ON SINGULARITY FOR THE m-HARMONIC FLOW

Masashi Misawa(*)
Department of Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1 Kurokami, Chuou-ku, Kumamoto-shi, Kumamoto 860-8555, Japan

Abstract We study local regularity and singularity for the evolution of $m-$harmonic maps on $\mathbb{R}^m$ into a smooth compact Riemannian manifold, called $m-$harmonic flow. For any initial data of finite $m-$energy, the global existence of the $m-$harmonic flow, which is regular except at most finitely many time-space points, is reported. The key ingredient is the uniform local regularity estimate for regular $m-$harmonic flows.

Mathematics Subject Classification (2000): 35B45, 35B65 (primary); 35D30, 35K59, 35K65 (secondary).

1 Introduction

Let $\mathcal{N}$ be a smooth compact Riemannian manifold of dimension n with metric h. We assume that, by Nash's embedding theorem, $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^l$ $(l > n)$. The $p-$harmonic map on $\mathbb{R}^m$ into $\mathcal{N}$ is prescribed by the nonlinear elliptic type system of 2nd-order partial differential equations

$$(1.1) \qquad \begin{cases} -\Delta_p u = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \end{cases}$$

with the $p-$Laplace operator

$$(1.2) \qquad \Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right).$$

Here the unknown map $u = (u^i)$, $i = 1, \dots, l$, is a vector-valued function defined on $\mathbb{R}^m$ with values into $\mathcal{N} \subset \mathbb{R}^l$, and $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, \dots, m$, $Du = (D_\alpha u^i)$ is the gradient of a map u, and $|Du|^2 = \sum_{\alpha=1}^m D_\alpha u \cdot D_\alpha u$ with an Euclidean inner product $\cdot$ in $\mathbb{R}^l$. The second fundamental form $A(u)(Du, Du)$ of $\mathcal{N} \subset \mathbb{R}^l$ is a vector field along the map $u \in \mathcal{N}$ with values into the orthogonal complement of the tangent space of $\mathcal{N}$ at u (if necessary, the manifold $\mathcal{N}$ is assumed to be orientable).

The $p-$harmonic map is a critical point of the $p-$energy $E(u)$

$$(1.3) \qquad E(u) := \int_{\mathbb{R}^m} \frac{1}{p} |Du|^p \, dx, \qquad p \geq 2$$

and satisfies its Euler-Lagrange equation (1.1). An approach to look for $p-$harmonic maps is to employ the gradient flow associated with the $p-$energy, called the $p-$harmonic flow, which are described by the evolutionary $p-$Laplacian system

$$(1.4) \qquad \begin{cases} \partial_t u - \Delta_p u = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \end{cases}$$

where $u = u(t, x)$ is defined on $\mathbb{R}^m_\infty = (0, \infty) \times \mathbb{R}^m$ with values onto $\mathbb{R}^l$, $\partial_t u = (\partial_t u^i)$ is the partial derivative on time.

(*)The work by M. Masashi was partially supported by the Grant-in-Aid for Scientific Research (C) No.$18K03375$ at Japan Society for the Promotion of Science

Now we recall a formal derivation of the Euler-Lagrange equation of (1.3) and the gradient flow. Let u be a smooth map from $\mathbb{R}^m$ to $\mathcal{N}$. Let ϕ be a smooth $\mathbb{R}^l-$vector valued function on $\mathbb{R}^m$ with compact support. Let $\Pi : \mathbb{R}^l \supset \mathcal{O}(\mathcal{N}) \to \mathcal{N} \subset \mathbb{R}^l$ be the nearest point projection from a tubular neighborhood $\mathcal{O}(\mathcal{N}) \subset \mathbb{R}^l$ of $\mathcal{N}$, to $\mathcal{N}$. For any sufficient small number τ, $|\tau| \ll |\phi|_\infty$, the map $u + \tau\phi$ has its value in $\mathcal{O}(\mathcal{N})$ and so, $\Pi(u + \tau\phi) \in \mathcal{N}$ is a admissible comparison map. The first variation (Gâteaux derivative) is computed by integration by parts as

$$\frac{d}{d\tau} E\left(\Pi(u+\tau\phi)\right)\Big|_{\tau=0} = \int_{\mathbb{R}^m} \left(-\Delta_p u + |Du|^{p-2} A(u)(Du, Du)\right) \cdot \phi \, dx. \tag{1.5}$$

Thus, the first variational formula (1.5)$= 0$ is the Euler-Lagrange equation (1.1) with the second fundamental form of $\mathcal{N} \subset \mathbb{R}^l$

$$A^i(u)(Du, Du) = \sum_{j,\,k=1}^{l} \frac{d^2\Pi^j}{du^i\, du^k}(u) Du^j \cdot Du^k, \quad i = 1, 2, \ldots, l.$$

For smooth maps $u \in C^\infty(\mathbb{R}^m, \mathcal{N})$ [†] , the gradient vector field $\nabla E(u)$ of the $p-$energy can be formally written as

$$\langle \nabla E(u), \phi \rangle^{\ddagger} = \frac{d}{d\tau} E\left(\Pi(u+\tau\phi)\right)\Big|_{\tau=0}$$

and then, by (1.5)

$$\nabla E(u) = -\Delta_p u + |Du|^{p-2} A(u)(Du, Du) \tag{1.6}$$

and so, the trajectory $\{u(t)\} \subset C^\infty(\mathbb{R}^m, \mathcal{N})$, $0 \le t < \infty$, of its negative direction gradient vector field is the solution to the differential equation

$$\frac{du}{dt} = -\nabla E(u) = \Delta_p u - |Du|^{p-2} A(u)(Du, Du). \tag{1.7}$$

Thus, the solutions of (1.4) may be naturally regarded as the steepest descent of the $p-$energy. A global in time solution to (1.4) for any initial data may converge to the critical points of the $p-$energy, the $p-$harmonic maps, as time tending to ∞. J. Eells and J. H. Sampson originally realized the heat flow method by (1.4) in the harmonic map case $p = 2$ in their pioneering work ([8], also [10]).

Theorem 1 *[14, 9] Suppose that the sectional curvature of the target manifold $\mathcal{N}$ is non-positive, Sect($\mathcal{N}$) ≤ 0. Then, for any smooth initial map between smooth compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$, there exists a unique global in time weak solution of the Cauchy problem on $\mathcal{M}$ for $p-$harmonic flow (1.4). The solution u and its gradient are Hölder continuous in time-space. The solution and its gradient uniformly converge to a weak solution and its gradient, respectively, of the $p-$harmonic map, as time tends to ∞, respectively, which are Hölder continuous.*

We call the weak solution which is locally continuous on time-space together with its gradient the *regular* solution. The curvature restriction on the target manifold in general is necessary for the global existence of regular solution of the $p-$harmonic flow. In fact, without any curvature restriction on the target manifold, we have some example of a blowing up solution at a finite time (see [3] in the case $p = m = 3$, also [2]). But, a global in time weak solution may exist.

† $C^\infty(\mathbb{R}^m, \mathcal{N})$ is a Banach manifold

‡ $\langle \nabla E(u), \cdot \rangle$ is a bounded linear functional on a tangent space $\bigcup_{u \in \mathcal{X}} C^\infty(\mathbb{R}^m, T_u(\mathcal{N}))$ of a Banach manifold $\mathcal{X} := C^\infty(\mathbb{R}^m, \mathcal{N})$.

