SETS DEFINABLE IN ORDERED ABELIAN GROUPS OF FINITE BURDEN

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Abstract. In this note, we survey some recent results on definable sets in ordered Abelian groups of finite burden, focusing on topological and arithmetical tameness properties. In the burden 2 case, and assuming definably completeness, definable discrete subsets of the universe can be characterized as those which are definable in an expansion which is elementarily equivalent to $(\mathbb{R}; <, +, \mathbb{Z})$. We end with some open questions and possible directions for future research.

1. Introduction

Since the beginning of the study of theories without the independence property (also known as NIP or dependent theories), researchers have sought tools to measure the complexity of formulas and types in the unstable NIP context. While the Morley rank of a nonalgebraic type in an ordered structure always has the value of ∞, there are other measures of complexity which can be usefully applied in NIP theories, such as dp-rank and burden (defined by Shelah [10] and clarified in later work by Adler [1]).

In this note, we will study some implications of having low dp-rank or low burden for sets definable in the following class of structures:

Definition 1.1. An *ordered Abelian group*, or *OAG*, is a structure $(G; <, +, ...)$ in an expansion of the language $\{<, +\}$ in which $+$ is the operation of an Abelian group and $<$ defines a total (strict) ordering which is invariant under addition:

$$\forall x, y, z \in G\ [x < y \to x + z < y + z].$$

Many interesting examples of unstable NIP theories are OAGs. For instance, by a theorem of Gurevich and Schmitt, any OAG in the language $\{<, +\}$ is NIP [8]. Also, any OAG which is o-minimal, or even weakly o-minimal, is NIP. More examples can be found in the survey article [7].

Most of the new results indicated here have been proved in detail in recent joint work with Alfred Dolich ([4] and [5]).

1.1. **Notation and definitions.** We mostly follow current standard notation in model theory: $\mathcal{M} = (M; <, \ldots)$ denotes a first-order structure $\mathcal{M}$ with universe M, a binary predicate symbol for $<$, and "…" means that there are possibly other non-logical symbols in the language. *Formulas* and *types* are as in first-order logic, and *definable* means definable by such a formula, possibly with parameters from some model of the background theory. Finite tuples such as $\overline{a}$ refer to elements from any elementary extension of the model, or can be considered to come from a sufficiently-saturated "monster" model.

Now we will briefly recall the basic definitions of dp-rank, dp-minimality, and burden which will be used below. The interested reader can consult the book of Simon [12] for a more extensive treatment of this topic.

Definition 1.2. [1] Let $p(\overline{x})$ be a partial type, possibly over parameters from a model. An *ict-pattern (of depth κ in $p(\overline{x})$)* is an array of formulas

$$\langle \varphi_i(\overline{x}; \overline{a}_{i,j}) \,:\, i < \kappa, j < \omega \rangle$$

such that for every function $\eta : \kappa \to \omega$, the set of formulas

$$p(\overline{x}) \cup \{\varphi_i(\overline{x}; \overline{a}_{i,j})^{\text{if } j = \eta(i)}\}$$

is consistent, where the notation "$\varphi(\overline{x}; \overline{b})^{\text{if } c=d}$" signifies $\varphi(\overline{x}; \overline{b})$ in case $c = d$, or the negation $\neg\varphi(\overline{x}; \overline{b})$ in case $c \neq d$.

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Definition 1.3. (1) A partial type $p(\overline{x})$ *has dp-rank less than* κ just in case there is **no** ict-pattern of depth κ in $p(\overline{x})$.

(2) The partial type $p(\overline{x})$ *has dp-rank* κ if it has dp-rank less than κ^+, but does not have dp-rank less than κ.

(3) A complete theory T *has dp-rank* κ if the partial type $x = x$ (in a single free variable) has dp-rank κ.

(4) A complete theory T is *dp-minimal* if it has dp-rank 1.

(5) A complete theory T is *strongly NIP* if it has dp-rank less than ω (that is, there is no ict-pattern with infinitely many rows).

The word "rank" in the terminology "dp-rank" can be misleading: the dp-rank of a type is not an ordinal-valued foundation rank (such as Morley rank), but rather a cardinal number, since the existence of an ict-pattern of depth κ depends only on the cardinality of the ordinal κ. The same applies to the definition of burden below.

Definition 1.4. [1] Let $p(\overline{x})$ be a partial type, possibly over parameters from a model. An *inp-pattern (of depth* κ *in* $p(\overline{x})$*)* is an array of formulas

$$\langle \varphi_i(\overline{x}; \overline{a}_{i,j}) : i < \kappa, j < \omega \rangle$$

and a natural number k_i for each $i < \kappa$ such that the following two conditions are satisfied:

(1) Each "row" $\{\varphi_i(\overline{x}; \overline{a}_{i,j}) : j < \omega\}$ is k_i-inconsistent (that is, the conjunction of any k_i of its formulas is inconsistent); and

(2) for every function $\eta : \kappa \to \omega$, the set of formulas

$$p(\overline{x}) \cup \{\varphi_i(\overline{x}; \overline{a}_{i,\eta(i)})\}$$

is consistent.

