# A FEW RESULTS ABOUT THE HYPERBOLIC ANDERSON MODEL

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The objective of the subsequent notes is to give an overview of the considerations and results contained in the two studies [7, 8], written in collaboration with with X. Chen, J. Song and S. Tindel. The reader is thus referred to these two publications for further details, and in particular for the proof of the assertions below.

## 1. Introduction

### 1.1. The model under consideration.

In both [7] and [8], our investigations focus on the so-called *hyperbolic Anderson model*, which is nothing but the "wave" version of the celebrated parabolic Anderson model. To be more specific, our objective is to offer a natural interpretation, as well as a wellposedness statement, for the equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u\,\dot{B}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d, \tag{1}$$

where:

- we assume for simplicity that $u(0,.) \equiv 1$ and $\partial_t u(0,.) \equiv 0$.
- $d \in \{1,2,3\}$.
- $\dot{B}$ is a *space-time fractional noise* of index $H = (H_0, ..., H_d) \in (0,1)^{d+1}$, as defined below.

**Definition 1.1.** *We call a* space-time fractional Brownian noise *of Hurst index* $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ *any centered Gaussian noise* $\dot{B}$ *on* $\mathbb{R}_+ \times \mathbb{R}^d$ *with covariance (formally) given by*

$$\mathbb{E}\big[\dot{B}(s,x)\dot{B}(t,y)\big] = |s-t|^{2H_0-2} \prod_{i=1}^d |x_i - y_i|^{2H_i-2}.$$

The fractional noise $\dot{B}$ can also be defined as a the space-time distributional derivative

$$\dot{B} := \frac{\partial^{d+1} B}{\partial t \partial x_1 \cdots \partial x_d}, \tag{2}$$

where $B$ refers this time to the space-time fractional Brownian field.

In the (very) specific situation where $H_0 = \ldots = H_d = \frac{1}{2}$, the above definition coincides with that of the classical space-time white noise. In fact, the space-time fractional noise is often considered as the most standard generalization of the white noise. Its use within stochastic differential models has been the topic of an abundant literature, including papers by many great authors such as M. Hairer, D. Nualart, M. Gubinelli and P. Friz, to cite but a few.

The consideration of a fractional noise in differential dynamics can be justified through many reasons, among which:

($i$) Fractional noise is a relevant model for *long-range dependence* phenomena. It is indeed a well-known fact that as soon as $H_i \neq \frac{1}{2}$, the disjoint increments of a fractional field (in the $i$-th direction) are no longer independent, but still exhibit some correlation which can be sharply quantified (in terms of $H_i$).

($ii$) By adjusting the values of $H_0, \ldots, H_d$, one can calibrate the (pathwise) regularity of the noise and thus fit with a given physical model, or adapt to some mathematical constraints. In brief, the smaller $H_i$, the more irregular $\dot{B}$ in the $i$-th direction.

(*iii*) Letting the indexes $H_0,\ldots,H_d$ vary allows us to study the influence of rough perturbations on the equation. The procedure typically goes as follows. When the indexes are close to 1 (that is, the noise is quite regular), the model under consideration can usually be treated with quite direct arguments, not too different from those used in smooth situations. Then, as the indexes decrease, direct interpretation becomes impossible, and the analysis must appeal to more sophisticated tools, such as expansion and renormalization procedures. In this context, a particularly interesting objective lies in the possibility to offer a "continuous" deformation of the model between the regular case ($H_0 = \ldots = H_d \approx 1$) and the classical space-time white-noise situation ($H_0 = H_1 = \ldots = H_d = \frac{1}{2}$), or at least the case of a white noise in time ($H_0 = \frac{1}{2}$).