Theorem 2 *[11] Let $p = m \geq 2$ and the initial data be in the set of Sobolev maps $W^{1,p}(\mathcal{M},\mathcal{N})$ between two smooth, compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$. Then, there exists a global in time weak solution of Cauchy problem on $\mathcal{M}$ for the $m-$harmonic flow. The solution and its gradient are Hölder continuous on time-space, except for at most finitely many time slices.*

In the case $p = m = 2$, the global in time existence as above is also shown for the initial-boundary value problem of the two-dimensional harmonic flow. Moreover, the solution is smooth except for at most finitely many points [19, 1].

We study the global existence and regularity of a weak solution of the Cauchy problem for the $m-$harmonic flow (1.4) with an initial data u_0 in $\mathbb{R}^m$

$$\begin{cases} \partial_t u - \operatorname{div}\left(|Du|^{m-2}Du\right) = |Du|^{m-2}A(u)(Du, Du) & \text{in } \mathbb{R}^m_\infty \\ u \in \mathcal{N} \end{cases} \tag{1.8}$$

The assertions in Theorem 2 may also hold true for the Cauchy problem (1.8) on the Euclidean space $\mathbb{R}^m$. Since the weak solution u is in $L^\infty(0,\infty; W^{1,m}(\mathbb{R}^m,\mathbb{R}^l))$ (see Definition 2 below), for any small positive ϵ_0, there exists a large positive number K_0 such that the $m-$energy of the solution outside $B(K_0, 0)$ is smaller than ϵ_0. Combining this fact and the local energy estimates(refer to Sect. 2.3 below), the proof in Theorem 2 can be extended into the Euclidean case. In this paper, we improve the regularity of the solution obtained in [11], and show that the solution has at most only *finitely many* singular points.

Before stating our main result, we recall the definitions of the nonlinear space of Sobolev maps from $\mathbb{R}^m$ into $\mathcal{N}$, and the weak solution for the Cauchy problem (1.8) for the $m-$harmonic flow.

Definition 1 (Sobolev maps) *The Sobolev maps on $\mathbb{R}^m$ are defined as follows:*

$$\begin{aligned} &\mathrm{W}^{1,m}\left(\mathbb{R}^m, \mathbb{R}^l\right) := \left\{ v \in \mathrm{L}^m\left(\mathbb{R}^m, \mathbb{R}^l\right) \,\middle|\, \exists \text{ a weak derivative } Dv \in \mathrm{L}^p\left(\mathbb{R}^m, \mathbb{R}^{ml}\right) \right\}; \\ &\mathrm{W}^{1,m}\left(\mathbb{R}^m, \mathcal{N}\right) := \left\{ v \in \mathrm{W}^{1,m}\left(\mathbb{R}^m, \mathbb{R}^l\right) \mid v \in \mathcal{N} \text{ almost everywhere in } \mathbb{R}^m \right\}; \\ &\|v\|_{\mathrm{W}^{1,m}(\mathbb{R}^m)} := \|v\|^p_{\mathrm{L}^m(\mathbb{R}^m)} + \|Dv\|_{\mathrm{L}^m(\mathbb{R}^m)}, \end{aligned} \tag{1.9}$$

where we assume that the origin $\in \mathcal{N}$ (since (1.8) is invariant under the parallel transformation).

Definition 2 (A weak solution) *Let $u_0 \in \mathrm{W}^{1,m}(\mathbb{R}^m,\mathcal{N})$. A map u is called a global weak solution of the Cauchy problem (1.8) if and only if u is a measurable vector-valued function defined on $\mathbb{R}^m_\infty := (0, \infty) \times \mathbb{R}^m$ with values into $\mathbb{R}^l$, satisfying the following four conditions :*

(D1) $u \in \mathrm{L}^\infty(0,\infty\,;\, \mathrm{W}^{1,m}(\mathbb{R}^m, \mathbb{R}^l))$, $\partial_t u \in \mathrm{L}^2(\mathbb{R}^m_\infty, \mathbb{R}^l)$;

(D2) $u \in \mathcal{N}$ *almost everywhere in* $\mathbb{R}^m_\infty$;

(D3) u *satisfies* (1.4) *in the sense of distributions, that is, for any smooth map* $\phi \in \mathrm{C}^\infty_0(\mathbb{R}^m_\infty, \mathbb{R}^l)$,

$$\int_{\mathbb{R}^m_\infty} \left\{ \partial_t u \cdot \phi + |Du|^{p-2}Du \cdot D\phi - |Du|^{p-2}\phi \cdot A(u)(Du, Du) \right\} dz = 0;$$

(D4) u *attains the initial data continuously in the Sobolev space*

$$|u(t) - u_0|_{\mathrm{W}^{1,m}(R^m,\, \mathbb{R}^l)} \to 0 \quad \text{as } t \to 0.$$

Our main theorem in this paper is the following.

Theorem 3 (A global existence and regularity for the $m-$harmonic flow) *Let $p = m \geq 3$ and let u_0 be any Sobolev map in $W^{1,m}(\mathbb{R}^m, \mathcal{N})$. Then, there exists a global weak solution u of the Cauchy problem for the $m-$harmonic flow with initial data u_0, satisfying the energy inequality*

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} E(u(t)) \leq E(u_0).$$

Moreover, the solution u is partial regular in the following sense : There exists at most finitely many time-space points Σ, given by

$$\{(T_l, x_{jl}) \in \mathbb{R}^m_\infty \,|0 < T_l < \infty,\, x_{jl} \in \mathbb{R}^m,\, l = 1, \dots, L < \infty;\, j = 1, \dots, J_l < \infty\},$$

$$\Sigma_l := \bigcup_{j=1}^{J_l} \{x_{jl}\}, \qquad \Sigma = \bigcup_{l=1}^{L} \Sigma_l \tag{1.10}$$

such that u and its gradient Du are locally in time-space continuous in the complement $\mathbb{R}^m_\infty \setminus \Sigma$; As $t \nearrow T_l$, $u(t), Du(t) \to u(T_l), Du(T_l)$ locally uniformly in $\mathcal{R}_l := \mathbb{R}^m \setminus \Sigma_l$; $u(t) \to u(T_l)$ in $W^{1,m}_{\text{loc}}(\mathcal{R}_l)$.

The theorem establishes that the singular set of a weak solution obtained in the case that $p = m$ actually consists of at most *finitely many* time-space points Σ. The proof is based on the local energy estimates similar as made by Hungerbühler, and the *small energy regularity* estimate available to the $p-$harmonic flow for $p > 2$ (see [16]). The small energy regularity estimate has been recently applied to the global existence of a *partial regular* weak solution to the Cauchy problem for $p-$harmonic flow (see [17]). The small energy regularity estimate is based on some monotonicity type estimates of local scaled energy.

2 Regularity estimates

In this section we state some energy estimates and regularity estimates available for regular solutions to the $m-$harmonic flow. The validity of these estimates have already been shown in [16], where the regularity of the $p-$harmonic flow is addressed in the case $p > 2$ (see also [15, 17]).

The regular solution of the $m-$harmonic flow is defined as follows:

Definition (regular solution) Let $\mathbb{R}^m_T = (0, T) \times \mathbb{R}^m$ for $0 < T \leq \infty$. A map u defined on $\mathbb{R}^m_T$ is a *regular solution* of (1.8) on $\mathbb{R}^m_T$ if and only if $u \in L^\infty(0, T\,; W^{1,m}(\mathbb{R}^m, \mathbb{R}^l))$, $\partial_t u \in L^2(\mathbb{R}^m_T, \mathbb{R}^l)$ such that $u \in \mathcal{N}$ almost everywhere in $\mathbb{R}^m_T$ and (1.8) is satisfied in the distributional sense. Furthermore, u and its gradient Du are locally continuous on $(0, T) \times \mathbb{R}^m$.

