ON LOCAL GEOMETRIC LANGLANDS
IN POSITIVE CHARACTERISTIC

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Contents

1. Introduction

1.1. Bezrukavnikov–Braverman introduced a positive characteristic version of the de Rham global geometric Langlands correspondence, with subsequent contributions by Chen–Zhu, Groechenig, and Travkin [BB07], [CZ15], [BTCZ16], [Gro16], [Tra16], [CZ17].

In the present note we would like to record some possible ingredients of the local counterpart, i.e., the local geometric Langlands correspondence in positive characteristic.

1.2. Let us indicate a basic motivation for us. In characteristic zero, many aspects of the representation theory of affine Lie algebras and related vertex algebras are controlled by local geometric Langlands, as first discovered by Feigin–Frenkel, Beilinson–Drinfeld, and Frenkel–Gaitsgory [FF91], [BD], [FG06b].

One would like a similar picture in positive characteristic. Our guiding principal here is that the added complexity in the representation theory matches the added complexity in the geometry of moduli spaces of connections on the spectral side. In particular, an anticipated central role of the double affine Hecke category throughout this representation theory e.g., in its manifestation as Iwahori equivariant crystalline D-modules on the affine flag variety, fits well into the correspondence.

1.3. We should stress from the outset that, given the naturality of the ideas discussed below, we expect many mathematicians have arrived at similar or overlapping pictures, although we are unaware of written material. In particular, we know this to be the case for Bezrukavnikov, Gaitsgory, and Raskin. In addition, the closely related 'very classical' local Langlands correspondence has been studied intensively in characteristic zero by Arinkin and Fedorov.

We apologize to anyone whose ideas or work has been wrongly omitted or unattributed. We also hope we have not done the correspondence itself too much of an injustice with whatever errors and misunderstandings we have introduced in our attempted transcription.

1.4. The organization of this note is as follows. In Section 2, we mention some basic expectations regarding the correspondence, and in Section 3 record some concrete consequences for categories of intertwining operators. Finally, in Section 4 we provide a similar discussion for the quantum Langlands correspondence.

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2. Statement of local Langlands

2.1. In this section, we give a rough formulation of the conjecture, and list some expected properties.

A reader primarily interested in representation theoretic applications may wish to skim Section 2.2 for mostly standard notation and Section 2.8 for the definition of the space $\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}}$, before proceeding directly to Sections 3 and 4.

2.2. **Setup and notation.**

2.2.1. Fix an algebraically closed field k of characteristic $p > 0$. In what follows, by a variety, scheme, or generally a prestack, we mean one over k.

Let $\mathcal{D}$ be a formal one dimensional disc, with closed point x, and $\mathcal{D}^\times$ the corresponding punctured disc. Let us denote by O and F the respective topological rings of functions, so that $\mathcal{D}$ is the formal spectrum of O, and F is the fraction field of O. Explicitly, if we choose a coordinate t on the disc, we have

$$O \simeq k[[t]] \quad \text{and} \quad F \simeq k((t)).$$

We denote the corresponding spaces of 1-forms by ω_O and ω_F, so that with respect the coordinate t we have

$$\omega_O \simeq k[[t]]dt \quad \text{and} \quad \omega_F \simeq k((t))dt.$$

For an affine scheme X, we write X_O and X_F for the corresponding arc and loop spaces.

2.3. Fix in addition a pinned reductive group (G, B, T, ψ). That is, G is a connected reductive group, $B \subset G$ is a Borel subgroup with unipotent radical N, $T \hookrightarrow B$ is a splitting of the abstract Cartan $T \simeq B/N$ into B, and ψ is a generic character

$$\psi : N \to \mathbb{G}_a,$$

i.e., in the open orbit under the action of T on characters of N. We denote the corresponding Langlands dual group by $(\check{G}, \check{B}, \check{T}, \check{\psi})$, and corresponding pinned Lie algebras by $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \psi)$ and $(\check{\mathfrak{g}}, \check{\mathfrak{b}}, \check{\mathfrak{t}}, \check{\psi})$, respectively.