With the above elements in mind, our global objective about (1) is to provide a treatment of the equation *for a class of indexes $H_i$'s as large as possible.*

Let us indicate that in the two strategies developed in the sequel, the equation will be handled in its mild form, namely (recall that we have assumed $u(0,.) \equiv 1$ and $\partial_t u(0,.) \equiv 0$):

$$u_t(x) = 1 + \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_s(y)\, dB_s(y), \tag{3}$$

where the notation $\mathcal{G}$ refers to the wave kernel on $\mathbb{R}^d$, characterized by its spatial Fourier transform

$$\mathcal{F}(\mathcal{G}_t)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0,\ \xi \in \mathbb{R}^d.$$

## 1.2. **Possible approaches to the problem.**

### 1.2.1. *A word on the white noise situation.*

Although it only represents a very specific example in our general fractional setting, the first situation one can think of - because of the many papers related to it - is when $\dot{B}$ is a white noise in time, or in other words when $H_0 = \frac{1}{2}$.

In this case, Itô integration theory, or more precisely its SPDE extension by Walsh, naturally offers us a powerful tool to interpret and solve the equation (3). This approach has been extensively studied in the literature, far beyond the linear Anderson model, and we can for instance refer the reader to fundamental works by A. Millet, M. Sanz-Solé, R. Dalang, D. Nualart, and many others.

Unfortunately, this standard treatment of the problem, based on martingale-type constructions, is known to collapse completely as soon as $H_0 \neq \frac{1}{2}$. One must then turn to alternative strategies.

### 1.2.2. *The fractional situation.*

Looking at the literature about differential equations driven by a fractional noise (including both SDEs and SPDEs), one can essentially identify two main strategies dealing with "fully fractional" situations, that is with cases where $H_0 \neq \frac{1}{2}$.

The first main approach, that we will refer to as the *stochastic* approach in the sequel, relies on the interpretation of the product against $dB$ as a Wick product, and the use of the related Skorohod integral. This is the strategy we have followed in our first study [7] about the model (3), and the one that we propose to develop in Section 2 below.

The second main approach is the so-called *pathwise* approach, which somehow draws a line between the fundamental stochastic objects at the core of the dynamics and the underlying deterministic machinery. These ideas can be found for instance in the theory of rough paths for standard differential equations (extending the Young integration procedure), or in the theory of regularity structures for stochastic parabolic equations. The pathwise approach is the method we have used in our second study [8] about (3), and the one we will elaborate on in Section 3 below.

## 2. Stochastic approach - Skorohod integral

We start with a brief account on the setting and results in [7] about the stochastic approach.

### 2.1. Skorohod interpretation of the model.

In the Skorohod setting, the equation (4) is recast into

$$u_t(x) = 1 + \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_s(y)\, d^\diamond B_s(y), \tag{4}$$

where the integral is understood in the Skorohod sense, that is as the stochastic integral derived from the so-called *Wick product* rule.

The Wick product (denoted by $\diamond$) is a standard tool in Gaussian analysis. At a basic level, it can be regarded as a conveniently rescaled version of the standard product: given two test-functions $h_1, h_2$ on $\mathbb{R}_+ \times \mathbb{R}^d$, one has for instance

$$\dot{B}(h_1) \diamond \dot{B}(h_2) = \dot{B}(h_1)\dot{B}(h_2) - \langle h_1, h_2 \rangle_{\mathcal{H}},$$

where the "correction" term $\langle h_1, h_2 \rangle_{\mathcal{H}}$ appeals to the Gaussian Hilbert space $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ generated by $\dot{B}$.

This correction term is actually designed in such a way that the integral built upon the Wick product rule, i.e. the Skorohod integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} v_s(y)\, d^\diamond B_s(y)$, can still be controlled in $L^2(\Omega)$, just as the standard Itô integral in the white noise situation. It holds indeed that

$$\mathbb{E}\Big[\Big| \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} v_s(y)\, d^\diamond B_s(y) \Big|^2\Big] \leq \mathbb{E}\Big[\|v\|_{\mathcal{H}}^2\Big] + \mathbb{E}\Big[\|Dv\|_{\mathcal{H}\otimes\mathcal{H}}^2\Big],$$

where $D$ stands here for the Malliavin derivative operator.

Such a stochastic control then paves the way for a stochastic treatment of equation (4), and at first provides us with a possible interpretation of the dynamics.