Just as ict-patterns give us the notion of dp-rank, inp-patterns yield the corresponding notion of burden:

Definition 1.5. (1) A partial type $p(\overline{x})$ *has burden less than* κ just in case there is **no** inp-pattern of depth κ in $p(\overline{x})$.

(2) The partial type $p(\overline{x})$ *has burden* κ if it has burden less than κ^+, but does not have burden less than κ.

(3) A complete theory T *has burden* κ if the partial type $x = x$ (in a single free variable) has burden κ.

(4) A complete theory T is *inp-minimal* if it has burden 1.

(5) A complete theory T is *strong* if it has burden less than ω (that is, there is no inp-pattern with infinitely many rows).

It turns out that in an NIP theory, the dp-rank of any type is bounded by $|T|^+$ and is equal to the burden (by [1], or see [12]). All partial types in a theory have burden less than ∞ if and only if the theory is NTP_2.

2. Topological properties of unary definable sets

One of the primary motivations for studying dp-minimal OAGs is that the sets definable in one free variable enjoy nice topological properties. Below, all topological notions (open, closed, dense, and so on) refer to the order topology on G, or for $X \subseteq G^n$, to the corresponding product topology.

We recall the following:

Theorem 2.1. *([6]) Suppose that* $\mathcal{G} = (G; <, +, \ldots)$ *is a dp-minmal, densely-ordered OAG and* $X \subseteq G$ *is definable. If* X *is infinite, then* X *is dense in some interval.*

Theorem 2.2. *([11]) Suppose that* $\mathcal{G} = (G; <, +, \ldots)$ *is a dp-minimal,* ***divisible*** *OAG and* $X \subseteq G$ *is definable. If* X *is infinite, then* X *has nonempty interior.*

In contrast, a divisible OAG of dp-rank 2 or higher may have definable unary sets which are infinite and discrete, or definable unary sets which are everywhere dense and codense, as the following example shows:

Example 2.3. Let $\mathcal{R}_{\mathbb{Z},\mathbb{Q}} = (\mathbb{R}; <, +, \mathbb{Z}, \mathbb{Q})$, the ordered group of the real numbers under addition, expanded by a unary predicate for the integers and a unary predicate for the rational numbers. It was shown in [2] that the complete theory $T_{\mathbb{Z},\mathbb{Q}}$ of $\mathcal{R}_{\mathbb{Z},\mathbb{Q}}$ has dp-rank 3, and that each of the reducts $(\mathbb{R}; <, +, \mathbb{Z})$ and $(\mathbb{R}; <, +, \mathbb{Q})$ has dp-rank 2.

A densely-ordered OAG which satisfies the conclusion of Theorem 2.2 – that any infinite definable subset of the universe has nonempty interior – is called *visceral*. Viscerality for OAGs, and more generally structures with visceral definable uniform topologies, were studied extensively in [3], where a cell decomposition result was obtained: every definable set is a finite union of sets which are definably homeomorphic to open sets via coordinate projections, and definable functions are "cellwise continuous," similar to the o-minimal case. Thus we consider visceral OAGs to be topologically tame.

Now suppose that $\mathcal{G}$ is a densely-ordered OAG which is **not** visceral, witnessed by $X \subseteq G$ which is definable, infinite, and has no interior. *A priori*, there are three possible cases:

(1) The set X contains infinitely many isolated points; or
(2) The set X is dense in some interval; or
(3) The set X contains only finitely many isolated points and is nowhere dense.

In Case (1), the set of all isolated points of X is definable, and so there is an infinite discrete set definable in $\mathcal{G}$. Similarly, in Case (2), there is a definable subset of G which is dense and codense in an interval (obtained by intersecting X with an appropriate interval). In Case (3), if F is the finite set of isolated points of X, then the topological closure of $X \setminus F$ is a definable set which is closed, infinite, nowhere dense, and contains no isolated points; we call a set with these four properties *Cantor-like*. Either Case (1) or Case (2) can occur in a dp-rank 2 structure, as shown by Example 2.3 above.

In the case of a divisible OAG with dp-rank 2, Case (1) cannot co-occur with Case (2) or Case (3):

Theorem 2.4. [5] *Let $\mathcal{G}$ be a divisible OAG of dp-rank at most 2. If there is a definable subset of G which is infinite and discrete, then there is no definable subset of G which is dense and codense in an interval, and furthermore there is no definable Cantor-like subset of G.*

Note that in a divisible OAG of *burden* 2 (instead of dp-rank 2), it is possible that there is both a definable infinite discrete set and a definable set which is dense and codense in an interval; see [5] for details of the construction.

