

Scaling limits for random processes from the point of view of group cohomology

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Women in Mathematics

Based on joint work with Kenichi Bannai and Yukio Kametani

① Introduction

② Hydrodynamic limit from geometric approach and main results

Motivation

Problem : Derive macroscopic dynamics from a microscopic stochastic process

General story

- $(S_t)_{t \geq 0}$: Microscopic stochastic process
- $(S_t^\varepsilon)_{t \geq 0}$: Properly **scaled** stochastic process in **space and time** with scaling parameter $\varepsilon > 0$
- $(\bar{S}_t)_{t \geq 0} = \lim_{\varepsilon \downarrow 0} (S_t^\varepsilon)_{t \geq 0}$: Macroscopic dynamics

Key ingredient for the convergence

Homogenization (averaging) in **space and time** : Microscopic state space has some homogeneity

Motivation

One particle model

- Example 1 : Random walk
 - $(S_t)_{t \geq 0}$: Discrete time/Continuous time simple random walk on \mathbb{Z}^d
 - $S_t^\varepsilon := \varepsilon S_{\varepsilon^{-2}t}$
 - (\bar{S}_t) : Brownian motion in \mathbb{R}^d with the diffusion matrix $A = (a_{jk})_{j,k=1}^d$,
 $a_{jk} = \frac{1}{d} \delta_{jk}$
- Example 2 : Diffusion process in \mathbb{R}^d with periodic coefficient
 - $G(x) = (g_{jk}(x))$: smooth positive definite, one-periodic \Leftrightarrow Riemannian metric on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, $m(x)dx$: Riemannian volume measure
 - $(S_t)_{t \geq 0}$: Diffusion process in \mathbb{R}^d with the generator
$$L = \frac{1}{2} \frac{1}{m(x)} \frac{\partial}{\partial x_j} \left(m(x) g^{jk}(x) \frac{\partial}{\partial x_k} \right)$$
 (Laplace-Beltrami operator)
 - $S_t^\varepsilon := \varepsilon S_{\varepsilon^{-2}t}$
 - (\bar{S}_t) : Brownian motion in \mathbb{R}^d with a constant diffusion matrix $A = (a_{jk})_{j,k=1}^d$ given by an implicit form and also a variational formula

Geometric interpretation

Example 1 : Random walk

- Microscopic geometric object : $(\mathbb{Z}^d, \mathbb{E}^d, \rho)$: weighted graph with the periodic weight $\rho : \mathbb{E}^d \rightarrow \mathbb{R}_{>0}$, $\rho(\pm e_j) = \frac{1}{2d}$
- Generator of the “Brownian motion” of the microscopic space :
$$Lf(x) = \sum_{e \in E_x} \rho(e)(f(te) - f(oe)) = \frac{1}{2d} \sum_{j=1}^d (f(x + e_j) + f(x - e_j) - 2f(x))$$
- Macroscopic geometric object : $(\mathbb{R}^d, G = (g_{jk}))$: Riemannian manifold with the **constant** metric $g_{jk} = d\delta_{jk}$
- Generator of “Brownian motion” of the macroscopic space :
$$Lf = \frac{1}{2d} \sum_{j=1}^d \frac{\partial^2}{\partial u_j^2} f = \frac{1}{2} \Delta_G f$$

Convergence of a geometric space with Riemannian structure and some homogeneity!

Motivation : Geometric interpretation

Example 2 : Diffusion process in \mathbb{R}^d with periodic coefficient

- Microscopic geometric object : $(\mathbb{R}^d, G = (g_{jk}(x)))$: Riemannian manifold with a periodic metric $g_{jk}(x)$
- Generator of the “Brownian motion” of the microscopic space :
$$L = \frac{1}{2} \frac{1}{m(x)} \frac{\partial}{\partial x_j} \left(m(x) g^{jk}(x) \frac{\partial}{\partial x_k} \right)$$
- Macroscopic geometric object $(\mathbb{R}^d, \bar{G} = (\bar{g}_{jk}))$: Riemannian manifold with a **constant** metric \bar{g}_{jk}
- Generator of “Brownian motion” of the macroscopic space :
$$Lf = \frac{1}{2} \sum_{j,k=1}^d \bar{g}^{jk} \frac{\partial^2}{\partial x_j \partial x_k} f = \frac{1}{2} \Delta_{\bar{G}} f$$

Convergence of a geometric space with Riemannian structure and some homogeneity!

Period matrix and the macroscopic diffusion coefficient

For both examples, $G = \mathbb{Z}^d$ acts on the microscopic geometric object, and $(H^1(X, \mathbb{R}))^G \cong H^1(G, \mathbb{R}) \cong \mathbb{Z}^d$ holds.

