in $\mathrm{L^p}$-spaces traces its history back to the Kolmogorov-Riesz theorem [8, 12], which is an $\mathrm{L^p}$-version of the Ascoli-Arzelà one.

Theorem 1.1 (Kolmogorov-Riesz). *A bounded set $F \subset \mathrm{L^p}(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space with the Euclidean metric and the Lebesgue measure, is relatively compact if and only if the following are satisfied.*

(1) *For every $\epsilon > 0$, there is $\delta > 0$ such that $\|\tau_a f - f\|_p < \epsilon$ for any $f \in F$ and any $a \in \mathbb{R}^n$ with $|a| < \delta$.*[1]

(2) *For each $\epsilon > 0$, there exists $r > 0$ such that $\|f\chi_{\mathbb{R}^n \setminus B(\mathbf{0},r)}\|_p < \epsilon$ for all $f \in F$.*[2]

Some generalizations in this direction are seen in [6]. Fixing $f \in \mathrm{L^p}(X)$ and $r > 0$, define the *average function* $A_r f : X \to \mathbb{R}$ of f by

$$A_r f(x) = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f\chi_{B(x,r)}(y) d\mu(y).$$

[1]The function $\tau_a f$ is defined by $\tau_a f(x) = f(x - a)$.

[2]For a subset $E \subset X$, we denote the characteristic function of E by χ_E.

Recently, the Kolmogorov-Riesz type criteria of compactness in $\mathrm{L^p}$-spaces and more generally, in Banach function spaces, have been obtained by use of average functions, refer to [4, 5]. It is said that X is *doubling* if the following condition holds.

- There is $\gamma \geq 1$ such that $\mu(B(x, 2r)) \leq \gamma\mu(B(x, r))$ for any $x \in X$ and any $r > 0$.[3]

In the paper [10], the following characterization is established.

Theorem 1.2. *Let X be doubling and let F be bounded in $\mathrm{L^p}(X)$. Suppose that for each $x \in X$ and each $r > 0$,*

$$\mu(B(x, r) \triangle B(y, r)) \to 0$$

as $y \to x$.[4] *Then F is relatively compact if and only if the following conditions are satisfied.*

(1) *For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|A_r f - f\|_p < \epsilon$ for each $f \in F$ and each $r \in (0, \delta)$.*

(2) *For every $\epsilon > 0$, there is a bounded subset $E \subset X$ such that $\|f\chi_{X \setminus E}\|_p < \epsilon$ for all $f \in F$.*

In the study of topologies on function spaces, typical convex subspaces of Hilbert space ℓ_2 and the Hilbert cube $\mathbf{Q}$ have been detected among them. Due to the efforts by R.D. Anderson [1] and M.I. Kadec [7], we have the following:

Theorem 1.3. *If X is infinite and separable, then $\mathrm{L^p}(X)$ is homeomorphic to ℓ_2.*

It is known that the subspace ℓ_2^f spanned by the canonical orthonormal basis of ℓ_2 is recognized among several function spaces as a factor. For instance, the author [9] investigated the topological type of the subspace $\mathrm{UC}(X) \subset \mathrm{L^p}(X)$ consisting of uniformly continuous functions.[5]

Theorem 1.4 (K)**.** *Suppose that X is a separable and locally compact metric measure space. If the subset $\{x \in X \mid \mu(\{x\}) \neq 0\}$ is not dense in X, then $\mathrm{UC}(X)$ is homeomorphic to the countable product of ℓ_2^f.*

Let $\mathrm{LIP}_b(X) \subset \mathrm{L^p}(X)$ be the subspace consisting of lipschitz functions with a bounded support. As an application of Theorem 1.2, the following corollary is obtained in [10].

Corollary 1.5. *Let X be non-degenerate, separable and doubling. If for every $x \in X$, the function*

$$(0, \infty) \ni r \mapsto \mu(B(x, r)) \in (0, \infty)$$

is continuous, then $(\mathrm{L^p}(X), \mathrm{LIP}_b(X))$ is homeomorphic to $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$.[6]

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[3]The number γ is called the *doubling constant*.

[4]Given subsets $A, B \subset X$, put $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

[5]R. Cauty [2] proved it in the case where $X = [0, 1]$ with the usual metric and the Lebesgue measure.