### 2.2. Skorohod approach: equivalent formulation.

Another standard tool in the Skorohod setting, at the core of our analysis in [7], is the so-called *chaos expansion* procedure: any functional $u(t,x)$ of $\dot{B}$ can be expanded as an infinite sum of multiple integrals

$$u(t,x) = \sum_{n \geq 0} I_n^B(f_n(.; t, x)), \tag{5}$$

for a suitable sequence $(f_n)_{n \geq 0}$ of deterministic kernels.

The Skorohod integration operator is then known to obey very simple algebraic rules with respect to the multiple integrals $(I_n^B)_{n \geq 1}$. Using these properties, one can (formally) inject the chaos expansion of the (future) solution $u$ into the mild equation (4) and derive an iterative relation for the corresponding kernels $(f_n)_{n \geq 0}$. Due to the linearity of our problem, this iterative relation happens to be explicitly solvable. Thus, if $u$ is a solution of (4), then for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the kernels $(f_n(.; t, x))_{n \geq 0}$ of the chaos expansion of $u(t,x)$ must be given by the expression

$$f_n(s_1, x_1, ..., s_n, x_n; t, x) = \frac{1}{n!} \mathcal{G}_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots \mathcal{G}_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}), \tag{6}$$

where $\sigma \in \mathfrak{S}_n$ is the permutation such that $0 < s_{\sigma(1)} < \ldots < s_\sigma(n) < t$.

Conversely, starting from the expression of $(f_n)_{n \geq 0}$ in (6), it turns out that, *provided we can guarantee the convergence of the right-hand side of* (5) *in* $L^2(\Omega)$, the resulting process $u$ is a solution of (4). In the end, these observations provide us with the following convenient wellposedness criterion:

**Proposition 2.1.** *Let* $(f_n)_{n \geq 1}$ *be the sequence given by* (6).

*Then the Skorohod equation* (4) *admits a unique solution if and only if for every* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\sum_{n \geq 0} n! \|f_n(.; t, x)\|_{\mathcal{H}^{\otimes n}} < \infty\,. \tag{7}$$

2.3. **Skorohod approach: main result.**

Once endowed with the general wellposedness criterion of Proposition 2.1 (valid for any Gaussian noise $\dot{B}$), our objective in the fractional setting becomes clear from a technical point of view. Namely, one must find - possibly optimal - conditions on the fractional noise, i.e. conditions on the indexes $H_0, \ldots, H_d$, so that the convergence in (7) holds true, for $(f_n)_{n\geq 0}$ given by (6).

With this objective in mind, and at the price of highly technical computations induced by the complex structure of $\mathcal{H}$, we have obtained the following - essentially optimal - condition.

**Theorem 2.2.** ***(Chen-D.-Song-Tindel, 21).***

*Consider the Skorohod wave equation*

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u \diamond \dot{B}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d,\ d \in \{1,2,3\},$$

*where $\dot{B}$ is a space-time fractional noise of index $(H_0, \ldots, H_d) \in (0,1)^{d+1}$, and assume that*

$$H_0, H_1, \ldots, H_d \geq \frac{1}{2}.$$

*Then the equation admits a unique solution* if and only if

$$H_0 + \sum_{i=1}^{d} H_i > d - \frac{1}{2}.$$

Note that our main wellposedness result in [7] applies in fact to a slightly more general class of Gaussian noises, and to more general initial conditions. The above statement for the fractional noise can be more specifically deduced from the combination of [7, Theorem 3.1] and [7, Remark 1.5].

Let us finally mention that Theorem 2.2 improves upon several previous results in the literature. The existence of a solution for (4) had for instance been established in [1, 4] under the more restrictive conditions

$$H_0, \ldots, H_d > \frac{1}{2}, \quad \sum_{i=1}^{d} H_i > d-1,$$

and more recently in [3] for the situation where

$$H_0 \approx 1, \quad H_1, \ldots, H_d > \frac{1}{2} \quad \text{and} \quad \sum_{i=1}^{d} H_i > \frac{d}{2},$$

which corresponds to the particular case of a time-independent noise.

