Real interpolation of B_σ spaces

Yoshihiro Sawano

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Abstract

Recently, B_σ spaces are defined by some authors in various context. The goal of this note is to provide an equivalent expression based on the work by Feichtinger as well as the one by Alvarez, Guzmán-Partida and Lakey. The description of the real interpolation of B_σ spaces is obtained as an application, which will supplement the results by Nakai and Sobukawa in their 2016 paper.

1 Introduction

Let E be a Banach lattice with the Fatou property. We recall the definition of the E-based B_σ space $B_\sigma(E)(\mathbb{R}^n)$.

Definition 1.1. Let E be a ball quasi-Banach function space. Then, for $\sigma \in [0,\infty)$, we define the *E-based nonhomogeneous B_σ space* $B_\sigma(E)(\mathbb{R}^n)$ and the *E-based homogeneous B_σ space* $\dot{B}_\sigma(E)(\mathbb{R}^n)$ as the sets of $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{B_\sigma(E)} \equiv \sup\limits_{r\ge 1} r^{-\sigma}\|f\chi_{[-r,r]^n}\|_E < \infty$ and $\|f\|_{\dot{B}_\sigma(E)} \equiv \sup\limits_{r>0} r^{-\sigma}\|f\chi_{[-r,r]^n}\|_E < \infty$, respectively.

We are interested in the case where $\sigma > 0$ since $B_0(E)(\mathbb{R}^n)$ and $\dot{B}_0(E)(\mathbb{R}^n)$ coincides with E thanks to the Fatou property of E.

We present our main result in this note based on the work by Feichtinger [3, Theorem 6] as well as the one by Alvarez, Guzmán-Partida and Lakey [1, (2.3)].

Theorem 1.2. *Let $\sigma > 0$ and let E be a Banach lattice with the Fatou property.*

1. *We have equivalence of norms:*

$$\|f\|_{B_\sigma(E)} \sim \|f\chi_{Q(1)}\|_E + \sup_{j\in\mathbb{N}} 2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n\setminus[-2^{j-1},2^{j-1}]^n}\|_E$$

for any measurable function f.

2. *We have equivalence of norms:*

$$\|f\|_{\dot{B}_\sigma(E)} \sim \sup_{j\in\mathbb{Z}} 2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n\setminus[-2^{j-1},2^{j-1}]^n}\|_E$$

for any measurable function f.

Thus, it makes sense to define E-based Herz spaces $K^\sigma_q(\mathbb{R}^n)$ (nonhomogeneous) and $\dot{K}^\sigma_q(\mathbb{R}^n)$ (homogeneous) by the norms

$$\|f\chi_{Q(1)}\|_E + \left\{\sum_{j=1}^{\infty}(2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n\setminus[-2^{j-1},2^{j-1}]^n}\|_E)^q\right\}^{\frac{1}{q}}$$

and

$$\left\{\sum_{j=-\infty}^{\infty}(2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n\setminus[-2^{j-1},2^{j-1}]^n}\|_E)^q\right\}^{\frac{1}{q}},$$

respectively.

As an application of Theorem 1.2, we prove the following real interpolation result:

Theorem 1.3. *Let $\sigma_1 > \sigma_0 > 0$, and let E adn E be a compatible couple of Banach lattices over $\mathbb{R}^n$ with the Fatou property.*

1. *With equivalence of norms, $(B_{\sigma_0}(E)(\mathbb{R}^n), B_{\sigma_1}(E)(\mathbb{R}^n))_{\theta,q} = K^{(1-\theta)\sigma_0+\theta\sigma_1}_{Eq}(\mathbb{R}^n)$.*
2. *With equivalence of norms, $(\dot{B}_{\sigma_0}(E)(\mathbb{R}^n), \dot{B}_{\sigma_1}(E)(\mathbb{R}^n))_{\theta,q} = \dot{K}^{(1-\theta)\sigma_0+\theta\sigma_1}_{Eq}(\mathbb{R}^n)$.*

Theorem 1.3 supplements [4, Theorem 3.1], where Nakai and Sobukawa worked in more general setting. In particular,

$$(\dot{B}_{\sigma_0}(E)(\mathbb{R}^n), \dot{B}_{\sigma_1}(E)(\mathbb{R}^n))_{\theta,q} = \dot{K}^{(1-\theta)\sigma_0+\theta\sigma_1}_{Eq}(\mathbb{R}^n),$$

the result for homogeneous spaces, can be found in [4].

