Elementary recursive complexity results in real algebraic geometry

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Real algebraic geometry

- solution of polynomial equalities and inequalities in R^k
- R: real closed field, totally ordered field, positive elements are square, IVT: Intermediate Value Theorem. If $P \in \mathrm{R}[X]$ $P(a)P(b) < 0, a < b$ then $\exists c \quad P(c) = 0$
- examples of real closed field : (such as $\mathbb{R}$ field of real numbers, $\mathbb{R}_{\mathrm{alg}}$ field of real algebraic numbers , and also and also non archimedean models such as $\mathbb{R}\langle\epsilon\rangle$ the field of Puiseux series
- $\mathrm{R}[i]$ is algebraically closed, using an algebraic proof due to Laplace of the Fundamental Theorem of Algebra.

Definitions
Modern algebra: non constructive proofs
Proof theory: primitive recursive degree bounds
Computer algebra: elementary recursive degree bounds
Discussion

Primitive recursive/elementary recursive

- primitive recursive functions obtained from 0, successor, chosing one coordinate, composition and recursion
- example: addition from successor, multiplication from addition, exponentiation from multiplication using recursion
- example: associate to n a tower of exponential whose height is n. $f(0) = 2$, $f(1) = 2^2$, $f(2) = 2^{2^2}$ ... easy to construct using recursion
- elementary recursive functions are functions obtained from addition, multiplication, substraction and division using chosing one coordinate, composition, finite summation and product. Typically: exponential function 2^n, doubly exponential function 2^{2^n}, a tower of exponentials of fixed height (example: 5 or 4).

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Positivity and sums of squares

- Is a polynomial with real coefficients taking only non negative values a sum of squares of polynomials?
- Yes if the number of variables is 1.
- Hint : decompose the polynomial in powers of irreducible factors: degree two factors (corresponding to complex roots) are sums of squares, degree 1 factors (corresponding to real roots appear with even degree)
- Yes if the degree is 2.
- Hint: a quadratic form taking only non negative values is a sum of squares of linear polynomials

Positivity and sums of squares

- Is a non-negative polynomial a sum of squares of polynomials?
- Yes if the number of variables is 1.
- Yes if the degree is 2.
- Also if the number of variables is 2 and the degree is 4
- No in all other cases.
- First explicit counter-example Motzkin '69

$$1 + X^4Y^2 + X^2Y^4 - 3X^2Y^2$$

takes only non negative values and is not a sum of squares of polynomials.

Motzkin's counter-example (degree 6, 2 variables)

$$M = 1 + X^4Y^2 + X^2Y^4 - 3X^2Y^2$$

- M takes only non negative values. Hint: arithmetic mean is always at least geometric mean.
- M is not a sum of squares. Hint : try to write it as a sum of squares of polynomials of degree 3 and check that it is impossible.
- Example: no monomial X^3 can appear in the sum of squares. Etc ...

Hilbert 17th problem

- Reformulation proposed by Minkowski.
- Question Hilbert '1900.
- Is a non-negative polynomial a sum of squares of rational functions ?
- Artin '27: Affirmative answer. Non-constructive.

Outline of Artin's proof

- Suppose P is not a sum of squares of the field rational functions.
- Sums of squares: proper cone of rational functions, and do not contain P (a cone contains squares, closed under addition and multiplication, a proper cone does not contain -1).
- Using Zorn's lemma, get a maximal proper cone of the field of rational functions which does not contain P. Such a maximal cone defines a total order on the field of rational functions.
- Every totally ordered field has a real closure.
- Taking the real closure of the field of rational functions for this order, get a field in which P takes negative values (when evaluated at the "generic point" = the point $(X_1, \ldots, X_k)$).
- Then P takes negative values over the reals. First instance of a transfer principle in real algebraic geometry. Based on Sturm's theorem, or Hermite quadratic form.

Definition (Hermite's Matrix)

Let $P, Q \in \mathrm{R}[X]$ with $\deg P = p \geq 1$. The Hermite's matrix $\mathrm{Her}(P; Q) \in \mathrm{R}^{p \times p}$ is the matrix defined for $1 \leq j_1, j_2 \leq p$ by

$$\mathrm{Her}(P; Q)_{j_1, j_2} = \mathrm{Tra}(Q(X) \cdot X^{j_1 + j_2 - 2})$$

where $\mathrm{Tra}(A(X))$ is the trace of the linear mapping of multiplication by $A(X) \in \mathrm{R}[X]$ in the R-vector space $\mathrm{R}[X]/P(X)$.