## 3. Pathwise approach - Young wave integral

Let us now report on the second possible strategy to handle the hyperbolic Anderson model, namely the pathwise strategy, along the considerations developed in [8].

3.1. **Pathwise approach.**

We are still interested in the mild hyperbolic Anderson model

$$u_t(x) = 1 + \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y) u_s(y)\, dB_s(y), \tag{8}$$

with $\dot{B}$ is a space-time fractional noise of index $H = (H_0, ..., H_d) \in (0,1)^{d+1}$, and would like to develop a pathwise approach to it.

What we mean here by *pathwise approach* can be loosely summed up as follows : we are looking for conditions on the sole *pathwise regularity* of $\dot{B}$ - in a suitable space of distributions - so that once these conditions are satisfied, the equation can be interpreted and solved with *deterministic* arguments only.

At this stage, let us recall that the theory of regularity structures, which provides powerful (pathwise) tools in the parabolic framework, does not apply in the wave setting, owing to the weaker regularization properties of the wave kernel (see Section 3.3 below for further details).

Our analysis in [8] relies on a first-order development of the equation. In other words, we are looking for an interpretation of the wave integral as the limit of Riemann sums, which somehow corresponds to the wave counterpart of the Young integral for SDEs.

In order to initiate this Young-type construction procedure, let us go back to the representation (2) of the fractional noise as a space-time derivative of a fractional Brownian field $B$. To be more specific, we write from now on

$$\dot{B} = \partial_t(\partial_x B), \quad \text{where } \partial_x B := \frac{\partial^{d+1} B}{\partial t \partial x_1 \cdots \partial x_d},$$

and accordingly rephrase equation (8) as

$$u_t = 1 + \int_0^t \mathcal{G}_{t-r}\big(u_r \, d(\partial_x B)_r\big), \tag{9}$$

where $\mathcal{G}$ now refers to the wave operator $\mathcal{G}_t f(x)$"="$\int_{\mathbb{R}^d} \mathcal{G}_t(x-y) f(y)\, dy$.

The aforementioned objective can then be refined along a new formulation. Namely, we would like to find out (almost sure) regularity conditions on the space derivative

$$\partial_x B : \Omega \times [0,T] \to \mathcal{B}^{-\alpha},$$

seen as a time process with values in a suitable space $\mathcal{B}^{-\alpha}$ of negative-order distributions, so that:

$(i)$ We can guarantee the convergence of the sequence of Riemann sums defined by

$$\mathcal{J}_t^{(n)} := \sum_{k=0}^{m-1} \mathcal{G}_{t-t_k^n}(u_{t_k^n} \{\partial_x B_{t_{k+1}^n} - \partial_x B_{t_k^n}\}), \quad \text{for } t \in (t_{m-1}^n, t_m^n]\,. \tag{10}$$

$(ii)$ We can use the limit to interpret and solve the equation (9).

### 3.2. **Previous works related to pathwise approach.**

Before going further, let us mention some previous findings about the pathwise treatment of stochastic wave equations.

3.2.1. *Additive noise models.* Some attention has been paid recently to models of wave equation with additive noise and polynomial nonlinearities, i.e. equations of the form

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u \pm u^p + \dot{B}, \tag{11}$$

with an additive fractional noise $\dot{B}$ and an integer $p \geq 1$. In this context, pathwise procedures are implemented for instance in [9, 10, 11], yielding existence results for (11) in case of a rough noise $\dot{B}$.

The strategy in this additive-noise situation essentially consists in an adaptation of the so-called "Da Prato-Debussche trick" to the wave framework. Unfortunately, such a method is clearly not available for the multiplicative case under consideration.