2.1 Preliminaries

The energy inequality is the fundamental global estimate for the $m-$harmonic flow.

Lemma 4 (Energy inequality) *Let $u_0 \in \mathrm{W}^{1,m}(\mathbb{R}^m, \mathcal{N})$ and u be a regular solution of (1.8) in $\mathbb{R}^m_T$ with a positive $T < \infty$. Then, it holds that, for any nonnegative $t_1 < t_2 \leq T$,*

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_{t_1,t_2})} + E(u(t_2)) \leq E(u(t_1)). \tag{2.1}$$

We need the so-called Bochner type estimate for the $m-$energy density.

Lemma 5 (Bochner type estimate) *Let u be a regular solution of (1.8) on $\mathbb{R}^m_T$ for a positive $T < \infty$. Then, it holds in $\mathbb{R}^m_T$ in the distribution sense that*

$$\partial_t \left(\frac{1}{2}|Du|^2\right) - D_\alpha \left(|Du|^{m-2}\mathcal{A}^{\alpha\beta} D_\beta \left(\frac{1}{2}|Du|^2\right)\right) \\ + C_1\, |Du|^{m-2}\left|D^2 u\right|^2 \le C_2\, \left(1 + |Du|^2\right)\, |Du|^m, \tag{2.2}$$

where

$$\mathcal{A}^{\alpha\beta} := \delta^{\alpha\beta} + (m-2)\frac{D_\alpha u \cdot D_\beta u}{|Du|^2}, \quad \left|D^2 u\right|^2 = D_\alpha D_\gamma u \cdot D_\alpha D_\gamma u,$$

with the summation convention over repeated indices, and the positive constants C_i $(i = 1, 2)$ depend on m and $\mathcal{N}$.

Let B_0 be a positive number, R be a positive number such that $R^{B_0} < \min\{1, T\}$ and (t_0, x_0) in the parabolic like envelope $\mathcal{P} := \left\{(t, x) \,:\, T - R^{B_0} < t \le T,\ |x|^{B_0} < t - (T - R^{B_0})\right\}$. In the following we use a time-space local cylinder. For $r, \tau > 0$ and $(t_0, x_0) \in (0, T] \times \mathbb{R}^m$, $Q(\tau, r)(t_0, x_0) = (t_0 - \tau, t_0) \times B(r, x_0)$, where $B(r, x_0)$ is an open ball in $\mathbb{R}^m$ with center x_0 and radius r. The time-space Lebesgue measure in $\mathbb{R} \times \mathbb{R}^m$ is denoted by $dz = dxdt$.

Lemma 6 (Gradient boundedness on small region) *Let u be a regular solution of (1.8) in $\mathbb{R}^m_T$. For some $(t_0, x_0) \in \mathcal{P}$, with $\rho_0 := \left((t_0 - (T - R^{B_0}))^{1/B_0} - |x_0|\right)/4$. Suppose that, for $B_0 > 0$, $r_0 > 0$, $C_1 > 0$ and $L > 0$,*

$$r_0 \le \rho_0/2; \quad L^{2-p}\,(r_0)^2 \le (\rho_0)^{B_0}; \quad r_0 \sup_{Q(L^{2-p}(r_0)^2,\, r_0)(t_0,\, x_0)} |Du| \le C_1. \tag{2.3}$$

Let $q > 1$ be a positive number. Then there exists a positive number C depending only on q, $m, p, \mathcal{N}$, but, independent of L, such that

$$\sup_{Q(L^{2-p}(r_0/2)^2,\, r_0/2)(t_0,\, x_0)} |Du|^2\, \mathcal{C}^q \le \frac{C\, L^{2-p}}{|Q\,(L^{2-p}(r_0)^2,\, r_0)|} \int_{Q(L^{2-p}(r_0)^2,\, r_0)(t_0,\, x_0)} |Du|^p\, \mathcal{C}^q\, dz + C\, L^2;$$

$$\mathcal{C}(t, x) := \left((t - (T - R^{B_0}))^{1/B_0} - |x|\right)_+; \quad q > 1. \tag{2.4}$$

2.2 Small energy regularity estimates

Now we will present the so-called small energy local regularity of the $m-$harmonic flow. This local regularity estimate has been recently shown to hold true for a regular solution of the $p-$harmonic flow with $p > 2$(see [17], also refer to [20]). Here we require the result in the case that $p = m \ge 3$.

Theorem 7 (Small energy regularity) *Let $m \ge 3$. Let B_0 and a_0 be positive numbers satisfying the conditions*

$$\frac{4(m-1)}{m} < B_0 < m; \quad \frac{B_0 - 2}{m-2} < a_0 \le 1. \tag{2.5}$$

Let u be a regular solution of (1.8) in $(0, T] \times \mathbb{R}^m$ for a positive $T < \infty$, satisfying the energy bound

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_T)} + \sup_{0<t<T} E(u) \le C_1 \tag{2.6}$$

for a positive number C_1 depending only on m and $\mathcal{N}$. Then, there exists a small positive numeber $R_0 < 1$, depending only on $\mathcal{N}$, m, B_0, a_0 and C_1, and the following holds true: Let γ_0 be any positive number

$$\gamma_0 = \frac{m(B_0 - 2)}{m - 2}. \tag{2.7}$$

If, for some small positive $R < \min\{R_0,\, T^{1/B_0}\}$,

$$\limsup_{r \searrow 0} r^{\gamma_0 - m} \int\limits_{\{t = T - R^{B_0}\} \times B(r,\,0)} |Du(t,\,x)|^m\, dx \le 1, \tag{2.8}$$

then, there holds

$$\sup_{\left(T - (R/4)^{B_0},\, T\right) \times B(R/4,\,0)} |Du| \le C_2\, R^{-a_0}, \tag{2.9}$$

where the positive constant C_2 depends only on γ_0, B_0, a_0, m, $\mathcal{N}$ and C_1.

Theorem 7 is just the so-called *small energy regularity* result for the $m-$harmonic flows, that is, under the condition (2.8) the gradients of regular solutions of the $m-$harmonic flow are uniformly locally bounded as in (2.9) and thus, uniformly locally continuous, by the fundamental regularity for the evolutionary $p-$Laplace operator (see [5]). The criterion as in (2.8) may be *almost optimal* on a scale order, comparing with the corresponding uniform regularity criterion for regular solutions of stationary $m-$harmonic maps (refer to [6, 18, 7]), because the exponent γ_0 can be chosen as close to m as possible, by the condtion of B_0 in (2.5).