2 Preliminary

Here we collect some preliminary facts, which are oritented the interpolation of vector-valued function spaces. First of all for $k > 0$ and a Banach space E, define the Banach space kE by

$$kE = E, \quad \|f\|_{kE} = k\|f\|_E \quad (f \in kE).$$

First we calcuate the K-functional of $(2^{a_0}E, 2^{a_1}E)$.

Lemma 2.1. *Let E be a Banach space. Then for any $f \in E$, $a_0, a_1 \in \mathbb{R}$ and $t > 0$,*

$$K(f,t;2^{a_0}E,2^{a_1}E) = \min(2^{a_0}, 2^{a_1}t)\|f\|_E.$$

Proof. By the decompositions $f = f + 0 = 0 + f$, we have $K(f,t;2^{a_0}E,2^{a_1}E) \leq \min(2^{a_0}, 2^{a_1}t)\|f\|_E$.

Let $\Phi : E \to \mathbb{C}$ be a bounded linear functional such that $\Phi(f) = \|f\|_E$ with norm 1. Choose any $g \in E$. Then we have

$$2^{a_0}\|g\|_E + 2^{a_1}t\|f-g\|_E \geq 2^{a_0}|\Phi(g)| + 2^{a_1}t|\Phi(g) - \|f\|_E| \geq \min(2^{a_0}, 2^{a_1}t)\|f\|_E,$$

since Φ has the norm 1. By taking the real part, we obtain

$$2^{a_0}\|g\|_E + 2^{a_1}t\|f-g\|_E \ge 2^{a_0}|\mathrm{Re}(\Phi(g))| + 2^{a_1}t|\mathrm{Re}(\Phi(g)) - \|f\|_E|.$$

Since the right-hand side is a linear function of $\mathrm{Re}(\Phi(g))$,

$$2^{a_0}\|g\|_E + 2^{a_1}t\|f-g\|_E \ge \min(2^{a_0}, 2^{a_1}t)\|f\|_E.$$

By taking the infimum over all $g \in 2^{a_0}E = E$, we obtain $K(f,t;2^{a_0}E, 2^{a_1}E) \ge \min(2^{a_0}, 2^{a_1}t)\|f\|_E$. □

The proof of the next lemma is a slight modification of the proof of [2, Theorem 5.6.1]. Here for the sake of convenience, we provide a detailed proof.

Lemma 2.2. *Let $\{E_j\}_{j=-\infty}^{\infty}$ be a sequence of Banach spaces. Then for $-\infty < s_0 < s_1 < \infty$,*

$$(\ell^\infty(\{2^{s_0 j}E_j\}_{j=-\infty}^{\infty}), \ell^\infty(\{2^{s_1 j}E_j\}_{j=-\infty}^{\infty}))_{\theta,q} = \ell^q(\{2^{((1-\theta)s_0+\theta s_1)j}E_j\}_{j=-\infty}^{\infty})$$

Proof. Let $\{f_j\}_{j=-\infty}^{\infty} \in (\ell^\infty(\{2^{s_0 j}E_j\}_{j=-\infty}^{\infty}), \ell^\infty(\{2^{s_1 j}E_j\}_{j=-\infty}^{\infty}))_{\theta,q}$. Then we have

$$\begin{aligned}
&K(\{f_j\}_{j=-\infty}^{\infty}, 2^k; \ell^\infty(\{2^{s_0 j}E_j\}_{j=-\infty}^{\infty}), \ell^\infty(\{2^{s_1 j}E_j\}_{j=-\infty}^{\infty}))\\
&= \inf_{\{g_j\}_{j=-\infty}^{\infty} \in \prod_{j=-\infty}^{\infty} E_j} \sup_{j\in\mathbb{Z}} (2^{s_0 j}\|g_j\|_{E_j} + 2^{s_1 j+k}\|f_j - g_j\|_{E_j})\\
&= \sup_{j\in\mathbb{Z}} \inf_{\{g_j\}_{j=-\infty}^{\infty} \in \prod_{j=-\infty}^{\infty} E_j} (2^{s_0 j}\|g_j\|_{E_j} + 2^{s_1 j+k}\|f_j - g_j\|_{E_j})\\
&= \sup_{j\in\mathbb{Z}} \min(2^{s_0 j}, 2^{s_1 j+k})\|f_j\|_{E_j}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&2^{-k\theta}\sup_{j\in\mathbb{Z}} \min(2^{s_0 j}, 2^{s_1 j+k})\|f_j\|_{E_j}\\
&= \sup_{j\in\mathbb{Z}} \min(2^{-((s_1-s_0)j+k)\theta}, 2^{((s_1-s_0)j+k)(1-\theta)})2^{(1-\theta)s_0 j+\theta s_1 j}\|f_j\|_{E_j}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&2^{(1-\theta)s_0 k+\theta s_1 k}\|f_k\|_{E_k}\\
&\le 2^{-k\theta}\sup_{j\in\mathbb{Z}} \min(2^{s_0 j}, 2^{s_1 j+k})\|f_j\|_{E_j}\\
&\le \left\{\sum_{j=-\infty}^{\infty} \left(\min(2^{-((s_1-s_0)j+k)\theta}, 2^{((s_1-s_0)j+k)(1-\theta)})2^{(1-\theta)s_0 j+\theta s_1 j}\|f_j\|_{E_j}\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