3.2.2. *Hyperbolic Anderson model.* As far as we know, the first pathwise developments for a wave equation with multiplicative noise can be found in [12], dealing with the one-dimensional case. Based on the specific expression of the wave kernel for $d = 1$, the strategy therein relies on a natural preliminary rotation of the model, that turns (8) into a more tractable equation in the plane $\mathbb{R}^2$. The downside of the method comes from the fact that the fractional noise injected in $\mathbb{R}^2$ no longer corresponds to a fractional noise for the original equation in $\mathbb{R}_+ \times \mathbb{R}$, but only to a "rotated" version of it. This drawback is one of our main motivations for investigating a more direct approach in [8].

The other reference about the hyperbolic Anderson model (understood in a pathwise sense) is the recent publication [2], focusing on a time-independent noise $\dot{B}$ in dimensions $d \in \{1,2\}$. The techniques in [2], based on chaos expansions for Stratonovich integrals, are thus specifically designed for the spatial-noise case, and cannot cover our general space-time noise.

As one can see, the study of wave equations in a rough setting is still wide open. Our contribution in [8] aimed at a better understanding of the Young regime within this landmark.

3.3. **About the kernel regularization effect.**

Let us now briefly evoke the new difficulties raised by the wave situation, in comparison with the widely-studied heat case.

To this end, recall first that the heat kernel $G$ (on $\mathbb{R}^d$) satisfies very nice properties in the scale of *space-time* Besov spaces. Roughly speaking, for any $\alpha \in \mathbb{R}$ and any test-function $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, it holds that

$$\big\| G *_{t,x} \varphi \big\|_{\mathfrak{B}^{\alpha+2}} \lesssim \big\| \varphi \big\|_{\mathfrak{B}^{\alpha}}, \tag{12}$$

where $*_{t,x}$ denotes the *space-time* convolution, and $\mathfrak{B}^\alpha$ is the Besov space of *space-time* distributions of regularity $\alpha$. This very clear and convenient expression of the "+2" regularization effect of the heat kernel is one of the starting points of the theory of regularity structures (among many other ingredients): thanks to it, time and space variables are somehow merged into a single space-time variable, and the model can then be expanded around this single variable.

Unfortunately, convenient *space-time* controls such as (12) are not available for the wave kernel $\mathcal{G}$, which - at least partially - explains why the regularity structures machinery no longer works in this case.

And yet, sharp estimates on the action of $\mathcal{G}$ also exist in the PDE literature. For instance, using Fourier analysis, it is easy to check that for all $\alpha \in \mathbb{R}$ and $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{N}\big[\mathcal{G} *_{t,x} f; L^\infty([0,T]; \mathcal{H}^{\alpha+1}(\mathbb{R}^d))\big] \lesssim \mathcal{N}[f; L^2([0,T]; \mathcal{H}^{\alpha}(\mathbb{R}^d))], \tag{13}$$

where $\mathcal{H}^\alpha$ is the standard Sobolev space of order $\alpha$ on $\mathbb{R}^d$. Nevertheless, such a control soon appears to be insufficient in order to interpret the integral $\int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big)$ at the core of our problem, for two fundamental reasons:

(*i*) Using inequality (13) would imply to handle the time-derivative $d(\partial_x B)_r$ in the scale $L^p([0,T],.)$, which, given the roughness of the fractional field $B$, is clearly not possible. The only (sharp) information at our disposal regarding the time process $t \mapsto \partial_x B_t$ involves its Hölder regularity.

(*ii*) Owing to the behaviour of the fractional field $B_t : x \mapsto B_t(x)$ as $|x| \to \infty$, the process $t \mapsto \partial_x B_t$ is only expected to live in a *weighted* Sobolev space $\mathcal{H}^\alpha_w(\mathbb{R}^d)$, for a suitable weight $w : \mathbb{R}^d \to \mathbb{R}_+$.

The desired pathwise construction of the wave integral thus requires us to exhibit a new control on the action of $\mathcal{G}$, in the vein of the classical Strichartz inequality, and taking the two above features (*i*) and (*ii*) into account.

3.4. **A new Strichartz-type estimate.**

We first introduce the class of weighted Sobolev spaces used in [8], and directly inspired by the developments of Rychkov in [13].

