Region colorings for spatial graphs

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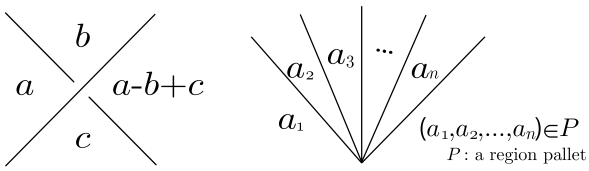
1 Dehn p-coloring

Let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ denote the cyclic group $\mathbb{Z}/p\mathbb{Z}$. A Dehn p-coloring for a spatial graph diagram is an assignment of an element(color) of $\mathbb{Z}_p$ to each region. At each crossing and vertex, the following conditions are satisfied:



2 Pallets of Dehn p-coloring

Definition 2.1. *Two elements* $\boldsymbol{a}, \boldsymbol{b} \in \bigcup_{n \in 2\mathbb{Z}_+} \mathbb{Z}_p^n$ *are equivalent* ($\boldsymbol{a} \sim \boldsymbol{b}$) *if* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are related by a finite sequence of the following transformations:*

(Op1) $(a_1, \ldots, a_n) \longrightarrow (a_2, \ldots, a_n, a_1)$,

(Op2) $(a_1, \ldots, a_n) \longrightarrow (a, a_2 + (-1)^2(a_1 - a), \ldots, a_i + (-1)^i(a_1 - a), \ldots, a_n + (-1)^n(a_1 - a))$ *for* $a \in \mathbb{Z}_p$,

(Op3) $(a_1, \ldots, a_n) \longrightarrow (a, a_1 - a_2 + a, \ldots, a_1 - a_i + a, \ldots, a_1 - a_n + a)$ *for* $a \in \mathbb{Z}_p$,

(Op4) $(a_1, \ldots, a_n) \longrightarrow (a_1, -a_1 + a_2 + a_3, a_3, \ldots, a_n)$.

A region pallet of $\mathbb{Z}_p$ *is a set* $P = \bigcup_{\lambda \in \Lambda} C_\lambda$ *for some* $\{C_\lambda\}_{\lambda \in \Lambda} \subset \bigcup_{n \in 2\mathbb{Z}_+} \mathbb{Z}_p^n / \sim$.

3 Results

Put $\boldsymbol{a} = (a_1, \ldots, a_n)$. We define $\tau_p : \bigcup_{n \in 2\mathbb{Z}_+} \mathbb{Z}_p^n \longrightarrow \mathbb{Z}$ by

$$\tau_p(\boldsymbol{a}) = \max \left\{ k \in \{1, \ldots, p\} \;\middle|\; \begin{array}{l} k | p, \\ a_1 + a_2 \equiv a_2 + a_3 \equiv \cdots \equiv a_n + a_1 \pmod{k} \end{array} \right\}.$$

Suppose p is an even integer. We define $\varepsilon_p : \bigcup_{n \in 2\mathbb{Z}_+} \mathbb{Z}_p^n \longrightarrow \mathbb{Z} \cup \{\infty\}$ by

$$\varepsilon_p(\boldsymbol{a}) = \begin{cases} 0 & \text{if } a_1 + a_2 \equiv \cdots \equiv a_n + a_1 \equiv 0 \pmod{2}, \\ 1 & \text{if } a_1 + a_2 \equiv \cdots \equiv a_n + a_1 \equiv 1 \pmod{2}, \\ \infty & \text{otherwise.} \end{cases}$$

We define $\mu_p : \bigcup_{n\in 2\mathbb{Z}_+} \mathbb{Z}_p^n \longrightarrow \mathbb{Z}$ by

$$\mu_p(\boldsymbol{a}) = E\big((a_1+a_2,\dots,a_n+a_1)\big) - O\big((a_1+a_2,\dots,a_n+a_1)\big),$$

where

$$E(\boldsymbol{a}) = \#\{i\in\{1,\dots,n\} \mid a_i \equiv 0 \pmod 2\}$$

and

$$O(\boldsymbol{a}) = \#\{i\in\{1,\dots,n\} \mid a_i \equiv 1 \pmod 2\}.$$

For $\tau \in \{1,\dots,p\}$ such that $\tau\equiv 0 \pmod 2$, $\tau|p$ and $\dfrac{p}{\tau} \equiv 0 \pmod 2$, define $\mu_{p,\tau} : \bigcup_{n\in 2\mathbb{Z}_+} \mathbb{Z}_p^n \longrightarrow \mathbb{Z}\cup\{\infty\}$ by