Hermite matrix easy to compute, its entries correspond to linear combination of the Newton sums (moments) of P.

Hermite method

Theorem (Hermite's Theory)

Let $P, Q \in \mathrm{R}[X]$ *with* $\deg P = p \geq 1$. *Then*

$$\mathrm{TaQu}(P, Q) = \mathrm{Si}(\mathrm{Her}(P; Q))$$

where

$$\mathrm{TaQu}(P, Q) := \sum_{x \in \mathrm{R} | P(x)=0} \mathrm{sign}(Q(x)),$$

$\mathrm{Si}(\mathrm{Her}(P; Q))$ *is the signature of the symmetric matrix* $\mathrm{Her}(P; Q)$. *Moreover* $\mathrm{Si}(\mathrm{Her}(P; Q))$ *is determined by the signs of the principal minors of* $\mathrm{Her}(P; Q)$.

Proof: uses complex conjugate roots.

Transfer principle

- A statement involving elements of R which is true in a real closed field containing R (such as the real closure of the field of rational functions for a chosen total order) is true in R.
- Not any statement, only "first order logic statement".
- Example of such statement

  $$\exists x_1 \ \ldots \ \exists x_k \ P(x_1, \ldots, x_k) < 0$$

  is true in a real closed field containing R if and only if it is true in R
- Special case of quantifier elimination.

Remaining problems

- Very indirect proof (by contraposition, uses Zorn).
- No hint on denominators: what are the degree bounds ?
- Artin notes effectivity is desirable but difficult.
- Effectivity problems : is there an algorithm checking whether a given polynomial is everywhere nonnegative?
- Can we use this algorithm to provide a representation as a sum of squares?

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Positivstellensatz (Krivine '64, Stengle '74)

- Find algebraic identities certifying that a system of sign condition is empty.
- In the spirit of Nullstellensatz.
  K a field, C an algebraically closed extension of **K**,
  $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$
  $P_1 = \ldots = P_s = 0$ no solution in C^k
  $\Longleftrightarrow$
  $\exists \quad (A_1, \ldots, A_s) \in \mathbf{K}[x_1, \ldots, x_k]^s \qquad A_1 P_1 + \cdots + A_s P_s = 1.$
- Grete Hermann, a female student from Hilbert has given in her Ph D dissertation an algorithmic proof of the classical Nullstellensatz, with elementary recursive complexity (doubly exponential in the number of variables).
- For real numbers, statement more complicated.

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Positivstellensatz

- $\mathbf{K}$ an ordered field, R a real closed extension of $\mathbf{K}$,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H}(x) : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases} \quad \text{no solution in } \mathrm{R}^k \quad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2\Big) \prod_{i \in I} P_i \quad (k_{I,j} > 0),$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x$$

such that $\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0.$

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Incompatibilities

$$\mathcal{H}(x) : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

$$\downarrow \mathcal{H} \downarrow : \qquad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

with

$S \in \left\{\prod_{i \in I_{\neq}} P_i^{2e_i}\right\}$ ← monoid associated to $\mathcal{H}$

$N \in \left\{\sum_{I \subset I_{\geq}} \left(\sum_j k_{I,j} Q_{I,j}^2\right) \prod_{i \in I} P_i\right\}$ ← cone associated to $\mathcal{H}$

$Z \in \langle P_i \mid i \in I_{=} \rangle$ ← ideal associated to $\mathcal{H}$

Positivstellensatz implies Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \iff P(x) < 0 \quad \text{no solution}$$

$$\iff \begin{cases} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{cases} \quad \text{no solution}$$

$$\iff \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2) P}_{\geq 0} = 0$$

$$\implies P = \frac{P^{2e} + \sum_i Q_i^2}{\sum_j R_j^2} = \frac{(P^{2e} + \sum_i Q_i^2)(\sum_j R_j^2)}{(\sum_j R_j^2)^2}.$$

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Positivstellensatz: proofs

- Classical proofs of Positivstellensatz based on Modern Algebra.
- Zorn's lemma and Tranfer principle, very similar to Artin's proof for Hilbert 17th problem.
- non-constructive
- no degree bounds

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Remaining problems

- Very indirect proof (by contraposition, uses Zorn).
- Effectivity is desirable but difficult.
- What are the degree bounds in the Positivstellensatz Identity ?
- Effectivity problems : is there an algorithm checking whether a given system of polynomial inequalities is empty?
- If the answer is yes, can we use this algorithm to construct a Positivstellensatz equality ?