Question 2.5. *Is there a divisible OAG of burden 2 with a definable Cantor-like set? And if so, is there a divisible OAG of burden 2 in which both a Cantor-like set and a set which is dense and codense in an interval are definable?*

3. Iterated difference sets in finite-burden OAGs

Next we will discuss results about algebraic properties of discrete sets $D \subseteq R$ definable in an OAG $\mathcal{R}$ of finite burden. Recall the following:

Theorem 3.1. [2] *If $\mathcal{R}$ is a divisible* ***Archimedean*** *OAG whose complete theory is strong, and if $D \subseteq R$ is definable, then D is a finite union of arithmetic progressions (sets of the form $\{a + bi : i \in \mathbb{N}\}$, where $a, b \in R$).*

Our more recent results were motivated by trying to generalize the Theorem above to the case of non-Archimedean OAGs. In the ensuing discussion, we will no longer assume the structure $\mathcal{R}$ is Archimedean, but we will use the stronger assumptions of finite burden and definable completeness (see below).

Definition 3.2. An ordered structure $\mathcal{R} = (R; <, \ldots)$ is *definably complete* if every nonempty definable subset of R which is bounded above (or below) has a supremum (or an infimum, respectively). A theory is definably complete if one of its models (equivalently, any of its models) is definably complete.

For instance, any expansion of $(\mathbb{R}, <)$ is definably complete. Our main reason for assuming definable completeness is that it guarantees the existence of well-defined successors of elements of a discrete definable set, as follows:

Definition 3.3. Let $D \subseteq R$ be a discrete definable subset of a definably complete OAG. If $a \in D$ and a is not the maximum element of D, then

$$S_D(a) = \min\{b \in D : b > a\},$$

which exists by definable completeness. If $a \in D$ is non-maximal, we define $\gamma_D(a) := S_D(a) - a$. Finally, the *difference set* of D is the set

$$D' = \{\gamma_D(a) \, : \, a \in D \text{ and } b \text{ is non-maximal}\}.$$

Fact 3.4. [4] *If D is a discrete definable subset in a definably complete OAG whose theory is strong, then D' is also discrete.*

The fact above allows us to define *iterated difference sets* $D^{(0)}, D^{(1)}, \ldots$ as follows: $D^{(0)} = D$, and for each $n \in \omega$, we recursively define $D^{(n+1)}$ to be the difference set of $D^{(n)}$, or in other words,

$$D^{(n+1)} = \left(D^{(n)}\right)'.$$

Then we have the following:

Theorem 3.5. [4] *Suppose that $\mathcal{R}$ is a definably complete OAG, $D \subseteq R$ is definable and discrete, and the burden of $\mathcal{R}$ is at most n. Then $D^{(n)}$ is finite. If we further assume that $\mathcal{R}$ is densely ordered and $n > 0$, then $D^{(n-1)}$ is finite.*

Note that as a particular case of Theorem 3.5, if $\mathcal{R}$ is densely ordered, definably complete, and has burden 2, then D' is finite.

We conjecture that for each $n \in \mathbb{N}$, there is an expansion $\mathcal{R}$ of a divisible OAG $\mathcal{R}$ of dp-rank $n+2$ in which there is an infinite definable discrete set $D \subseteq R$ such that $D^{(n)}$ is infinite. For $n = 0$ a known example is $\langle \mathbb{R}; <, +, \mathbb{Z} \rangle$, and for $n = 1$ we conjecture that an example could be constructed along the following lines: let $\mathcal{R}_1$ be an ω-saturated elementary extension of the structure $\langle \mathbb{R}; <, +, 0, 1 \rangle$, let C be some positive Dedekind cut in the "standard model" $\langle \mathbb{R}; <, +, 0, 1 \rangle$, and let $D_1 \subseteq R_1$ be an infinite discrete set with $|D_1'| = 1$ which is contained in C and has a least and a greatest element. Then we conjecture that there is an infinite discrete set $D \subseteq R_1$ such that:

(1) $D' = D_1$;
(2) For every $\mathbb{Z}$-chain $\mathcal{Z}(a) \subseteq D$, we have $|\mathcal{Z}(a)'| \in \{1, 2\}$;
(3) If $|\mathcal{Z}(a)'| = 2$ then there is some $b \in D'$ such that $\mathcal{Z}(a)' = \{b, S_{D'}(b)\}$;
(4) If $a_1, a_2 \in D$ and $a_1 < a_2$, then no element of $\mathcal{Z}(a_1)'$ is greater than an element of $\mathcal{Z}(a_2)'$;
(5) The structure $\mathcal{R}_2 = \langle R_1; <, +, D \rangle$ is definably complete; and
(6) The structure $\mathcal{R}_2$ has dp-rank 3.