$$\mu_{p,\tau}(\boldsymbol{a}) = \begin{cases} \left|\mu_{\frac{p}{\tau}}\left(\left(\dfrac{a_1-a_1}{\tau},\dfrac{a_2-a_2}{\tau},\dots,\dfrac{a_{2j-1}-a_1}{\tau},\dfrac{a_{2j}-a_2}{\tau},\dots,\dfrac{a_n-a_2}{\tau}\right)\right)\right| & \text{if } \tau_p(\boldsymbol{a})=\tau,\\ \infty & \text{otherwise.}\end{cases}$$

We have the following theorem.

Theorem 3.1. *(1) Let $n=2$.*

(i) When p is an odd integer, we have

$$\mathbb{Z}_p^2/\!\sim\; = \left\{\mathbb{Z}_p^2\right\}.$$

(ii) When p is an even integer, we have

$$\mathbb{Z}_p^2/\!\sim\; = \{\eta_\varepsilon \mid \varepsilon\in\{0,1\}\},$$

where

$$\eta_\varepsilon = \{\boldsymbol{a}\in\mathbb{Z}_p^2 \mid \varepsilon_p(\boldsymbol{a}) = \varepsilon \pmod 2\}.$$

(2) Let n is an even integer greater than 2.

(i) When p is an odd integer, we have

$$\mathbb{Z}_p^n/\!\sim\; = \{\delta_\tau \mid \tau\in\{1,\dots,p\} \text{ s.t. } \tau|p\},$$

where

$$\delta_\tau = \{\boldsymbol{a}\in\mathbb{Z}_p^n \mid \tau_p(\boldsymbol{a}) = \tau\}.$$

(ii) When p is an even integer, we have

$$\begin{aligned}\mathbb{Z}_p^n/\!\sim\; = &\left\{\alpha_{\tau,\mu} \;\middle|\; \begin{array}{l}\tau\in\{1,\dots,p\} \text{ s.t. } (\tau|p)\wedge\big(\tau\equiv 1 \pmod 2\big);\\ \mu\in\mathbb{Z} \text{ s.t. } (-n<\mu<n)\wedge\big(\mu\equiv 0 \pmod 2\big)\wedge\big(\frac{n-|\mu|}{2}\equiv 0 \pmod 2\big)\end{array}\right\}\\ &\bigcup\left\{\beta_{\tau,\varepsilon} \;\middle|\; \begin{array}{l}\tau\in\{1,\dots,p\} \text{ s.t. } (\tau|p)\wedge\big(\tau\equiv 0 \pmod 2\big)\wedge\big(\frac{p}{\tau}\equiv 1 \pmod 2\big);\\ \varepsilon\in\{0,1\}\end{array}\right\}\\ &\bigcup\left\{\gamma_{\tau,\varepsilon,\mu} \;\middle|\; \begin{array}{l}\tau\in\{1,\dots,p\} \text{ s.t. } (\tau|p)\wedge\big(\tau\equiv 0 \pmod 2\big)\wedge\big(\frac{p}{\tau}\equiv 0 \pmod 2\big);\\ \varepsilon\in\{0,1\};\\ \mu\in\mathbb{Z} \text{ s.t. } (0\le\mu<n)\wedge\big(\mu\equiv 0 \pmod 2\big)\wedge\big(\frac{n-|\mu|}{2}\equiv 0 \pmod 2\big)\end{array}\right\},\end{aligned}$$

where

$$\alpha_{\tau,\mu} = \{\boldsymbol{a}\in\mathbb{Z}_p^n \mid \tau_p(\boldsymbol{a})=\tau,\ \mu_p(\boldsymbol{a})=\mu\},$$
$$\beta_{\tau,\varepsilon} = \{\boldsymbol{a}\in\mathbb{Z}_p^n \mid \tau_p(\boldsymbol{a})=\tau,\ \varepsilon_p(\boldsymbol{a})=\varepsilon\},$$

and

$$\gamma_{\tau,\varepsilon,\mu} = \{\boldsymbol{a}\in\mathbb{Z}_p^n \mid \tau_p(\boldsymbol{a})=\tau,\ \varepsilon_p(\boldsymbol{a})=\varepsilon,\ \mu_{p,\tau}(\boldsymbol{a})=\mu\}.$$