ERGODIC OPTIMIZATION AND ITS RELATION TO THERMODYNAMIC FORMALISM

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Let $T : X \to X$ be a continuous map on a compact metric space X. For a continuous function $\varphi : X \to \mathbb{R}$ we consider the ergodic maximum

$$\beta(\varphi) := \sup_{\mu \in \mathcal{M}_T(X)} \int_X \varphi \, d\mu \tag{1}$$

where $\mathcal{M}_T(X)$ is the space of all T-invariant Borel probability measures on X with the weak*-toplogy. An invariant measure attaining the supremum of (1) is called *maximizing measure* of φ. The study of maximizing measures is called *Ergodic optimization* and has been attracted interest by many authors for a decade (See for a nice survey [Jen19]).

On the other hand we are also interested in the topological pressure of φ

$$\mathcal{P}(\varphi) := \sup_{\mu \in \mathcal{M}_T(X)} \left(h_\mu + \int_X \varphi \, d\mu \right) \tag{2}$$

where h_μ is a metric entropy of μ. An invariant measure attaining (2) is called *equilibrium measure* and is a main object in *thermodynamic formalism*, which has been a powerful tool in ergodic theory and dynamical system and its applications to other topics.

In this paper we will illustrate a maximizing measure naturally appears as the "zero temperature limit" of equilibrium measures. In this paper we assume the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous, which ensures the existence of equilibrium measures.

Let $\beta > 0$ and μ_β be an equilibrium measure of $\beta\varphi$. We consider the following problem:

Does (μ_β) converges as β goes to infinity?

This is called *zero temperature limit problem* [BLL13]. It is easy to see that any accumulation point of (μ_β) should be a maximizing meausre of φ, i.e., letting μ_∞ be an accumulation point of (μ_β), we have

$$\beta(\varphi) = \int_X \varphi \, d\mu_\infty. \tag{3}$$

Set $\mathcal{M}_{\max}(\varphi)$ be the set of all maximizing measures of φ. Then (3) implies that (μ_β) converges if $\mathcal{M}_{\max}(\varphi)$ is a singleton. If $\mathcal{M}_{\max}(\varphi)$ is not a singleton, what we should check next is entropy of maximizing measures. Indeed, we obtain that any accumulation point of (μ_β) should attains the maximum entropy among all maximizing measures, i.e.,

$$\sup_{\mu \in \mathcal{M}_{\max}(\varphi)} h_\mu = h_{\mu_\infty}.$$

This yields that (μ_β) converges if there exists unique $\mu \in \mathcal{M}_{\max}(\varphi)$ with the maximum entropy $\sup_{\mu \in \mathcal{M}_{\max}(\varphi)} h_\mu$.

Taking account of the above observation, itt is natural to ask when $\mathcal{M}_{\max}(\varphi)$ is a singleton. This is studied by Jenkinson and for generic $\varphi \in C(X)$, $\mathcal{M}_{\max}(\varphi)$ is a singleton [Jen06]. Hence (μ_β) converges for generic $\varphi \in C(X)$. On the other hand, there remains many functions whose set of maximizing measures is far from singleton: If (X, T) satisfies the specification property, there exists a dense subset $\mathcal{D} \subset C(X)$ s.t. for every $\varphi \in \mathcal{D}$, $\mathcal{M}_{\max}(\varphi)$ contains uncountably many ergodic elements [Shi18].

It is also natural to expect (μ_β) converges for φ with nice continuity. Now we restrict our attention to symbolic dynamics. Let $X = A^{\mathbb{N}}$ with a finite set A and a metric $d(x, y) = 2^{-n}$ where $n = \inf\{i \in \mathbb{N} : x_i \neq y_i\}$. Let $T : X \to X$ be the left shift. A function $\varphi : X \to \mathbb{R}$ is a *locally constant* if there exists $N \in \mathbb{N}$ s.t. $\varphi(x) = \varphi(y)$ if $d(x, y) = 2^{-n}$ for $n \geq N$. For a locally constant function φ, (μ_β) converges, which is proved by different authors in different ways [Bré08, Lep14, CGU11]. However, for Lipschitz functions we may have non-convergence. Chazottes and Hochman first gives a Lipschitz function for which (μ_β) does not converges [CH10]. Another construction is given by Coronel and Rivera-Letelier [CRL15]. The situation in higher dimensional is more complicated [CH10, CS20, Ved20].

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