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Quantifier elimination

- Classical proofs of Positivstellensatz based on Modern Algebra.
- Constructive proofs use a quantifier elimination algorithm over the reals.
- What is quantifier elimination ?
- High school mathematics

$$\exists \quad x \quad ax^2 + bx + c = 0, a \neq 0$$

$\Longleftrightarrow$

$$b^2 - 4ac \geq 0, a \neq 0$$

- Valid for any formula, due to Tarski, using Tarski-queries and induction on the number of variables, algorithm !
- Deciding emptyness of a system of inequalities is algorithmic.

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Strategy of Lombardi

- For every system of sign conditions with no solution, find a simple algorithmic proof of the fact there is no solution, based on quantifier elimination
- Use this proof to construct an algebraic incompatibility and control the degrees for the Positivstellensatz.
- Uses notions introduced by Henri Lombardi.
- Key concept : weak inference.

Quantifier elimination methods

- Many existing methods
- The older ones have a primitive recursive complexity : Tarski, Seidenberg, Cohen-Hormander.
- The one chosen by Henri Lombardi for a constructive proof of Positivstellensatz is Cohen-Hormander algorithm as explained in [BCR].

Degree of an incompatibility

$$\mathcal{H}(x) : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

$$\downarrow \mathcal{H} \downarrow : \qquad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2\Big) \prod_{i \in I} P_i, \qquad Z = \sum_{i \in I_{=}} Q_i P_i$$

the degree of $\mathcal{H}$ is the maximum degree of

$$S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad Q_{I,j}^2 \prod_{i \in I} P_i \ \ (I \subset I_{\geq}, j), \qquad Q_i P_i \ \ (i \in I_{=}).$$

Example:

$$\begin{cases} x & \neq & 0 \\ y - x^2 - 1 & \geq & 0 \\ xy & = & 0 \end{cases} \qquad \text{no solution in } \mathbb{R}^2$$

$$\downarrow \ x \neq 0, \ \ y - x^2 - 1 \geq 0, \ \ xy = 0 \ \downarrow:$$

$$\underbrace{x^2}_{>0} + \underbrace{x^2(y - x^2 - 1) + x^4}_{\geq 0} + \underbrace{(-x^2 y)}_{=0} = 0.$$

The degree of this is incompatibility is 4.

Weak Inference

(in the particular case we need) $\mathcal{F},\mathcal{G}$ systems of sign conditions $\mathbf{K}[u]$ and $\mathbf{K}[u,t]$. A weak inference

$$\mathcal{F}(u) \quad \vdash \quad \exists\, t\ \mathcal{G}(u,t)$$

is a construction which for every system of sign condition $\mathcal{H}$ in $\mathbf{K}[v]$ with $v \supset u$ not containing t and every incompatibility

$$\downarrow \mathcal{G}(u,t),\ \mathcal{H}(v) \downarrow_{\mathbf{K}[v,t]}$$

produces an incompatibility

$$\downarrow \mathcal{F}(u),\ \mathcal{H}(v) \downarrow_{\mathbf{K}[v]} .$$

From right to left.