Period matrix

- In the class of one-forms of microscopic geometric objects, there is a topological basis $d\theta_1, \dots, d\theta_d \in (H^1(X, \mathbb{R}))^G$.
- Once we introduce a Riemannian structure, which induces an inner product $\langle \cdot, \cdot \rangle$ in the class of one-forms, there is a harmonic basis $H_1, \dots, H_d \in (H^1(X, \mathbb{R}))^G$ so that $\langle d\theta_j, H_k \rangle = \delta_{jk}$.
- The change-of-basis matrix from $d\theta_1, \dots, d\theta_d$ to H_1, \dots, H_d is called a period matrix.

Geometric interpretation of the macroscopic diffusion matrix

For these examples and more general random walks on periodic lattices, the macroscopic diffusion matrix A is the inverse of the period matrix. In other words, the period matrix is the “Riemannian” metric of the macroscopic geometric object.

Motivation

Our goal : Generalize these ideas to the case for a microscopic system with many particles!

Interacting particle systems

- Example 3 : Exclusion processes
 - $(\eta_t)_{t \geq 0}$: Continuous time Markov process on $\{0, 1\}^{\mathbb{Z}^d}$
 - $Lf(\eta) = \sum_{x, y \in \mathbb{Z}^d} r_{x, y}(\eta)(f(\eta^{x, y}) - f(\eta))$ where $\eta^{x, y}$ is obtained from by exchanging η_x and η_y
 - $\pi : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \mathcal{M}(\mathbb{R}^d) : \langle \pi(\eta), f \rangle := \sum_{x \in \mathbb{Z}^d} \eta_x f(x)$
 - $\pi^\varepsilon : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \mathcal{M}(\mathbb{R}^d) : \langle \pi^\varepsilon(\eta), f \rangle := \varepsilon^d \sum_{x \in \mathbb{Z}^d} \eta_x f(\varepsilon x)$
 - $S_t^\varepsilon := \pi^\varepsilon(\eta_{\varepsilon^{-2}t})$.
 - (\bar{S}_t) : Deterministic dynamic given by $\bar{S}_t = \rho(t, u)du$ where $\rho(t, u)$ is the solution of the diffusion equation

$$\partial_t \rho = \sum_{j, k=1}^d \partial_{u_j} (D_{jk}(\rho) \partial_{u_k} \rho).$$

Can we construct a good microscopic geometric object and understand $D_{jk}(\rho)$ as a period matrix? \Rightarrow Yes!

- The convergence of Markov processes \doteq The convergence of the generator + tightness + the existence and uniqueness of the process with the generators
- The scaling limit of random walks on general periodic lattices (crystal lattices) are not trivial as the case for \mathbb{Z}^d and the discrete harmonic analysis plays a role to describe the macroscopic diffusion matrix.
- The scaling limit like Example 2 is called the homogenization problem. There have been many studies on this topic.
- The scaling limit like Example 3 is called **the hydrodynamic limit**. There have been many studies on this topic too, but there was not a universal framework to unify different models.
- We introduced **a universal framework** for the microscopic geometric object.
- The role of the group action was not understood well in the theory of the hydrodynamic limit. (Even not for the one-particle case.)
- By introducing a general framework and its geometric interpretation, we also obtain **new hydrodynamic limits for specific models**.

① Introduction

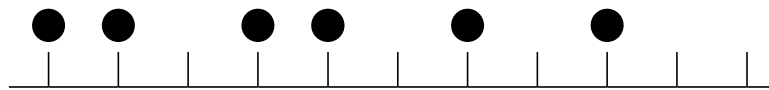
② Hydrodynamic limit from geometric approach and main results

Typical example : Exclusion process on \mathbb{Z}^d

- $\{0, 1\}^{\mathbb{Z}^d}$: State space = Configuration space
- $\eta = (\eta_x) \in \{0, 1\}^{\mathbb{Z}^d}$, η_x : number of particle at $x \in \mathbb{Z}^d$
- Exclusion process : Continuous time Markov process $\{\eta(t)\}_{t \geq 0}$ with the generator L

$$Lf(\eta) = \sum_{x,y \in \mathbb{Z}^d} r_{x,y}(\eta) \eta_x (1 - \eta_y) \{f(\eta^{x,y}) - f(\eta)\}$$

- Jump rate : $r_{x,y} : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \mathbb{R}_{\geq 0}$: “frequency of jump from x to y ”



Typical example : Exclusion process on \mathbb{Z}^d

We always assume :