The proof of Theorem 7 is based on the so-called monotonicity estimate of a scaled local energy. Here we state a new *monotonicity* type estimate of a *localized* scaled $p-$energy, without the proof. Let u be a regular solution of (1.4) in $\mathbb{R}^m_T$ for a positive $T < \infty$. Let us define our localized scaled $p-$energy in the following way: Let $(t_0,\, x_0)$ be in the parabolic like envelope

$$\left\{(t,\, x) \in (0, \infty) \times \mathbb{R}^m \;:\; \min\{T,\, 1\} > t \ge |x|^{B_0}\right\}; \quad B_0 > 2.$$

The localized scaled energy is defined as

$$E_\pm(r) = \frac{1}{\Lambda^p} \int\limits_{\{t = t_0 \pm \Lambda^{2-m} r^2\} \times \mathbb{R}^m} \frac{1}{m} |Du(t, x)|^m\, \mathcal{B}_\pm(t_0, x_0; t, x)\, \mathcal{C}^q(t, x)\, dx \tag{2.10}$$

for a scale radius r, $0 \le r \le \min\{1,\, T^{1/B_0}\}$, where $\Lambda = \Lambda(r)$ is a function of a scale radius r, defined as

$$\Lambda = \Lambda(r) = r^{\frac{B_0 - 2}{2 - m}}; \quad B_0 > \tfrac{4(m-1)}{m}. \tag{2.11}$$

The *forward* or *backward* in time Barenblatt like function, denoted by $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively, are defined as

$$\mathcal{B}_\pm(t_0,\, x_0\,;\, t,\, x) = \frac{1}{(\mp t_0 \pm t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{2\, (\mp t_0 \pm t)^{\frac{1}{B_0}}}\right)^{\frac{m}{m-1}}\right)_+^{\frac{m-1}{m-2}}, \quad \mp t < \mp t_0. \tag{2.12}$$

The localized function $\mathcal{C}$ is used as

$$\mathcal{C}(t,\, x) := \left(t^{1/B_0} - |x|\right)_+; \quad q > 2. \tag{2.13}$$

We call $E_+(r)$ and $E_-(r)$ the forward and backward localized scaled $p-$energy, respectively.

Our monotonicity type estimates of the scaled energys are the followings.

Lemma 8 (Monotonicity estimate for the backward localized scaled $p-$energy) *Let $q > 2$. Let u be a regular solution of (1.8) in $\mathbb{R}^m_T$ for a positive $T < \infty$. Then, there holds for all positive numbers $r < \rho \le \min\left\{1,\ (t_0/2)^{1/B_0}\right\}$*

$$E_-(r) \le E_-(\rho) + C\ (\rho^\mu - r^\mu) + C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\mathcal{C}^{\tilde{q}}(t)\,|Du(t)|^{2(m-1)}\|_{L^\infty\left(B((t_0-t)^{1/B_0},\,x_0)\right)}\,dt, \tag{2.14}$$

where $\tilde{q} = \min\{q-2,\ q(m-1)/m\}$, B_0 as in (2.11), and the positive exponent μ depends only on $\mathcal{N}$, m and B_0, and the positive constant C depends only on the same ones as μ and q.

Lemma 9 (Monotonicity estimate for the forward localized scaled $p-$energy) *Let $q > 2$. Let u be a regular solution of (1.8) on $\mathbb{R}^m_T$ for a positive $T < \infty$. Then, there holds for all positive numbers $r < \rho \le \min\left\{1,\ (T-t_0)^{1/B_0}\right\}$*

$$E_+(\rho) \le E_+(r) + C\ (\rho^\mu - r^\mu) + C \int_{t_0+r^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{\tilde{q}}(t)\,|Du(t)|^{2(m-1)}\|_{L^\infty\left(B((t-t_0)^{1/B_0},\,x_0)\right)}\,dt, \tag{2.15}$$

where $\tilde{q} = \min\{q-2,\ q(m-1)/m\}$, B_0 as in (2.11), and the positive constants μ and C have the same dependence as those in Lemma 8.

2.3 Second derivative and gradient L^q-estimates

We here state some local regularity estimates available for a regular solution to (1.8) in $\mathbb{R}^m_T$, $0 < T < \infty$, which are crucial for the proof of the main theorem. Let the local $m-$energy be defined for any point $x_0 \in \mathbb{R}^m$, any positive $r \le 1$ and $t \ge 0$ as

$$\Psi(r; t, x_0) := \int_{B(r,\,x_0)} |Du(t)|^m dx. \tag{2.16}$$

Put the local in time-space regions: For $t_0 \in (0,T)$ and $\tau > 0$, $\mathcal{I}/2 := (t_0 - \tau/4, t_0) \subset \mathcal{I} := (t_0 - \tau/2, t_0) \subset 2\mathcal{I} := (t_0 - \tau, t_0) \subset (0,T)$ and, for nonnegative $R \le 1$, $\mathcal{B}/2 := B(R/2, 0)$ $\mathcal{B} := B(R,0)$ and $2\mathcal{B} := B(2R, 0)$.

The second derivative estimate for the $m-$harmonic flow has been obtained in [11, Lemma 5, p. 601].

Lemma 10 (Second derivative estimate) *Let u be a regular solution of (1.8) in $\mathbb{R}^m_T$ for a positive $T < \infty$. There exists a positive constant $\epsilon_0 = \epsilon_0(m, \mathcal{N})$ such that, if, for some small positive numbers $t_0 \le T$, $R \le 1$ and $\tau \le 1$,*

$$\sup_{t\in 2\mathcal{I}} \Psi(2R; t, x_0) \le \epsilon_0, \tag{2.17}$$

then, there holds

$$\int_{\mathcal{I}\times\mathcal{B}} (|Du|^{m-2}|D^2u|^2 + |Du|^{m+2})dz \le C\left\{R^m + E(u_0) + E(u_0)\,\tau\left(1+\frac{1}{R^2}\right)\right\}, \tag{2.18}$$

where the positive constant C depends only on m and $\mathcal{N}$.

Proof of Lemma 10. Let $\eta = \eta(x)$ be a smooth function defined for $x \in \mathbb{R}^m$ such that $\eta = 1$ in $\mathcal{B}$, $\eta = 0$ outside the closure of $2\mathcal{B}$ and $|D\eta| \le C/R$. Let $\sigma = \sigma(t)$ be a smooth function on $(-\infty, t_0]$ such that $\sigma = 1$ in $\mathcal{I}$, $\sigma = 0$ outside the closure of $2\mathcal{I}$ and $|\partial_t\sigma| \le C/\tau$.

Let u be a regular solution in $\mathbb{R}^m_T$ of the $m-$harmonic flow. We proceed the estimations by use of the Bochner estimate (2.2) in Lemma 5. We use a test function $\sigma\eta^m$ in the weak form of (2.2), where the admissibility of the test function is shown by the usual approximation argument with the Steklov averaging on time. Thus, we have

$$\begin{aligned}
&\int_{2\mathcal{B}} \frac{1}{2}|Du|^2\eta^m\sigma dx + C_1 \int_{2\mathcal{I}\times 2\mathcal{B}} |Du|^{m-2}|D^2u|^2\eta^m\sigma dz \\
&= \int_{2\mathcal{I}\times 2\mathcal{B}} \Big(\frac{1}{2}|Du|^2\eta^m\partial_t\sigma + |Du|^{m-2}\mathcal{A}^{\alpha\beta}D_\beta\frac{1}{2}|Du|^2 D_\alpha\eta^m\sigma \\
&\qquad + C_2\,|Du|^m\eta^m\sigma + C_2\,|Du|^{m+2}\eta^m\sigma\Big)\,dz.
\end{aligned}\tag{2.19}$$