If we take the ℓ^q-norm over $k \in \mathbb{Z}$, then we obtain

$$\{f_j\}_{j=-\infty}^{\infty} \in \ell^q(\{2^{((1-\theta)s_0+\theta s_1)j}E_j\}_{j=-\infty}^{\infty}).$$

We can reverse the argument to have

$$(\ell^\infty(\{2^{s_0 j}E_j\}_{j=-\infty}^{\infty}), \ell^\infty(\{2^{s_1 j}E_j\}_{j=-\infty}^{\infty}))_{\theta,q} \supset \ell^q(\{2^{((1-\theta)s_0+\theta s_1)j}E_j\}_{j=-\infty}^{\infty}).$$

Thus, the proof is complete. □

3 Proof of Theorem 1.2

We concentrate on the nonhomogeneous space $B_\sigma(E)(\mathbb{R}^n)$. One inequality $\gtrsim$ is clear. So, we have only to prove $\lesssim$. We remark that

$$\|f\|_{B_\sigma(E)} \sim \sup_{j\in\mathbb{N}} 2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n}\|_E.$$

With this remark in mind, we fix j. We estimate

$$\begin{aligned}
&2^{-j\sigma}\|f\chi_{[-2^j,2^j]^n}\|_E \\
&\le 2^{-j\sigma}\|f\chi_{[-1,1]^n}\|_E + 2^{-j\sigma}\sum_{k=1}^{j}\|f\chi_{[-2^k,2^k]^n\setminus[-2^{k-1},2^{k-1}]^n}\|_E \\
&\le 2^{-j\sigma}\|f\chi_{[-1,1]^n}\|_E + 2^{-j\sigma}\sum_{k=1}^{j}2^{k\sigma}\sup_{l\in\mathbb{N}}2^{-l\sigma}\|f\chi_{[-2^l,2^l]^n\setminus[-2^{l-1},2^{l-1}]^n}\|_E \\
&\lesssim \|f\chi_{[-1,1]^n}\|_E + \sup_{l\in\mathbb{N}}2^{-l\sigma}\|f\chi_{[-2^l,2^l]^n\setminus[-2^{l-1},2^{l-1}]^n}\|_E,
\end{aligned}$$

as required.

4 Proof of Theorem 1.3

This time we concentrate on the homogeneous space $B_\sigma(E)(\mathbb{R}^n)$; the non-homogeneous space can be handled similarly but the proof is less simplified. Set

$$E_j = \{f \in E \,:\, \mathrm{supp}(f) \subset [-2^j,2^j]^n \setminus [-2^{j-1},2^{j-1}]^n\} \quad (j \in \mathbb{Z}).$$

We also set

$$\ell^\infty(\{2^{-j\sigma}E\}_{j\in\mathbb{Z}}) = \left\{\{f_j\}_{j=-\infty}^{\infty} \,:\, f_j \in E \quad (j \in \mathbb{Z}), \quad \sup_{j\in\mathbb{Z}} 2^{-j\sigma}\|f_j\|_E\right\}$$

Thanks to Theorem 1.2, the mapping

$$f \in \dot{B}_\sigma(E)(\mathbb{R}^n) \mapsto \{f\chi_{[-2^j,2^j]^n\setminus[-2^{j-1},2^{j-1}]^n}\}_{j=-\infty}^{\infty} \in \ell^\infty(\{2^{-j\sigma}E_j\}_{j\in\mathbb{Z}})$$

is an isomorphism. We have

$$(\ell^\infty(\{2^{-j\sigma_0}E_j\}_{j\in\mathbb{Z}}), \ell^\infty(\{2^{-j\sigma_1}E_j\}_{j\in\mathbb{Z}}))_{\theta,q} = \ell^\infty(\{2^{-j((1-\theta)\sigma_0-\theta\sigma_1)}E_j\}_{j\in\mathbb{Z}})$$

thanks to Lemma 2.2. Thus, by Theorem 1.2 once again, we obtain the desired result.

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