- **Translation invariant** : $r_{x,y}(\eta) = r_{0,y-x}(\tau_{-x}\eta)$
- **Locality of interaction** : $r_{x,y}$ are local functions
- **Finite range** : $\exists R > 0$ s.t. $r_{x,y} \equiv 0$ if $\|x - y\| := \sum_{i=1}^d |x_i - y_i| > R$
- **Non degenerate** \Rightarrow the density of particles ρ characterizes the invariant measures $\{\nu_\rho\}$

Reversible or Mean-zero case : Expected HDL equation

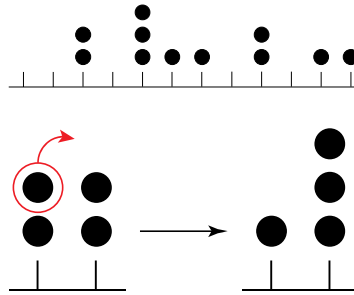
$$\partial_t \rho = \nabla \cdot D(\rho) \nabla \rho = \sum_{i,j=1}^d \partial_{u_i} (D_{ij}(\rho) \partial_{u_j} \rho)$$

Rigorous results :

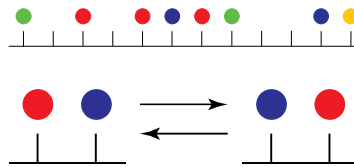
- **Symmetric (not necessarily nearest neighbor) simple** :
 $r_{x,y}(\eta) = r_{y,x}(\eta) = c_{x,y}$, $D_{ij}(\rho) = D_{ij} = \sum_{x \in \mathbb{Z}^d} c_{0,x} x_i x_j$
- **Reversible and nearest neighbor** (Funaki-Uchiyama-Yau (product measure), Varadhan-Yau (non-product measure with mixing condition)) : $D(\rho)$ is given by a variational formula

Other typical microscopic models

Generalized exclusion process : state space $\{0, 1, 2, \dots, \kappa\}^{\mathbb{Z}^d}$



Multi-color (species) exclusion process : state space $\{0, 1, 2, \dots, \kappa\}^{\mathbb{Z}^d}$



Open problems of hydrodynamic limits (before our work)

Specific models

- Multi-species exclusion process $\{0, 1, 2, \dots, \kappa\}^{\mathbb{Z}^d}$
- Energy exchange model $\mathbb{R}_+^{\mathbb{Z}^d}$: Mesoscopic model obtained from some deterministic model

General extensions

- **Finite range interaction (not nearest neighbor)** models on \mathbb{Z}^d , where the underlying graph is $(\mathbb{Z}^d, \mathbb{E}_R^d := \{(x, y) : |x - y| \leq R\})$
- Models on **crystal lattices**, such as hexagonal lattice, diamond lattice...
- Stationary measures which are not product (except for the exclusion process)

Main result 1 : Framework of microscopic models

Microscopic models are defined by **geometric data** and **stochastic data**

- **Geometric (spatial/topological) data : the triple (S, ϕ, \mathcal{X})**
 - Local state space (Set S) (ex. $\{0, 1\}, \{0, 1, 2\}, \mathbb{N}, \mathbb{R}, \mathbb{R}_+$)
 - Local interaction (Map $\phi : S \times S \rightarrow S \times S$) (ex. $\phi(s_1, s_2) = (s_2, s_1)$)
 - Underlying spatial space (Graph $\mathcal{X} = (X, E)$) (ex. $(\mathbb{Z}^d, \mathbb{E}^d), (\mathbb{Z}^d, \mathbb{E}_R^d)$, triangular lattice, diamond lattice)
- **Stochastic (spatial/metric) data**
 - Speed of local interaction $r : \Phi \rightarrow \mathbb{R}_{>0}$ (ex. $r_{x,y}(\eta)$)
 - Equilibrium measures : μ (ex. Bernoulli product measures ν_ρ)

Symmetry data also plays an essential role

- **Symmetry data : G**
 - Symmetry of the underlying space space (Group G acting on \mathcal{X}) (ex. $G \cong \mathbb{Z}^d$)

Topological structure constructed by geometric data

Suppose the triple (S, ϕ, \mathcal{X}) is given.