Each term in the right hand side of (2.19) is estimated as follows: The second integral term is estimated above by the Cauchy inequality as

$$\begin{aligned}
&C\int_{2\mathcal{I}\times 2\mathcal{B}} |Du|^{m-1}|D^2u|\eta^{m-1}|D\eta|\sigma\,dz \\
&\le \delta\int_{2\mathcal{I}\times 2\mathcal{B}} |Du|^{m-2}|D^2u|^2\eta^m\sigma\,dz + \frac{C}{\delta\,R^2}\int_{2\mathcal{I}\times 2\mathcal{B}} |Du|^m\eta^{m-2}\,\sigma\,dz,
\end{aligned}\tag{2.20}$$

of which the first term containing the second derivative is absorbed into the second term of the left hand side, if δ is chosen to be small.
The last integral term in the right hand side of (2.19) is estimated by the Hölder inequality and the Sobolev inequality $W^{1,2}_0(2\mathcal{B}) \hookrightarrow L^{2m/(m-2)}(2\mathcal{B})$ and thus, is bounded above by

$$\begin{aligned}
C_2\int_{2\mathcal{I}\times 2\mathcal{B}} |Du|^{m+2}\eta^m\sigma\,dz &\le \int_{2\mathcal{I}}\left(\int_{2\mathcal{B}}|Du|^m\,dx\right)^{\frac{2}{m}}\left(\int_{2\mathcal{B}}|Du|^{\frac{m^2}{m-2}}\eta^{\frac{m^2}{m-2}}\,dx\right)^{\frac{m-2}{m}}\sigma\,dt \\
&\le \sup_{t\in\mathcal{I}}\left(\int_{2\mathcal{B}}|Du(t)|^m\,dx\right)^{\frac{2}{m}}\int_{2\mathcal{I}}\left(C\int_{2\mathcal{B}}\left|D\left(|Du|^{\frac{m}{2}}\,\eta^{\frac{m}{2}}\right)\right|^2\,dx\right)\sigma\,dt \\
&\le C\,(\epsilon_0)^{\frac{2}{m}}\int_{2\mathcal{I}\times 2\mathcal{B}}\left(|Du|^{m-2}|D^2u|^2\eta^m + \frac{1}{R^2}|Du|^m\eta^{m-2}\right)\sigma\,dz,
\end{aligned}\tag{2.21}$$

where the first second derivative term is absorbed into the second term in the left hand side of (2.19), if the positive number ϵ_0 is small.

Gathering the above estimates we get

$$\begin{aligned}
&\int_{2\mathcal{B}}|Du|^2\eta^m\sigma dx + \int_{2\mathcal{I}\times 2\mathcal{B}}|Du|^{m-2}|D^2u|^2\eta^m\sigma dz \\
&\le C\int_{2\mathcal{I}\times 2\mathcal{B}}\left\{\frac{1}{\tau}|Du|^2\eta^m + |Du|^m\left(\frac{C}{R^2}\eta^{m-2} + \eta^{m-2}\right)\sigma\right\}\,dz \\
&\le C\left\{R^m + E(u_0) + E(u_0)\,\tau\left(1+R^{-2}\right)\right\},
\end{aligned}\tag{2.22}$$

where the energy inequality (2.1) in Lemma 4 is used, and thus, the desired estimation (2.18) follows from (2.21) and (2.22) □

The L^q-estimate of gradient follows from Lemma 10, of which the proof is similar to [11, Lemma 7, p. 603].

Lemma 11 (Gradient L^q- estimate) *Let u be a regular solution to (1.8) in $\mathbb{R}^m_T$ with a positive $T < \infty$. There exists a positive constant $\epsilon_0 = \epsilon_0(m, \mathcal{N})$ such that, if, for some small positive numbers $t_0 \leq T$, $R \leq 1$ and $\tau \leq 1$,*

$$\sup_{t \in 2\mathcal{I}} E(2R; t, x_0) \leq \epsilon_0, \tag{2.23}$$

then, it holds true that

$$\begin{aligned} &\sup_{t \in \mathcal{I}/2} \int_{\mathcal{B}/2} |Du|^{m+2} dx + \int_{\mathcal{I}/2 \times \mathcal{B}/2} |D|Du|^m|^2 dz \\ &\leq C\left(E(u_0) + R^m\right)\left(\frac{1}{\tau} + \frac{1}{R^m}\right) + CE(u_0)\frac{1}{R^m}\left(1 + \frac{1}{R^2}\right), \end{aligned} \tag{2.24}$$

where the positive constant C depends only on m and $\mathcal{N}$.

Proof of Lemma 11. As before, let u be a regular solution in $\mathbb{R}^m_T$ of the $m-$harmonic flow. The cut-off functions are changed a little. Let $\eta = \eta(x)$ be a smooth function defined for $x \in \mathbb{R}^m$ such that $\eta = 1$ in $\mathcal{B}/2$, $\eta = 0$ outside the closure of $\mathcal{B}$ and $|D\eta| \leq C/R$. Let $\sigma = \sigma(t)$ be a smooth function on $(-\infty, t_0]$ such that $\sigma = 1$ in $\mathcal{I}/2$, $\sigma = 0$ outside the closure of $\mathcal{I}$ and $|\partial_t \sigma| \leq C/\tau$. By use of a test function $\sigma\eta^m|Du|^m$ in the weak form of (2.2), where the admissibility of the test function is shown by the usual approximation argument with the Steklov averaging on time, we have

$$\begin{aligned} &\int_{\mathcal{B}} \frac{1}{m+2}|Du|^{m+2}\eta^m\sigma dx + C_1 \int_{\mathcal{I}\times\mathcal{B}} |Du|^{2(m-1)}|D^2u|^2\eta^m\sigma\, dz \\ &= \int_{\mathcal{I}\times\mathcal{B}} \Big(\frac{1}{m+2}|Du|^{m+2}\eta^m\partial_t\sigma + |Du|^{2(m-1)}\mathcal{A}^{\alpha\beta}D_\beta\frac{1}{2}|Du|^2 D_\alpha\eta^m\sigma \\ &\qquad + C_2\,|Du|^{2m}\eta^m\sigma + C_2\,|Du|^{2m+2}\eta^m\sigma\Big)\, dz. \end{aligned} \tag{2.25}$$

Each term in the right hand side of (2.25) will be estimated in the following: By the Cauchy inequality, the second integral term is bounded above by

$$\begin{aligned} &C\int_{\mathcal{I}\times\mathcal{B}} |Du|^{2(m-1)+1}|D^2u|\,\eta^{m-1}\,|D\eta|\,\sigma\, dz \\ &\leq \delta \int_{\mathcal{I}\times\mathcal{B}} |Du|^{2(m-1)}|D^2u|^2\,\eta^m\,\sigma\, dz + \frac{C}{\delta}\int_{\mathcal{I}\times\mathcal{B}} |Du|^{2m}\,\eta^{m-2}\,|D\eta|^2\, dz, \end{aligned} \tag{2.26}$$

of which the first term with a small number δ is absorbed into the second term in the left hand side of (2.25) and, by Young's inequalities with a small positive number δ', the second integral term is estimated above as

$$\frac{C}{\delta}\int_{\mathcal{I}\times\mathcal{B}} \left(\delta'\,|Du|^{2m+2}\,\eta^m + C(\delta'^{-1})\,|Du|^{m+2}\,|D\eta|^m\right)\sigma\, dz. \tag{2.27}$$

By Young's inequality, the third and forth terms are bounded by

$$C \int\limits_{\mathcal{I}\times\mathcal{B}} \left(1 + |Du|^{2m+2}\right) \eta^m \, \sigma \, dz. \tag{2.28}$$