**Definition 3.1.** *Let $w : \mathbb{R}^d \to \mathbb{R}_+$ be a weight given by one of the two expressions:*

$$w(x) = w_\mu(x) := e^{-\mu|x|} \quad \text{for } \mu > 0, \qquad \text{or} \qquad w(x) = P(x) := \big(1 + |x|^{d+1}\big)^{-1}. \tag{14}$$

*For $f : \mathbb{R}^d \to \mathbb{R}$ and $p > 1$, we set*

$$\|f\|_{L^2_w} := \left( \int_{\mathbb{R}^d} |f(x)|^p\, w(x)\, dx \right)^{1/p}.$$

*Then for all $-\infty < s \leq 1$, all $1 < p < \infty$ and $1 < q \leq \infty$, we define the space $\mathcal{B}^{s,w}_{p,q}$ as the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm*

$$\|f\|_{\mathcal{B}^{s,w}_{p,q}(\varphi_0)} := \left( \sum_{j \geq 0} 2^{jsq} \big\|\varphi_j * f\big\|^q_{L^p_w} \right)^{\frac{1}{q}}. \tag{15}$$

*where $\varphi_j(x) = 2^{dj}\varphi(2^j x)$, for $\varphi$ suitably chosen in $\mathcal{D}(\mathbb{R}^d)$ (see* [13, Section 2.2]*).*

*Whenever $w(x) = w_\mu(x) = e^{-\mu|x|}$ for some $\mu > 0$, we write $L^p_\mu := L^p_{w_\mu}$ and $\mathcal{B}^{s,\mu}_{p,q} := \mathcal{B}^{s,w_\mu}_{p,q}$.*

Our main technical result toward the construction of the wave integral can now be stated as follows.

**Proposition 3.2.** ***(Chen-D.-Song-Tindel, 22).*** *Let $d \in \{1,2\}$, $T > 0$, $\mu_* > 0$, and set*

$$\rho_d := \begin{cases} 1 & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d = 2. \end{cases} \tag{16}$$

*Then for all $-\infty < \alpha \leq 0$, $0 < \mu \leq \mu_*$, $1 < p < \infty$, $1 < q \leq \infty$, $\kappa \in [0, \rho_d]$ and $0 \leq s < t \leq 1$, one has*

$$\|\{\mathcal{G}_t - \mathcal{G}_s\} f\|_{\mathcal{B}^{\alpha+\kappa,\mu}_{p,q}} \lesssim |t-s|^{\rho_d - \kappa} \|f\|_{\mathcal{B}^{\alpha,\mu}_{p,q}}\,, \tag{17}$$

*for some proportional constant that depends only on $\mu_*$.*

In light of (17), the parameter $\rho_d$ can somehow be interpreted as the maximal regularization effect one can obtain through $\mathcal{G}$ *in either the time or the space direction.* On the one hand, maximizing time (i.e. Hölder) regularity in (17) reduces to taking $\kappa = 0$, which indeed provides us with a $\rho_d$-Hölder control. On the other hand, maximizing space regularity in (17) clearly consists in taking $\kappa$ maximal, that is $\kappa = \rho_d$.

The value of $\rho_d$ in (16) can be shown to be optimal when $d = 1$. On the other hand, it is not clear to us - at this stage - whether inequality (17) is optimal for $d = 2$, or if it could be improved up to $\rho_2 = 1$ using another strategy or another scale of Besov spaces. Note also that we have not been able to exhibit a similar regularization result for $d = 3$. This being said, one should not forget that, in contrast with the classical Strichartz inequality, the estimate (17) holds true for a general class of *weighted* Besov spaces, which may account for the limited value of $\rho_2$ in the statement, or its difficult extension to $d = 3$.

The above quantification of the regularization effect of $\mathcal{G}$ is certainly the main ingredient of the proof of the central Proposition 3.4 below.

### 3.5. Main pathwise results.

Let us introduce what will later becomes our class of integrands.