- The data (S, ϕ) defines the space of **conserved quantities** $\text{Consv}^\phi(S)$, which is a subspace of function $\{f : S \rightarrow \mathbb{R}\}$
- The data (S, ϕ, \mathcal{X}) defines **a graph structure (S^X, Φ)** , which we call a configuration space with transition structure :
 $\Phi = \{(\eta, \eta^e) : \eta \in S^X, e \in E\}$.
- We introduce **a uniform cohomology** on the graph (S^X, Φ)
 - $C_{\text{unif}}^0(S^X)$: set of uniform functions
 - $C_{\text{unif}}^1(S^X)$: set of uniform one forms
 - $\partial : C_{\text{unif}}^0(S^X) \rightarrow C_{\text{unif}}^1(S^X)$: differential (usual graph differential)
 - $Z_{\text{unif}}^1(S^X)$: set of uniform closed forms
 - $\partial C_{\text{unif}}^0(S^X)$: set of uniform exact forms
 - $H_{\text{unif}}^0(S^X) := \ker \partial$
 - $H_{\text{unif}}^1(S^X) := Z_{\text{unif}}^1(S^X) / \partial C_{\text{unif}}^0(S^X)$

Main result 2 : Characterization of “smooth” cohomology

- Assumption 1
 - (S, ϕ) is **irreducibly quantified** (\sim the dynamics is non-degenerate)
 - \mathcal{X} is **transferable** ($(\mathbb{Z}^d, \mathbb{E}_R^d)$, $d \geq 2$ satisfies the condition)
- Assumption 2
 - (S, ϕ) is **simple** ($\text{Consv}^\phi(S)$ is the one-dimensional space, and some more)
 - \mathcal{X} is **weakly transferable** ($(\mathbb{Z}^d, \mathbb{E}_R^d)$, $d \geq 1$ satisfies the condition)

Theorem (Bannai-Kametani-S)

Under the assumptions 1 or 2

$$H_{\text{unif}}^0(S^X) \cong \text{Consv}^\phi(S), \quad H_{\text{unif}}^1(S^X) \cong \{0\}.$$

- \mathcal{X} must be an infinite graph under the assumption.

Main result 3 : De Rham cohomology for S^X/G

Assume that a group G acts freely on the locale \mathcal{X} .

- Action of G on functions and forms are naturally induced.
 - $\mathcal{E} := \partial(C_{\text{unif}}^0(S^X)^G)$: set of G -invariant uniform exact forms
 - $\mathcal{C} := Z_{\text{unif}}^1(S^X)^G$: set of G -invariant uniform closed forms
 - $H^1(G, \text{Consv}^\phi(S))$: **the first group cohomology of G** with coefficients in $\text{Consv}^\phi(S)$

Theorem (Bannai-Kametani-S)

Under the assumptions 1 or 2

$$\mathcal{C}/\mathcal{E} \cong H^1(G, \text{Consv}^\phi(S)).$$

In particular, if $G \cong \mathbb{Z}^d$, then

$$\mathcal{C} \cong \mathcal{E} \oplus \bigoplus_{k=1}^d \text{Consv}^\phi(S).$$

Main result 4 : A version of Hodge-Kodaira theorem

- Using **the stochastic data**, an inner product is defined on $(C_{\text{unif}}^1(S^X))^G$. (analogy to Riemannian metric)
 - $\mathcal{E}_{L^2} := \overline{\partial(C_{\text{unif}}^0(S^X))^G}$: completion of set of G -invariant uniform exact forms
 - $\mathcal{C}_{L^2} := Z_{L^2}^1(S^X)^G$: set of G -invariant L^2 closed forms
- Assume : S is a finite set and $G \cong \mathbb{Z}^d$, and the induced measure on S^X is product.

Theorem (Bannai-S)

Under the assumptions 1 or 2, and several essential assumptions including above,

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^d \text{Consv}^\phi(S).$$

New interpretation of the macroscopic diffusion matrix

There are two natural decomposition of closed forms

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^d \text{Consv}^\phi(S) : \text{ topological (Varadhan's) decomposition}$$

$$\mathcal{C}_{L^2} \cong \mathcal{E}_{L^2} \oplus \bigoplus_{k=1}^d \text{Consv}^\phi(S) : \text{ orthogonal decomposition}$$

Diffusion matrix

Macroscopic diffusion matrix $D(\rho) =$ Transition matrix of two different decomposition under the measure $\nu_\rho =$ The inverse of the period matrix

Summary

- The decomposition of the space of $(H^1)^G(X)$ or $(H^1)^G(S^X)$ is the key to understand the macroscopic diffusion matrix
- The **group cohomology of G** play an essential role as $(H^1)^G(S^X)$ can be computed easily once we prove $H^1(S^X) = \{0\}$
- Uniform functions and uniform forms are “smooth functions” on S^X/G
- Hodge-Kodaira theorem is generalized to configuration spaces
- The macroscopic diffusion matrix is the inverse of the period matrix universally.