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Weak inferences: case by case reasoning

$$A \neq 0 \quad \vdash \quad A<0 \quad \vee \quad A>0$$

$\downarrow \mathcal{H},\ A<0 \downarrow \leftarrow$ degree δ_1

$$\underbrace{A^{2e_1}S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0$$

$\downarrow \mathcal{H},\ A>0 \downarrow \leftarrow$ degree δ_2

$$\underbrace{A^{2e_2}S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1}S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2}S_2 + N_2 + Z_2 = -N_2' A$$

$$A^{2e_1+2e_2}S_1S_2 + N_3 + Z_3 = -N_1'N_2'A^2$$

$$\underbrace{A^{2e_1+2e_2}S_1S_2}_{>0} + \underbrace{N_1'N_2'A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

Weak inferences: case by case reasoning

Starting from two incompatibilities

$\downarrow \mathcal{H},\ \ A<0 \downarrow \leftarrow$ degree δ_1 $\qquad$ $\downarrow \mathcal{H},\ \ A>0 \downarrow \leftarrow$ degree δ_2

$$\underbrace{A^{2e_1}S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2}S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

we constructed (by making a product) a new incompatibility

$$\underbrace{A^{2e_1+2e_2}S_1S_2}_{>0} + \underbrace{N_1'N_2'A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$$\downarrow \mathcal{H},\ \ A \neq 0 \downarrow \leftarrow \text{degree } \delta_1+\delta_2$$

List of statements needed into weak inferences form

- Many simple weak inferences of that kind are combined to obtain more interesting weak inferences.
- In particular: IVT, the Intermediate Value Theorem, has to be transformed into a weak inference
- Finally Henri Lombardi proved primitive recursive degree bounds for Positivstellensatz, hence of the Hilbert 17 th problem Lombardi '90.
- There are prior or other contributions for the 17 th problem only. • Kreisel '57 - Daykin '61 - - Schmid '00
- All these constructive proofs $\rightsquigarrow$ primitive recursive degree bounds k and $d = \deg P$.

Sign determination

- R a real closed field (such as $\mathbb{R}$, $\mathbb{R}_{\mathrm{alg}}$, $\mathbb{R}\langle\epsilon\rangle$)
- a univariate non zero polynomial P and a list of other univariate polynomials $Q_1, \ldots, Q_s$ all in $\mathrm{R}[X]$
- find the list of non-empty sign conditions (i.e. elements of $\{0, 1, -1\}^s$) realized by $Q_1, \ldots, Q_s$ at the real roots of P (i.e. roots in R)
- variant: compute also the corresponding cardinalities

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Special case 1: real root counting

- a univariate non zero polynomial $P \in \mathrm{R}[X]$
- decide whether P has a real root (i.e. a root in R) or not
- variant: compute also the number of roots of P in R
- using Hermite's method

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Special case 2: Tarski query

- a univariate non zero polynomial $P \in \mathrm{R}[X]$ and another polynomial $Q \in \mathrm{R}[X]$
- decide the signs of Q at the roots of P in R (variant: count the cardinalities)
- tool : Tarski-query

$$\mathrm{TaQu}(P, Q) := \sum_{x \in \mathrm{R} | P(x)=0} \mathrm{sign}(Q(x))$$

- using Hermite's method

Special case 2: Tarski query

$c(P = 0, Q = 0)$ is the number of roots of P in R where $Q = 0$ etc

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c(P=0, Q=0) \\ c(P=0, Q>0) \\ c(P=0, Q<0) \end{bmatrix} = \begin{bmatrix} \mathrm{TaQu}(P,1) \\ \mathrm{TaQu}(P,Q) \\ \mathrm{TaQu}(P,Q^2) \end{bmatrix}$$

Compute three Tarski-queries, then compute three cardinals and decide which are the non-empty sign conditions.
Using Hermite's method.

General case

- Tarski-queries are considered as black-boxes
- compute Tarski-queries of P and products of the Q_i or their squares (using Hermite's method for example)
- solve a linear system,
- compute the cardinals of sign conditions at the roots of P,
- gives sign determination

Naive algorithm

Order the elements of $\{0, 1, -1\}^s$ lexicographically and consider the elements of $\{0, 1, 2\}^s$ as coding all natural numbers smaller than $3^s - 1$.

- Perform the 3^s products of the Q_i and Q_i^2
- Compute the 3^s corresponding Tarski-queries, which defined a vector t
- Define the $3^s \times 3^s$ matrix of signs M whose columns are indexed by $\{0, 1, -1\}^s$ and rows are indexed by $\{0, 1, 2\}^s$, the σ, α entry being the sign taken by $Q_1^{\alpha_1} \ldots, Q_s^{\alpha_s}$ at σ.
- solve the linear system $M \cdot c = t$ where c is the unknown
- keep the non-zero elements of c which are the cardinals of the non-empty sign conditions

Naive algorithm

Example for $s = 2$, the matrix of signs is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and is invertible.