**Definition 3.3.** *We fix two parameters $a_0 \geq 0$, $a_1 > 0$, and set $\mu_t := a_0 + a_1 t$.*

*Then for all $T > 0$ and $\gamma, \kappa \in [0,1]$, we define $\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)$ as the set of functions $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ for which the following norm is finite:*

$$\|u\|_{\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)} := \sup_{s \in [0,T]} \|u_s\|_{\mathcal{B}^{\kappa,\mu_s}_{2,\infty}} + \sup_{0 \leq s < t \leq T} \frac{\|u_t - u_s\|_{\mathcal{B}^{\kappa,\mu_t}_{2,\infty}}}{|t-s|^\gamma}\,. \tag{18}$$

We are finally in a position to state our main result about the interpretation and control of the Young wave integral in a rough setting.

**Proposition 3.4.** *Assume that $d \in \{1,2\}$ and fix two times $0 < T \leq T_0$, as well as $p > d+1$.*

*Let $\gamma, \theta$ be two time regularity parameters and $\kappa, \alpha$ be two space regularity parameters such that the following conditions are satisfied:*

*(i) The coefficients $\gamma, \theta, \kappa, \alpha$ all sit in the interval $[0,1]$, and we have*

$$\kappa + \alpha + \gamma + (1 - \rho_d) < \theta\,, \quad \gamma + \theta > 1\,, \quad \kappa > \alpha + \frac{d}{p}\,, \quad \gamma < 1 - \frac{d+1}{p}\,. \tag{19}$$

*(ii) The process $u$ is an element of $\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)$, and $\dot{b}$ belongs to $\mathcal{C}^\theta([0,T]; \mathcal{B}^{-\alpha,P}_{p,\infty})$.*

*Then the sequence $\{\mathcal{J}^{(n)}; n \geq 1\}$ of Riemann sums defined by (10) converges in $\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)$. We denote*

$$\lim_{n\to\infty} \mathcal{J}^{(n)}_t =: \int_0^t \mathcal{G}_{t-r}(u_r\, d\dot{b}_r)\,. \tag{20}$$

*Moreover, the Young integral (20) verifies*

$$\left\| \int_0^{\cdot} \mathcal{G}_{\cdot-r}(u_r\, d\dot{b}_r) \right\|_{\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)} \leq \|\mathcal{G}_{\cdot}(u_0\, (\dot{b}_T - \dot{b}_0))\|_{\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)} + c_{T_0} \|\dot{b}\|_{\mathcal{C}^\theta([0,T];\mathcal{B}^{-\alpha,P}_{p,\infty})} \|u\|_{\mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)}\,, \tag{21}$$

*where $c_{T_0} > 0$ does not depend on $T$, $u$ and $\dot{b}$.*

Thus, not only does the above statement provide us with an interpretation of the Young wave integral, but it also offers a stable control on the construction, which in turn can be used to set up a fixed point procedure for the related equation.

**Theorem 3.5.** *In the above setting, the (Young) hyperbolic Anderson model*

$$u_t = 1 + \int_0^t \mathcal{G}_{t-r}\big(u_r\, d\dot{b}_r\big), \quad t \in [0,T],$$

*admits a unique solution $u \in \mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)$, for any $T > 0$.*

3.6. **Application.**

At this point, it should be noticed that the results in Section 3.5 consist of *purely deterministic properties*, which apply to any driving path $\dot{b} \in \mathcal{C}^\theta([0,T];\mathcal{B}^{-\alpha,P}_{p,\infty})$.

Therefore, going back to our stochastic model (9), it remains us to check that the driving process $t \mapsto \partial_x B_t$ can indeed be injected - in an almost sure way - into the scale $\in \mathcal{C}^\theta([0,T];\mathcal{B}^{-\alpha,P}_{p,\infty})$, for suitable values of $\theta, \alpha, p$ depending on the indexes $H_0,\ldots,H_d$ of $B$. This is precisely the topic of the following statement, the proof of which relies on stochastic arguments and on moments estimates for the fractional field. Note that, in contrast with the results of Section 3.5, the subsequent regularity property holds true in any dimension $d \geq 1$.