Looking at (2.27) and (2.28), we estimate the $(2m+2)-$powered integral term as follows: By the Hölder inequality and the Sobolev inequality, $W_0^{1,2}(\mathcal{B}) \hookrightarrow L^{2m/(m-2)}(\mathcal{B})$, we get

$$\begin{aligned}
\int\limits_{\mathcal{I}\times\mathcal{B}} |Du|^{2m+2}\, \eta^m \, \sigma \, dz \quad &\le \int\limits_{\mathcal{I}} \left(\int\limits_{\mathcal{B}} |Du|^{\frac{2m^2}{m-2}} \, \eta^{\frac{m^2}{m-2}} \, dx \right)^{\frac{m-2}{m}} \left(\int\limits_{\mathcal{B}} |Du|^m \, dx \right)^{\frac{2}{m}} \sigma \, dt \\
&\le \sup_{t\in\mathcal{I}} \left(\int\limits_{\mathcal{B}} |Du(t)|^m \, dx \right)^{\frac{2}{m}} \int\limits_{2\mathcal{I}} C \left(\int\limits_{\mathcal{B}} \left| D \left(|Du|^m \, \eta^{\frac{m}{2}} \right) \right|^2 \, dx \right) \sigma \, dt \\
&\le C \sup_{t\in\mathcal{I}} \left(\int\limits_{\mathcal{B}} |Du(t)|^m \, dx \right)^{\frac{2}{m}} \\
&\qquad \times \int\limits_{\mathcal{I}\times\mathcal{B}} \left(|Du|^{2(m-1)} |D^2 u|^2 \, \eta^m + |Du|^{2m} \, \eta^{m-2} \, |D\eta|^2 \right) \sigma \, dz,
\end{aligned} \tag{2.29}$$

of which the last term is estimated as in (2.27), leading to

$$\begin{aligned}
\int\limits_{\mathcal{I}\times\mathcal{B}} |Du|^{2m+2}\, \eta^m \, \sigma \, dz \quad &\le \; C\,(\epsilon_0)^{\frac{2}{m}} \int\limits_{\mathcal{I}\times\mathcal{B}} \Big(|Du|^{2(m-1)} |D^2 u|^2 \, \eta^m \\
&\qquad + \delta' \, |Du|^{2m+2} \, \eta^m + C(\delta'^{-1}) \, |Du|^{m+2} \, |D\eta|^m \Big) \, \sigma \, dz,
\end{aligned} \tag{2.30}$$

where $0 < \epsilon_0 \le 1$ and, $\delta' > 0$ is chosen so small that, in the inequality in (2.30), the $(2m+2)$ powered integral of gradient is absorbed into in the first term. In this way, we have

$$\int\limits_{\mathcal{I}\times\mathcal{B}} |Du|^{2m+2}\, \eta^m \, \sigma \, dz \le C\,(\epsilon_0)^{\frac{2}{m}} \int\limits_{\mathcal{I}\times\mathcal{B}} \left(|Du|^{2(m-1)} |D^2 u|^2 \, \eta^m + |Du|^{m+2} \, |D\eta|^m \right) \sigma \, dz, \tag{2.31}$$

of which the second derivative term with a small positive ϵ_0 is absorbed into the second term in the left hand side of (2.25).

Finally, plugging (2.31) into (2.27) and (2.28) and gathering the resulting inequalities in (2.25) we have

$$\begin{aligned}
&\int\limits_{\mathcal{B}} |Du|^{m+2} \eta^m \sigma dx + \int\limits_{2\mathcal{I}\times 2\mathcal{B}} |Du|^{2(m-1)} |D^2 u|^2 \eta^m \sigma \, dz \\
&\le C \int\limits_{\mathcal{I}\times\mathcal{B}} \left(|Du|^{m+2} \eta^m \partial_t \sigma + |Du|^{m+2} \, |D\eta|^m \, \sigma \right) \, dz,
\end{aligned} \tag{2.32}$$

where the last $(m+2)-$powered integral of gradients is estimated by (2.18) in Lemma 10.

□

3 Finite singularity

In this section we consider the weak solution to the $m-$harmonic flow (1.8) obtained by Hungerbühler. This weak solution is regular except at most finitely many time slices $\{T_l\}$,

$0 < T_l < \infty$, $l = 1, \ldots, L < \infty$. Let (T_{l-1}, T_l), $l = 1, \ldots, L < \infty$, be taken arbitrarily and fixed. Thus, this weak solution is locally regular in $\mathbb{R}^m_{T_{l-1},T_l} = (T_{l-1}, T_l) \times \mathbb{R}^m$, by the partial regularity result in [11, Theorem 10, page 624]. We study the regularity of this weak solution around T_l. Hereafter, by parallel transformation on time, let $T := T_l - T_{l-1}$ and we suppose that u is a weak solution of the $m-$harmonic flow (1.8), and locally regular in $\mathbb{R}^m_T = (0,\, T) \times \mathbb{R}^m$. In the construction of the regular solution u on $\mathbb{R}^m_T$ of (1.8) (see [11, Theorem 6, Theorem 8, Theorem 9, pp. 620-623]), the usual *continuous induction* on time is used to extend the solution into the maximal existence time-interval $(0, T)$ and thus, the regular solution u on $\mathbb{R}^m_T$ is obtained from passing to the limit of some sequence of regular solutions. In the following we shall make some integral estimates, holding true uniformly on the sequence of regular solutions.

The proof of Theorem 3 is based on the local regularity criterion.

Theorem 12 (Small $m-$energy regularity) *Let B_0 be any positive number satisfying $4(m-1)/m < B_0 < m$. Suppose that u be a regular solution of (1.8) on $\mathbb{R}^m_T$ for a positive T above. Then, there exist positive numbers $\epsilon_0 < 1$ and $R_0 < 1$ depending only on m and $\mathcal{N}$ such that, if, for some positive $R \leq R_0$,*

$$\liminf_{t \nearrow T} \Psi(2R; t, x_0) \;\leq\; \frac{\epsilon_0}{4}, \tag{3.1}$$

then it holds true for some positive $\bar{R} < R$ that u and its gradient Du are uniformly Hölder continuous in $Q((\bar{R}/8)^{B_0}, \bar{R}/8)(T, 0)$, with a Hölder exponent and constant depending only on $\bar{R}$, B_0, m and $\mathcal{N}$.

Theorem 12 implies the condition for a regular weak solution of (1.8) to be regular up to T. The proof is based on Theorem 7. Now we shall study how to derive the criterion (2.8) from (3.1), by use of some local integral estimates of the second derivatives and gradients in Lemmata 10 and 11. Before going to the detail of proof, we verify the finiteness of singularity from Theorem 12

Proposition 13 *Suppose that u be a regular solution of (1.8) on $\mathbb{R}^m_T$ for a positive T above. If u is not regular around (T, x_0) for some $x_0 \in \mathbb{R}^m$, then there holds, for any positive $R \leq R_0$,*

$$\liminf_{t \nearrow T} \Psi(2R; t, x_0) > \frac{\epsilon_0}{4}. \tag{3.2}$$

Proposition 14 (Finite singularity) *It hold true that the singular set $\mathcal{S} \subset \mathbb{R}^m$ at T is contained in the following set:*

$$\mathcal{S} \subset \Sigma := \left\{ x_0 \in \mathbb{R}^m \;\middle|\; \liminf_{t \nearrow T} \Psi(2R; t, x_0) > \epsilon_0/4 \text{ for any positive } R \leq R_0 \right\} \tag{3.3}$$

and thus, the singular set $\mathcal{S}$ at T consists of at most finitely many points.