4. Discrete sets in the burden 2 case

Throughout this section, we will assume that $\mathcal{R}$ is a divisible, definably complete OAG of burden 2 in which an infinite unary discrete set D is definable. By Theorem 3.5 above, this implies that D' is finite, and we will now give a finer analysis of the structure of D. In fact, we can show that D is "almost" a finite union of arithmetic sequences, similarly to the conclusion of Theorem 3.1 in the Archimedean case.

We say that a discrete definable subset E of R is *pseudo-arihtmetic* if $|E'| = 1$. Our first result is:

Theorem 4.1. [4] *If $D \subseteq R$ is definable and discrete, then D is a finite union of points and infinite pseudo-arithmetic sets.*

In fact, even more can be said about the structure of discrete unary definable sets:

Theorem 4.2. [4] *There is a discrete subgroup G of R such that $(R; <, +, G) \equiv (\mathbb{R} :<, +, \mathbb{Z})$ (as in Example 2.3 above) and such that* any *discrete $D \subseteq R$ which is definable in $\mathcal{R}$ is also definable in the structure $(R; <, +, G)$.*

Note that the conclusion of the Theorem above only applies to unary definable sets (that is, definable subsets of R) and not to definable subsets of R^n for $n > 1$. For example, the structure $(\mathbb{R}; <, +, \sin)$ has dp-rank 2, and the infinite discrete unary set $\{k\pi \, : \, k \in \mathbb{Z}\}$ is definable, but the definable structure induced on $\mathbb{R}^2$ is more complicated than that of a model of the theory of $(\mathbb{R}; <, +, \mathbb{Z})$ due to the definable function $\sin(\cdot)$.

It turns out that the complete theory of the sturcutre $(\mathbb{R}; <, +, \mathbb{Z})$ has quantifier elimination in the expanded language with constants for 0 and 1, unary function symbols for multiplication by each $\lambda \in \mathbb{Q}$, and a unary function symbol for the "floor" function $x \mapsto \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer k such that $k \leq x$ (see [9]). This, combined with Theorem 4.2 and the elimination of quantifiers for Presburger arithmetic, can be used to describe the definable unary sets in $\mathcal{R}$ in a fairly precise way; see [4] for details.

5. More open questions

In our paper [4], we showed that if $\mathcal{R}$ is a definably complete OAG whose theory is strong and $D \subseteq R$ is discrete and definable, then in any infinite chain of successive elements $a_0, a_1, a_2, \ldots$ of D, there are only finitely many Archimedean classes represented by the differences $a_{i+1} - a_i$. We conjecture that the answer to the following question is "yes," but we do not know how to prove it:

Question 5.1. *Suppose that $\mathcal{R}$ is a definably complete OAG whose theory is strong and $D \subseteq R$ is definable and discrete. Are there only finitely many Archimedean classes represented in the difference set D'?*

Question 5.2. *Can a reasonable cell decomposition theorem be proven for definably complete OAGs of burden at most 2? Or more generally, for definably complete OAGs of finite burden?*

References

1. Hans Adler, *Strong theories, burden, and weight*, available at http://www.logic.univie.ac.at/∼adler/docs/strong.pdf, 2007.
2. Alfred Dolich and John Goodrick, *Strong theories of ordered abelian groups*, Fundamenta Mathematicae **23** (2017), 269–296.
3. ———, *Tame topology over definable uniform structures*, Notre Dame Journal of Formal Logic **to appear** (2022).
4. ———, *Discrete sets definable in strong expansions of ordered Abelian groups*, arXiv:2208.06929, 2023.
5. ———, *Topological properties of definable sets in ordered Abelian groups of burden 2*, Mathematical Logic Quarterly (2023).
6. John Goodrick, *A monotonicity theorem for dp-minimal densely ordered groups*, Journal of Symbolic Logic **75** (2010), no. 1, 221–238.
7. ———, *Definable sets in dp-minimal ordered Abelian groups*, RIMS Kôkyûroku (Model theoretic aspects of the notion of independence and dimension) (2022), no. 2218.
8. Yuri Gurevich and P. H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Transactions of the American Mathematical Society **284** (1984), no. 1, 171–182.
9. Chris Miller, *Expansions of dense linear orders with the intermediate value property*, The Journal of Symbolic Logic **66** (2001), 1783–1790.
10. Saharon Shelah, *Strongly dependent theories*, Israel Journal of Mathematics **204** (2014), no. 1, 1–83, arXiv:math/0504197v3.
11. Pierre Simon, *On dp-minimal ordered structures*, The Journal of Symbolic Logic **76** (2011), no. 2, 448–460.
12. ———, *A guide to NIP theories*, Cambridge University Press, 2015.

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