Naive algorithm

Rows are numbered from 0 to 8. The row of number 4 (fifth row) is the sign of the polynomial Q_1Q_2 on the list of signs (since 4 is written 1+3 in basis 3)

$$\begin{array}{cccccccccc} \mathrm{sign}(Q_1) & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ \mathrm{sign}(Q_2) & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ \mathrm{sign}(Q_1Q_2) & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{array}$$

The correctness is proved by induction on s.
The number of calls to the Tarski-query black box is exponential in s.

Improved algorithm

- Notice that the number of non-empty sign conditions is at most the number $r \leq d$ of real roots
- Remove non-empty sign conditions at each induction step
- Use the special structure of the matrix to solve the linear system in quadratic time
- **Prove that the $Q_1^{\alpha_1} \ldots, Q_s^{\alpha_s}$ whose Tarski-query is computed in the algorithm have at most $\log_2 d$ non zero entries.**

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Complexity

- The total number of calls to the Tarski-query blackbox is $3sd$
- The Tarski-query blackbox is called for P and polynomials of degree at most $2d \log_2 d$

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Real algebraic numbers

- Real algebraic numbers can be characterized by the signs they give to their derivatives (Thom encodings) : easy by induction on the degree
- Thom encodings can be computed by sign determination
- No numerical approach needed, valid on any real closed field
- Once we know the Thom encodings, sign determination gets simplified, only products of (a few) derivatives and one of the other polynomial (or its square) are used.

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Sign determination and quantifier elimination

- Eliminating one variable corresponds (basically) to parametric sign determination
- $P, Q_1, \ldots, Q_s$ are polynomials in parameters u and main variable X
- compute polynomials in the parameters u whose signs fix the list of non-empty sign conditions realized by $Q_1[u][X], \ldots, Q_s[u][X]$, at the real roots of $P[u][X]$

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Sign determination and quantifier elimination

- Tarski's proof of quantifier elimination is basically naive sign determination
- Complexity primitive recursive

There are much better quantifier elimination methods

- Cynlindrical algebraic decomposition is doubly exponential
- Polynomial when the number of variables is fixed
- Uses the notion of connected component of a sign condition

More recent methods doubly exponential in the number of blocks of quantifiers and polynomial when this number if fixed. Use even more geometry (critical points ...).

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Sign determination and quantifier elimination

- New purely algebraic quantifier elimination method using sign determination and Thom encodings
- Complexity elementary recursive
- Polynomial in the number of polynomials when the number of variables is fixed but **NOT** in the degree of the polynomials
- Does not need the notion of a connected component of a sign condition

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Elementary recursive degree bounds for Positivstellensatz

- strategy: transform a simple proof that a system of inequalities has no solution into the construction of an algebraic identity
- turn the preceding ingredients : computation of signature of Hermite quadratic form, Thom encodings, sign determination into construction of algebraic identities
- control the degree of these identities
- not having to deal sign connected components of sign conditions is crucial

(Joint work with Daniel Perrucci and Henri Lombardi)

Construct specific algebraic identities expressing that

- a real polynomial of odd degree has a real root
- a real polynomial has a complex root (by Laplace's algebraic proofof the Fudamental Theorem of Algebra)
- Tarski queries computed by Hermite quadratic forms
- the Sylvester's inertia law for quadratic forms is valid
- realizable sign conditions for a family of univariate polynomials at the roots of a polynomial, fixed by sign of minors of Hermite quadratic forms (uses Thom's encoding, and sign determination),
- realizable sign conditions for $\mathcal{P} \subset \mathbf{K}[x_1, \ldots, x_k]$ are fixed by list of non empty sign conditions for $\mathrm{Proj}(\mathcal{P}) \subset \mathbf{K}[x_1, \ldots, x_{k-1}]$: efficient projection method using only algebra

and at the end produce a sum of squares, with elementary recursive complexity (tower of five exponentials) !

How is produced the sum of squares ?