Proof of Proposition 14. It follows from Proposition 13 that the singular set $\mathcal{S}$ at T of a solution u is contained in the set Σ as in (3.3). We shall estimate the size of Σ given in (3.3) in terms of Hausdorff measure. For any positive number K, let $B(K) := B(K, 0)$ be a ball and $B(K) \cap \Sigma$ be taken. By Vitalli's covering lemma, for any positive $R \leq R_0$, there exist finitely many balls $\{B(2R, x_i)\}$ with $\{x_i\} \subset B(K) \cap \Sigma$, $i = 1, \ldots, L$, such that the balls $B(2R, x_i)$, $i = 1, \ldots, L$, are disjoint each other and the family of balls $\{B(10R, x_i)\}$,

$i = 1, \dots, L$, covers $B(K) \cap \Sigma$, that is $B(K) \cap \Sigma \subset \bigcup_{i=1}^{L} B(10R, x_i)$. By the definition of Σ in (3.3),

$$\frac{\epsilon_0}{4} < \liminf_{t \nearrow T} \Psi(2R; t, x_i) \quad \text{for all positive } R \le R_0 \text{ and any } i = 1, \dots, L. \tag{3.4}$$

Hence, summing up (3.4) over $i = 1, \dots, L$ yields

$$\begin{aligned} L\frac{\epsilon_0}{4} &< \sum_{i=1}^{L} \left(\liminf_{t \nearrow T} \Psi(2R; t, x_i) \right) \\ &\le \liminf_{t \nearrow T} \left(\sum_{i=1}^{L} \Psi(2R; t, x_i) \right) \\ &= \liminf_{t \nearrow T} \int_{\bigcup_{i=1}^{L} B(2R; t, x_i)} |Du(t)|^m \, dx \\ &\le E(u_0), \end{aligned} \tag{3.5}$$

where the energy inequality (2.1) in Lemma 4 is used in the last inequality. The number L of balls of the covering depends on the radius K of $B(K)$ firstly fixed, and bounded uniformly on K by (3.5). Taking the supremum on $K > 0$ in (3.5) yields

$$\sup_{K>0} L(K) \le E(u_0).$$

Therefore, the finiteness of Σ is verified by (3.5). □

Proof of Theorem 3. Now let us prove Theorem 3. Here it is only verified that the solution is regular outside Σ.

Lemma 15 *Suppose that, for some positive $R \le R_0$,*

$$\liminf_{t \nearrow T} \Psi(2R; t, x_0) \le \frac{\epsilon_0}{4}.$$

Then, there exists a sequence $\{t_k\}$ of times $t_k \nearrow T$ as $k \to \infty$ such that

$$\Psi(2R; t_k, x_0) \le \frac{\epsilon_0}{2}. \tag{3.6}$$

Lemma 16 *Let B_0 be any number satisfying $4(m-1)/m < B_0 < m$. Let $R \le R_0$ be a positive number and $\{t_k\}$, $t_k \nearrow T$, be a sequence of times, obtained in Lemma 15. Choose $t_k > \max\left\{T/2,\, T - R^{B_0}\right\}$ and, let $t_0 := t_k$ and $\bar{R} := (T - t_0)^{1/B_0}$. Then there exists a positive number $\tau \le \min\{t_0/2, R\}$ such that, for any t, $t_0 - \tau \le t \le t_0$,*

$$\Psi(2R; t, x_0) \le \epsilon_0. \tag{3.7}$$

Proof of Lemma 16. Let B_0 be a fixed number satisfying $4(m-1)/m < B_0 < m$. We can choose t_k, $T > t_k > \max\left\{T/2,\, T - R^{B_0}\right\}$. Put $t_0 = t_k$ and thus, from (3.6) in Lemma 15 it follows that

$$\Psi(2R; t_0, x_0) \le \frac{\epsilon_0}{2}. \tag{3.8}$$

Since Du is locally continuous in $(0, T) \times \mathbb{R}^m$, it holds for a small positive $\tau < t_0/2$ that, for any $t \in [t_0 - \tau, t_0]$,

$$\Psi(2R; t, x_0) \le \Psi(2R; t_0, x_0) + \frac{\epsilon_0}{2}. \tag{3.9}$$

The desired claim (3.7) follows from (3.8) and (3.9). □

Lemma 17 *Let t_0 be as in Lemma 16. Suppose that, for any t, $t_0 - \tau \le t \le t_0$,*

$$\Psi(2R; t, x_0) \le \epsilon_0. \tag{3.10}$$

Then there holds that

$$\sup_{t \in \mathcal{I}/2} \int_{\mathcal{B}/2} |Du|^{m+2}\, dx + \int_{\mathcal{I}/2 \times \mathcal{B}/2} |D|Du|^m|^2\, dz$$
$$\le C\,(E(u_0) + R^m)\left(1 + \frac{1}{R^m}\right) + CE(u_0)\frac{\tau}{R^m}\left(1 + \frac{1}{R^2}\right), \tag{3.11}$$

where the positive constant C depends only on m and $\mathcal{N}$, and the notations are used: $\mathcal{I}/2 := (t_0 - \tau/4, t_0)$ and $\mathcal{B}/2 := B(R, 0)$.

Proof of Lemma 17. The solution u is regular in $(0, T) \times \mathbb{R}^m$. Under the hypothesis (3.10), we can apply the second derivative estimates in Lemma 10 to have (2.18). Then, by use of (2.18), moreover, we obtain the L^q-estimate as in (2.24) in Lemma 11. Therefore, the desired estimate (3.11) follows from the above argument. □

Proof of Theorem 12. Now the proof of Theorem 12 will be given as follows. We shall show that the condition (2.8) in Theorem 7 holds true for the weak solution u of (1.8), which is locally regular in $(0, T) \times \mathbb{R}^m$.

Let B_0 and γ_0 be the positive numbers given in (2.5) and (2.7) in Theorem 7. Let R be the positive number as in (3.1), that is just (3.10) in Lemma 17, and thus, the estimation (3.11) is available with a time-interval $\mathcal{I}/2 = (t_0 - \tau/4, t_0)$. Using the boundedness of the first integral in (3.11) and the Hölder inequality, we have, for any small positive number $r \le R$ and any time $t \in \mathcal{I}/2$,

$$r^{\gamma_0 - m} \int_{B(r,0)} |Du(t)|^m\, dx \le \left(\int_{B(R,0)} |Du(t)|^{m+2}\, dx \right)^{\frac{m}{m+2}} |B(r)|^{\frac{2}{m+2}}\, r^{\gamma_0 - m}$$
$$= C_0\, r^{\gamma_0 - \frac{m^2}{m+2}}\, E(u_0), \tag{3.12}$$

where C_0 is given by the number as in the right hand side of (3.11),

$$\left\{ C\,(E(u_0) + R^m)\left(\frac{1}{\tau} + \frac{1}{R^m}\right) + CE(u_0)\frac{1}{R^m}\left(1 + \frac{1}{R^2}\right) \right\}^{\frac{m}{m+2}} \tag{3.13}$$

with the positive constant C depending only on m and $\mathcal{N}$. Here we can choose B_0 as

$$m > B_0 > \frac{m^2 + 4}{m + 2} \iff m > \gamma_0 = \frac{m(B_0 - 2)}{m - 2} > \frac{m^2}{m + 2}, \tag{3.14}$$

because

$$m > \frac{m^2 + 4}{m + 2} \iff m > 2$$

and thus, from (3.12) with the choice B_0 as in (3.14), it follows that, for any time $t \in \mathcal{I}/2$,

$$\limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{B(r,0)} |Du(t)|^m\, dx = 0, \tag{3.15}$$

which yields the condition (2.8) on any time-section $t \in \mathcal{I}/2$ in Theorem 7. For any positive $\epsilon < \tau/8$, we use (3.15) at a time-section $t = t_0 - \epsilon$ to obtain the gradient boundedness from (2.9): Let $\bar{R} := (T - t_0)^{1/B_0}$ be as in (3.7) in Lemma 16. For any positive $\epsilon < \tau/8$, there holds

$$\sup_{\left(T-\epsilon-(\bar{R}/4)^{B_0},\, T-\epsilon\right)\times B(\bar{R}/4,\, 0)} |Du| \le C\,\bar{R}^{-a_0}, \qquad \frac{B_0-2}{m-2} < a_0 \le 1, \tag{3.16}$$

where the positive constant C depends only on γ_0, B_0, a_0, m and $\mathcal{N}$.