2.4. **Naive formulation of the conjecture.**

2.4.1. We will now give a naive formulation of the conjecture. Since we see some, but we suspect not all, of the necessary technical corrections, we do not attempt to record a non-naive statement here.

In characteristic zero, the analogous conjecture is due to Frenkel–Gaitsgory [FG06b], [Fre07], with subsequent contributions by Arinkin, Beraldo, Raskin, Yang, and others, as recorded in [Gai16] and the notes [ABC$^+$18]. In what follows, any statement which also makes sense in characteristic zero has a direct antecedent there, and we refer once and for all to *loc. cit.* for discussion and references. The non-expert reader might also consult the survey [Dhi22].

2.4.2. On the automorphic side, we would like to consider the 2-category of categories equipped with a strong action of G_F. Let us indicate the meaning of a naive version of this 2-category.[1]

On any scheme X of finite type, one has its category D-mod(X) of crystalline D-modules. If X is smooth and affine, this is the unbounded derived category of modules for its algebra of differential operators without divided powers. For general X, a convenient model is PD-crystals, i.e., quasicoherent sheaves which know how to grow along divided power nil-thickenings.[2]

The definition extends naturally to placid ind-schemes such as X_O or X_F. As in the work of Raskin in characteristic zero [Ras15], one now has dual categories D-mod(X) and D-mod$^!(X)$ which come equipped with functoriality for $*$-pushforwards and !-pullbacks, respectively.

In particular, the category D-mod(G_F) is monoidal under convolution, i.e., pushforward along multiplication, and the naive version of the desired 2-category is

$$\text{D-mod}(G_F)\text{-mod}.$$

In down to earth terms, objects are 1-categories $\mathcal{C}$ equipped with a convolution action

$$\text{D-mod}(G_F) \otimes \mathcal{C} \to \mathcal{C},$$

and morphisms are D-mod(G_F)-equivariant functors.

Example 2.1. If G_F acts on a placid ind-scheme X, then one has a convolution action of D-mod(G_F) on D-mod(X).

2.4.3. On the spectral side, we will consider sheaves of categories on the space of local systems. Let us briefly recall the meaning of a naive version of this 2-category.

If $X = \operatorname{Spec} A$ is an affine scheme, then a quasicoherent sheaf of categories on X is the same as an A-linear 1-category, and for general X one proceeds by gluing.

Consider the space of $\check{G}$-local systems on the punctured disc in the de Rham sense, i.e. the moduli space of $\check{G}$-connections $(\mathcal{P}_{\check{G}}, \nabla)$ on $\mathcal{D}^\times$. Explicitly, this may be presented as the prestack quotient of connections on the trivial bundle modulo gauge equivalence

$$\text{LocSys}_{\check{G}}(\mathcal{D}^\times) := \{d + \check{\mathfrak{g}} \otimes \omega_F\}/\check{G}_F.$$

A naive version of the desired 2-category is then quasicoherent sheaves of categories on this space, i.e.,

$$\text{2-QCoh}(\text{LocSys}_{\check{G}}(\mathcal{D}^\times)).$$

[1]In what follows, by a category or 1-category we mean a cocomplete, presentable k-linear stable ∞-category. Such categories along with the functors between them commuting with colimits and Lurie tensor product form a symmetric monoidal $(\infty, 2)$-category which we denote by StabCat$_{cont}$. By a 2-category we mean a cocomplete StabCat$_{cont}$-linear $(\infty, 2)$-category.

[2]Let us mention in passing that, unlike in characteristic zero, the category of D-modules on a derived scheme now differs from that of its classical truncation, and this plays a basic role, e.g., in base change.

Example 2.2. If $X \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$ is a map of prestacks, then $\mathrm{QCoh}(X)$ is naturally (the global sections of) a sheaf of categories over $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$.