Suppose that P takes always non negative values. The proof that

$$P \geq 0$$

is transformed, step by step, in a proof of the weak inference

$$\vdash \quad P \geq 0.$$

Which means that if we have an initial incompatibility of $\mathcal{H}$ with $P \geq 0$, we know how to construct a final incompability of $\mathcal{H}$ itself

Going right to left.

How is produced the sum of squares ?

In particular $P < 0$, i.e. $P \neq 0, -P \geq 0$, is incompatible with $P \geq 0$, since

$$\underbrace{P^2}_{>0} + \underbrace{P \times (-P)}_{\geq 0} = 0$$

is an initial incompatibility of $P \geq 0, P \neq 0, -P \geq 0$!
Hence, taking $\mathcal{H} = [P \neq 0, -P \geq 0]$ we know how to construct an incompatibility of $\mathcal{H}$ itself !

$$\underbrace{P^{2e}}_{>0} + \underbrace{\sum_i Q_i^2 - (\sum_j R_j^2) P}_{\geq 0} = 0$$

which is the final incompatibility we are looking for !!
We expressed P as a sum of squares of rational functions !!!

Elementary recursive Hilbert 17 th problem

A non negative polynomials of degree d in k variables can be represented as a sum of squares of rational functions with elementary recursive degree bound:

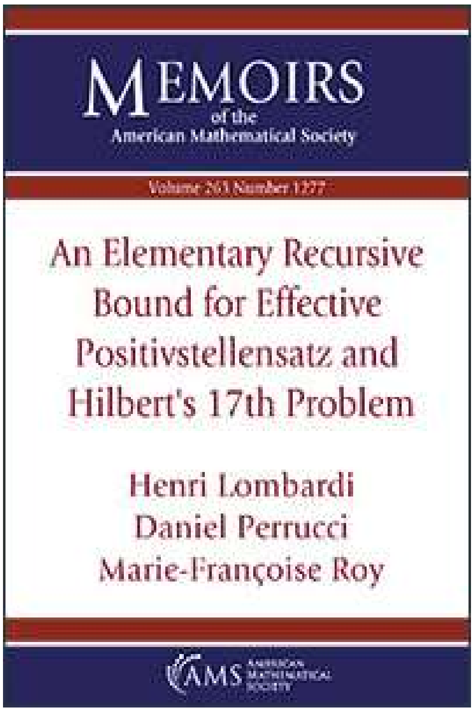
$$2^{2^{2^{d^{4^k}}}}.$$

[LPR]
and similar results for Positivstellensatz and Real Nullestellensatz

Discussion

- Why a tower of five exponentials ?
- outcome of our method ... no other reason ...
- the existence of a real root for an univariate polynomials of degree d already gives a construction of algebraic identities with two level of exponentials
- the proof of Laplace starts from a polynomial of degree d and produces a polynomial of degree d^d : triple exponential for the construction of algebraic identities corresponding to the fundamental theorem of algebra
- our projection method based only on algebra then gives univariate polynomials of doubly exponential degrees (eliminating variables one after the other using Hermite's method)
- finally : a tower of 5 exponentials
- long paper, appeared in Memoirs of the AMS ...

If you want to read more



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References

[BGP] Blekherman G., Gouveia J. and Pfeiffer J. Sums of Squares on the Hypercube Manuscript. arXiv:1402.4199. [BCR] J. Bochnak , M. Coste , M.-F. Roy. Real algebraic geometry. Second edition in english.Ergebnisse der Mat., vol. 36. Berlin Heidelberg New York: Springer (1998)

[BPR] S. Basu, R. Pollack, M.-F. Roy, Algorithms in real algebraic geometry, Algorithms and Computation in Mathematics, 10, Second edition. *Springer-Verlag, Berlin*, 2006.

[PR] D. Perrucci, M.-F. Roy. Elementary recursive quantifier elimination based on Thom encoding and sign determination. Annals of Pure and Applied Logic, Volume 168, Issue 8, August 2017, Pages 1588-1604 (preliminary version, arXiv:1609.02879v2) .

[LPR] H. Lombardi, D. Perrucci, M.-F. Roy, *An elementary recursive bound for effective Positivstellensatz and Hilbert 17-th problem* (preliminary version, arXiv:1404.2338).

(and many other references there)