For any positive $\epsilon < \tau/8$, we use a changing variable $s = t + \epsilon$, $y = x$ and put $w_\epsilon(s, y) := u(s - \epsilon, x)$. Then, there holds, letting $w = w_\epsilon$,

$$\begin{aligned} &\partial_s w - \Delta_p w = |Dw|^{p-2} A(w)(Dw, Dw), \quad w \in \mathcal{N} \\ &\text{in } Q := \left(T - (\bar{R}/4)^{B_0},\, T\right) \times B(\bar{R}/4,\, 0) \end{aligned} \tag{3.17}$$

and, with the same a_0 as in (3.16),

$$\sup_Q |Dw| \le C\,\bar{R}^{-a_0}. \tag{3.18}$$

Hence, by (3.17) and (3.18), the solution w is a weak solution of the evolutionary $p-$Laplace equation

$$\partial_s w - \Delta_p w = f; \qquad f \text{ is uniformly bounded in } Q \tag{3.19}$$

and thus, w and Dw are uniformly continuous in $Q := \left(T - (\bar{R}/8)^{B_0},\, T\right) \times B(\bar{R}/8,\, 0)$ (see [5, Theorem 1.1', Remark 3.1, p. 256] [12, Theorem 1, p. 390], also [4, 13]). Therefore, by the Arzella-Ascoli theorem, we can take a subsequence of w_ϵ, denoted by the same notation, and the limit function w_∞ such that, as $\epsilon \searrow 0$,

$$w_\epsilon \longrightarrow w_\infty, \quad Dw_\epsilon \longrightarrow Dw_\infty \qquad \text{uniformly in } Q \tag{3.20}$$

and the limit function w_∞ and its gradient Dw_∞ are uniformly continuous in Q. From (3.20) for $u(s - \epsilon, y) = w_\epsilon(s, y)$ in Q, it follows that, for any $s \in \left[T - (\bar{R}/8)^{B_0},\, T\right]$, $u(s - \epsilon) = w_\epsilon(s)$ converges to $w_\infty(s)$ uniformly on $B(\bar{R}/8,\, 0)$, as $\epsilon \searrow 0$. By the energy inequality (2.1) in Lemma 4 and $u \in \mathcal{N}$ in $\mathbb{R}^m_T$, it holds by the compactness of the Sobolev embedding $W^{1,p}(B(2R,\, 0)) \hookrightarrow L^{mp/(m-p)}(B(2R,\, 0))$ that, for any q, $1 \le q < mp/(m-p)$, $u(t)$ converges to $u(T)$ strongly in $L^q(B(2R,\, 0))$ as $t \nearrow T$ and thus, almost everywhere in $B(2R,\, 0)$. Therefore, from the facts above it is verified that $u(T) = w_\infty(T)$ almost everywhere in $B(\bar{R}/8,\, 0)$ and thus, the solution u is continuously on time-space extended up to the time T in $B(\bar{R}/8,\, 0)$. The proof is complete. □

Funding : The work by M. Masashi was partially supported by the Grant-in-Aid for Scientific Research (C) No.18K03375 at Japan Society for the Promotion of Science.

References

[1] K.-C. Chang, Heat flow and boundary value problem for harmonic maps, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **6**, no. 5 (1989), 363-395.

[2] K.-C. Chang, W. -Y. Ding, R. Ye, Finite-time blow up of the heat flow of harmonic maps from surfaces, *J. Differential Geom.* **36**, no. 2 (1992), 507-515.

[3] C.-N. Chen, L. F. Cheung, Y. S. Choi, C. K. Law, On the blow-up of heat flow for conformal 3−harmonic maps, *Trans. AMS* **354**, no. 12, (2002) 5087-5110.

[4] H. J. Choe, Hölder continuity of solutions of certain degenerate parabolic systems, *Nonlinear Anal.* **8**(3), (1992) 235-243.

[5] E. DiBenedetto, Degenerate Parabolic Equations, Universitext, New York, NY: Springer-Verlag. xv, 387 (1993).

[6] F. Duzaar, M. Fuchs, On removable singularities of $p-$harmonic maps, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **7**, no. 5, (1990) 385-405.

[7] F. Duzaar, G. Mingione, The $p-$harmonic approximation and the regularity of $p-$harmonic maps, *Calc. Var. Partial Differential Equations*, **20**, (2004) 235-256.

[8] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Am. J. Math.* **86** (1964), 109-169.

[9] A. Fardoun, R. Regbaoui, Heat flow for $p-$harmonic maps between compact Riemannian manifolds, *Indiana Univ. Math. J.* **51**, no. 6, (2002), 1305-1320.

[10] R. Hamilton, Harmonic maps of manifolds with boundary$m-$harmonic flow, Lect. Notes in Math. **471**, Springer-Verlag, Berlin-New York, 1975. 593-631.

[11] N. Hungerbühler, $m-$harmonic flow, *Ann. Scuola Norm. Sup. Pisa CI. Sci.* **24** (1997), 593-631.

[12] C. Karim, M. Misawa, Gradient Hölder regularity for nonlinear parabolic systems of $p-$Laplacian type, *Differential Integral Equations* **29** (2016), no. 3-4, 201-228.

[13] M. Misawa, Local Hölder regularity of gradients for evolutional $p-$Laplacian systems, *Ann. Mat. Pura Appl. (IV)* **181**, (2002) 389-405.

[14] M. Misawa, Existence and regularity results for the gradient flow for $p-$harmonic maps, *Electron J. Differ. Equ.* **36** (1998), 1-17.

[15] M. Misawa, Regularity for the evolution of $p-$harmonic maps, *J. Differential Equations* **264**, (2018) 1716-1749.

[16] M. Misawa, Local regularity and compactness for the $p-$harmonic map heat flows, *Adv. Calc. Var.* (2017), DOI: 10.1515/acv-2016-0064.

[17] M. Misawa, Global existence and partial regularity for the $p-$harmonic flow, *Calc. Var.* **58:54** (2019), DOI: 10.1515/acv-2016-0064.

[18] R.. Schoen, Analytic aspects of the harmonic map problem, Seminar on Nonlinear Partial Differential Equations (S. S. Chern editor), *MSRI Publications* **2**, (1984), 321-358, Springer-Verlag, New-York.

[19] M. Struwe, On the evolution of harmonic maps of Riemannian surfaces, *Comment. Math. Helv.* **60**, (1985), no.4, 558-581.

[20] M. Struwe, On the evolution of harmonic maps in higher dimensions, *J. Differential Geometry* **28**, (1988) 485-502.