[6]For spaces $Y_1 \supset Y_2$ and $Z_1 \supset Z_2$, the pair (Y_1, Y_2) is homeomorphic to (Z_1, Z_2) provided that there exists a homeomorphism $f : Y_1 \to Z_1$ such that $f(Y_2) = Z_2$.

2 A characterization of compactness in $\mathrm{L^p}(X)$

Combining Hölder's inequality with the Fubini-Tonelli theorem, we can prove the following lemma, which means that A_r is bounded as an operator on $\mathrm{L^p}(X)$.

Lemma 2.1. *Suppose that X is a doubling metric measure space with the doubling constant γ and that $f \in \mathrm{L^p}(X)$. Then $\|A_r f\|_p \le \gamma^{1/p}\|f\|_p$ for any $r > 0$.*

We will show the "only if" part of Theorem 1.2.

Sketch of proof. Since F is relatively compact, and hence it is totally bounded, it can be approximated by finitely many bounded functions $f_i \in \mathrm{L^p}(X)$, $1 \le i \le n$, with a bounded support E_i. According to Lemma 2.1, we can see that for each $f \in F$,

$$\begin{aligned}\|A_r f - f\|_p &\le \|A_r f - A_r f_i\|_p + \|A_r f_i - f_i\|_p + \|f_i - f\|_p \\ &\le \|A_r f_i - f_i\|_p + (\gamma^{1/p} + 1)\|f_i - f\|_p,\end{aligned}$$

where γ is the doubling constant of X. By virtue of the Lebesgue differentiation theorem and the dominated convergence theorem, $\|A_r f_i - f_i\|_p \to 0$ as $r \to 0$, and hence the condition (1) holds. On the other hand, letting $E = \bigcup_{x \in \bigcup_{i=1}^n E_i} B(x, 1)$, observe that

$$\|f\chi_{X\setminus E}\|_p \le \|f\chi_{X\setminus E} - f_i\chi_{X\setminus E}\|_p + \|f_i\chi_{X\setminus E}\|_p \le \|f - f_i\|_p.$$

Therefore the condition (2) follows immediately. □

From now on, consider the following conditions between metrics and measures on X:

($\star$) For each $x \in X$, $(0,\infty) \ni r \mapsto \mu(B(x,r)) \in (0,\infty)$ is continuous;

($*$) For any $x \in X$ and any $r \in (0,\infty)$, $\mu(B(x,r) \triangle B(y,r)) \to 0$ as $y \to x$;

($\dagger$) For every $r \in (0,\infty)$, $X \ni x \mapsto \mu(B(x,r)) \in (0,\infty)$ is continuous.

As is easily observed, the implications $(\star) \Rightarrow (*) \Rightarrow (\dagger)$ hold. Measures of closed balls with the same radius, whose centers are belonging to a bounded subset of a doubling metric measure space are lower bounded.

Lemma 2.2. *Let X be doubling. For every bounded subset $E \subset X$ and for every positive number $r > 0$, $\inf\{\mu(B(x,r)) \mid x \in E\} > 0$.*

It is a key idea in proving the "if" part of Theorem 1.2 to approximate a function by a simple function, while it is significant to approximate a function by a "line graph" in the proof of the Ascoli-Arzelà one. The following proposition implies that the operator A_r on $\mathrm{L^p}(X)$ is compact when X is bounded and doubling, and $p > 1$.

Proposition 2.3. *Let X be a doubling metric measure space satisfying the property ($*$) and let $F \subset \mathrm{L^p}(X)$ be a bounded subset. Suppose that $E \subset X$ is a bounded set and that $r > 0$. Then $\{(A_r f)\chi_E \mid f \in F\}$ is relatively compact in $\mathrm{L^p}(X)$ when $p > 1$. Additionally, if for each $\epsilon > 0$, there exists $\delta > 0$ such that $\|A_r f - f\|_1 < \epsilon$ for every $f \in F$ and every $r \in (0,\delta)$, then the above holds even if $p = 1$.*

Sketch of proof. By Hölder's inequality, we get that for any $x, y \in X$,

$$|A_r f(x) - A_r f(y)| \leq \frac{1}{\mu(B(x,r))} \|f\|_p (\mu(B(x,r) \triangle B(y,r))^{1/q} + \left| \frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(y,r))} \right| \|f\|_p (\mu(B(y,r)))^{1/q}.$$