**Proposition 3.6.** *Let $B$ be a fractional field on $\mathbb{R}_+ \times \mathbb{R}^d$, with Hurst indexes $H_0, H_1, \ldots, H_d \in (0,1)^{d+1}$. Then, almost surely, we have*

$$\partial_x B \in \mathcal{C}^\theta([0,T];\mathcal{B}^{-\alpha,P}_{p,\infty})$$

*provided the parameters $\mu, \theta, \alpha$ satisfy*

$$\mu > 0, \quad \theta \in (0, H_0) \quad \text{and} \quad \alpha > d - H_+, \quad \text{where } H_+ := \sum_{i=1}^d H_i.$$

We can finally combine the above regularity result with the deterministic statement of Theorem 3.5 so as to deduce the desired wellposedness property for our fractional wave Anderson model.

**Theorem 3.7.** ***(Chen-D.-Song-Tindel, 22).***

*Let $d \in \{1,2\}$ and consider a fractional field $B$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with Hurst indexes $H_0, H_i \in (0,1)$ satisfying*

$$\begin{cases} H_0 + H_1 > \frac{3}{2}, & \text{if } d = 1, \\ H_0 + H_1 + H_2 > \frac{11}{4}, & \text{if } d = 2. \end{cases}$$

*Then, almost surely, the (Young) hyperbolic Anderson model*

$$u_t = 1 + \int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big), \quad t \in [0,T],$$

*admits a unique solution $u \in \mathcal{E}^{\gamma,\kappa}_{2,\infty}(T)$, for any $T > 0$.*

## 4. Pespectives

The previous results naturally leave many questions open.

4.1. **About the Skorohod approach.**

As can be seen in the statement of Theorem 2.2, our results in the Skorohod setting are restricted to the situation where the Hurst indexes $H_i$ are all equal or greater than $\frac{1}{2}$.

It would of course be interesting to extend these considerations to rougher situations where at least part of these indexes would be smaller than $\frac{1}{2}$. It is however a well-known fact that the Gaussian space generated by a fractional Brownian motion of index smaller than $\frac{1}{2}$ behaves very differently from the $\frac{1}{2}+$ situation, due to new singularity phenomena to take into account. Such a difference can for instance be observed through the special treatment given to the indexes $H_i < \frac{1}{2}$ in [6, Theorem 3.14].

### 4.2. About the pathwise approach.

We cannot deny that, as technical as our developments in [8] may be, the range of our results in the pathwise framework is quite limited so far, regarding both the space dimension ($d \in \{1, 2\}$) and the coefficients $H_i$ (see the restrictions in Theorem 3.7).

In fact, the Young construction is usually considered as a first step in the pathwise analysis, or otherwise stated as a first-order expansion of the model. We now hope that our considerations can be generalized to rougher noises at the price of higher-order expansion procedures.

The strategy in this direction may find its inspiration in the recent developments of [11] for nonlinear equations with additive noise. In particular, it could involve paracontrolled-type structures.

### 4.3. About the comparison between both approaches.

In the series of articles [5, 6], we started a line of research aiming at a comparative study between the Skorohod and pathwise settings for the *parabolic* Anderson model in rough environments. At the core of our project in the aforementioned papers lies the following observation, which still holds true in the wave setting: while the pathwise solution might be seen as more physically relevant, the Skorohod solution often offers more possibilities in terms of quantitative analysis (moments, asymptotics, ...). In [5, 6], we were thus able to transfer some nontrivial information about moments of the stochastic heat equation from the Skorohod to the pathwise "Stratonovich" equation. The key to this transfer lies in a comparison of the two solutions through their respective Feynman-Kac representations.

Although the Feynman-Kac formula is known to be a less efficient tool in the wave setting, it might still help us to implement a similar procedure and provide a sharp comparison between the Skorohod solution given by Theorem 2.2 and the pathwise solution given by Theorem 3.7 (or its future extensions at higher orders).

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