In the case where $p = 1$, if $y \in B(x,1)$ and $s > 0$, then according to Lemma 2.2,

$$\|f\|_1 \leq \|A_s f - f\|_1 + \frac{1}{\inf\{\mu(B(z,s)) \mid z \in B(x,r+1)\}} \|f\|_1 \mu(B(x,r) \triangle B(y,r)).$$

By the assumption, $\mu(B(x,r) \triangle B(y,r)) \to 0$ as $y \to x$, and hence $\mu(B(y,r)) \to \mu(B(x,r))$. Thus the following equicontinuity of average functions on F is valid.

- For every $x \in X$ and every $\epsilon > 0$, there is $\delta > 0$ such that if $d(x,y) < \delta$, then $|A_r f(x) - A_r f(y)| < \epsilon$ for all $f \in F$.

The boundedness of average functions on F follows from the one of F.

- For each $x \in X$, $\{A_r f(x) \mid f \in F\}$ is bounded in $\mathbb{R}$.

Combining the above equicontinuity and the boundedness with the Vitali covering theorem (cf. [14, Theorem 6.20]), we can show that $\{A_r f \mid f \in F\}$ is approximated by a finitely many collection of simple functions, which implies that $\{A_r f \mid f \in F\}$ is totally bounded. □

We shall prove the "if" part of Theorem 1.2.

Sketch of proof. For a bounded subset $E \subset X$ and $r > 0$,

$$\|f - (A_r f)\chi_E\|_p \leq \|f\chi_E - (A_r f)\chi_E\|_p + \|f\chi_{X \setminus E}\|_p \leq \|f - A_r f\|_p + \|f\chi_{X \setminus E}\|_p.$$

It follows from the conditions (1) and (2) that F can be approximated by the subset $\{(A_r f)\chi_E \mid f \in F\}$, that is totally bounded in $\mathrm{L}^{\mathrm{p}}(X)$ due to Proposition 2.3. Hence F is relatively compact. □

3 The topological structure of $\mathrm{LIP}_b(X)$

To prove Corollary 1.5, we shall use the following criterion, which was proven by D. Curtis, T. Dobrowolski and J. Mogilski [3].

Theorem 3.1 (Curtis-Dobrowolski-Mogilski)**.** *Let C be a σ-compact convex set in a completely metrizable linear space, whose closure $\mathrm{cl}\, C$ is an AR and not locally compact. Then $(\mathrm{cl}\, C, C)$ is homeomorphic to $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$ if C contains an infinite-dimensional locally compact convex set.*

The density of $\mathrm{LIP}_b(X) \subset \mathrm{L}^\mathrm{p}(X)$ follows from the doubling property and the condition ($\star$) of X.

Proposition 3.2. *If X is a doubling metric measure space satisfying the property ($\star$), then $\mathrm{LIP}_b(X)$ is dense in $\mathrm{L}^\mathrm{p}(X)$.*

Sketch of proof. To approximate a function of $\mathrm{L}^\mathrm{p}(X)$, by virtue of the Vitali covering theorem, we can choose a lipschitz function $g : X \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} a_i & \text{if } x \in B(x_i, \delta_i), i = 1, \cdots, n, \\ \frac{-a_i(d(x,x_i)-\delta_i)}{r_i - \delta_i} + a_i & \text{if } x \in B(x_i, r_i) \setminus B(x_i, \delta_i), i = 1, \cdots, n, \\ 0 & \text{if otherwise,} \end{cases}$$

where for each $i = 1, \cdots, n$, $x_i \in X$, $a_i \neq 0$, $0 < \delta_i < r_i$, and $\{B(x_i, r_i) \mid i = 1, \cdots, n\}$ is pairwise disjoint. □

Fix any point $x_0 \in X$. Given a positive integer n, put

$$L(n) = \{f \in \mathrm{LIP}(X) \mid \|f\|_p \leq n, \mathrm{lip}\, f \leq n \text{ and } \mathrm{supp}\, f \subset B(x_0, n)\}.^7$$

Note that $\mathrm{LIP}_b(X) = \bigcup_{n\in\mathbb{N}} L(n)$. Applying Theorem 1.2, we can prove the following:

Proposition 3.3. *Suppose that X is a doubling metric measure space which satisfies the property ($*$). Then each $L(n)$ is compact.*

Sketch of proof. By the definition of $L(n)$, it is bounded and satisfies the condition (2) of Theorem 1.2. Observe that $L(n)$ is closed. Taking any $f \in L(n)$ and any $r \in (0, 1)$, since f is n-lipschitz, we have that for each $x \in X$,

$$\begin{aligned} |A_r f(x) - f(x)| &\leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| d\mu(y) \\ &\leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} n d(x,y) d\mu(y) \leq nr. \end{aligned}$$

Then $\|A_r f - f\|_p \leq nr(\mu(B(x_0, n+1)))^{1/p}$, which implies that the condition (1) of Theorem 1.2 holds. □

It will be shown that each $L(n)$ is homeomorphic to $\mathbf{Q}$ by using the following characterization [13].

Theorem 3.4 (Toruńczyk)**.** *A compact AR Y is homeomorphic to $\mathbf{Q}$ if and only if the following is satisfied.*

($\sharp$) *Let n be any positive integer and $f : [0,1]^n \times \{0,1\} \to Y$ be any continuous function. For every $\epsilon > 0$, there is a continuous function $g : [0,1]^n \times \{0,1\} \to Y$ such that g is ϵ-close to f and $g([0,1]^n \times \{0\}) \cap g([0,1]^n \times \{1\}) = \emptyset$.*[8]

[7] The symbols $\mathrm{lip}\, f$ and $\mathrm{supp}\, f$ stand for the lipschitz constant and the support of f, respectively.
[8] This property is called the *disjoint cells property.*

Recall that for a metric space $Y = (Y, d_Y)$, for functions $f : Z \to Y$ and $g : Z \to Y$, and for a positive number $\epsilon > 0$, f is *ϵ-close* to g if $d_Y(f(z), g(z)) \leq \epsilon$ for all $z \in Z$.

Lemma 3.5. *Let X be a doubling metric measure space such that $\mu(\{x_0\}) = 0$, and let n be a positive integer. For any $\epsilon > 0$, there exist maps $\Phi : L(n) \to \{f \in L(n) \mid \operatorname{lip} f < n\}$ and $\Psi : L(n) \to \{f \in L(n) \mid \operatorname{lip} f = n\}$ which are ϵ-close to the identity map on $L(n)$.*

Sketch of proof. Let $\Phi(f) = (1 - \epsilon/n)f$ for each $f \in L(n)$. Due to the map Φ, we may assume that there exists $m < n$ such that $\|f\|_p \leq m$ and $\operatorname{lip} f \leq m$ for every $f \in L(n)$. Taking $0 < r_2 < r_1$ appropriately, for each $f \in L(n)$, we can define a function $\psi(f) : X \setminus (B(x_0, r_1) \setminus B(x_0, r_2)) \to \mathbb{R}$ by

$$\psi(f)(x) = \begin{cases} f(x_0) + n(r_2 - d(x, x_0)) & \text{if } x \in B(x_0, r_2), \\ f(x) & \text{if otherwise.} \end{cases}$$

Applying McShane's lipschitz extension [11], let

$$\Psi(f)(x) = \begin{cases} \psi(f)(x) & \text{if } x \in X \setminus (B(x_0, r_1) \setminus B(x_0, r_2)), \\ \inf_{a \in X \setminus (B(x_0, r_1) \setminus B(x_0, r_2))} (\psi(f)(a) + nd(a, x)) & \text{if otherwise,} \end{cases}$$

which is the desired. □

It is a direct consequence of the above lemma that every $L(n)$ satisfies the property ($\sharp$) of Theorem 3.4.

Proposition 3.6. *Let X be a doubling metric measure space with $\mu(\{x_0\}) = 0$. If the condition ($*$) holds, then $L(n)$ is homeomorphic to $\mathbf{Q}$ for any positive integer n.*

Now we shall show Corollary 1.5.

Sketch of proof. By Propositions 3.2 and 3.6, we can prove that $\mathrm{LIP}_b(X) \subset \mathrm{L}^p(X)$ is a dense σ-compact subset containing topological copies of $\mathbf{Q}$. According to Theorem 3.1, the pair $(\mathrm{L}^p(X), \mathrm{LIP}_b(X))$ is homeomorphic to $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$. □

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