2.4.4. We may now state the naive correspondence.

Conjecture 2.3. There is an equivalence of 2-categories

$$\mathbb{L} : \text{D-mod}(G_F)\text{-mod} \simeq \text{2-QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)).$$

We expect this to literally hold only after appropriately renormalizing both sides of the conjecture, and to depend on a choice of a square root of the canonical bundle ω_O, cf. Section 2.9 below.

For experts, we should note that we expect the automorphic side needs renormalization already for finite dimensional groups, unlike in characteristic zero. However, some of the necessary ideas already appear in the work of Raskin on renormalizing *weak* actions of infinite dimensional groups in characteristic zero [Ras20], in particular to ensure good formal properties of the operation of taking invariants.

Let us now state some expected properties of the desired equivalence $\mathbb{L}$.

2.5. **Matching p-centers.**

2.5.1. A basic additional feature of life in positive characteristic is the presence of p-centers. We now record what form this takes in the present setting, essentially following the global case [BB07].

2.5.2. Let us denote by W the finite Weyl group of G and $\check{G}$. We denote by $\mathcal{A}$ the local Hitchin base

$$\mathcal{A} := (\mathfrak{t}^*/\!/W)_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^\times = (\check{\mathfrak{t}}/\!/W)_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^\times,$$

where $\omega_F^\times$ denotes the punctured canonical bundle $\omega_F \backslash 0$.

2.5.3. For a scheme X, we write $\mathrm{Fr} : X \to X^{(1)}$ for the relative Frobenius. With this, the desired common p-center is $\mathcal{A}^{(1)}$. Let us spell this out on both sides of the equivalence.

2.5.4. Let $\mathcal{C}$ be a $\text{D-mod}(G_F)$-module. We may consider its center either as an abstract category or as a G_F-module, i.e., the (equivariant) Hochschild cochains

$$\mathrm{HH}^*(\mathcal{C}) := \mathrm{End}_{\mathrm{End}_{\mathrm{StabCat}_{cont}}(\mathcal{C})}(\mathrm{id}_\mathcal{C}) \quad \text{and} \quad \mathrm{HH}^*_{G_F}(\mathcal{C}) := \mathrm{End}_{\mathrm{End}_{\text{D-mod}(G_F)\text{-mod}}(\mathcal{C})}(\mathrm{id}_\mathcal{C}).$$

Recalling the monoidal unit of $\text{D-mod}(G_F)$ is the delta distribution δ_e at the identity element, we obtain tautological maps

$$\begin{array}{ccc} \mathrm{Hom}_{\text{D-mod}(G_F/^{\mathrm{Ad}}G_F)}(\delta_e, \delta_e) & \xrightarrow{\mathrm{Oblv}} & \mathrm{Hom}_{\text{D-mod}(G_F)}(\delta_e, \delta_e) \\ \downarrow & & \downarrow \\ \mathrm{HH}^*_{G_F}(\mathcal{C}) & \xrightarrow{\mathrm{Oblv}} & \mathrm{HH}^*(\mathcal{C}). \end{array}$$

In particular, we obtain maps in degree zero of the form

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{A}^{(1)}) & \xrightarrow{\mathrm{Oblv}} & \mathrm{Fun}((\mathfrak{g}^* \otimes \omega_F)^{(1)}) \\ \downarrow & & \downarrow \\ \mathrm{HH}^*_{G_F}(\mathcal{C}) & \xrightarrow{\mathrm{Oblv}} & \mathrm{HH}^*(\mathcal{C}). \end{array} \tag{2.4}$$

For this reason, every such $\mathcal{C}$ may be tensored by sheaves of categories over $\mathcal{A}^{(1)}$ to obtain other D-mod(G_F)-modules. I.e., D-mod(G_F)-mod is naturally a module 2-category for 2-QCoh$(\mathcal{A}^{(1)})$.

2.5.5. Now let $\mathcal{S}$ be a sheaf of categories over $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$. Taking the characteristic polynomial of the p-curvature of a connection defines a map

$$\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times) \to \mathcal{A}^{(1)},$$

cf. Section 3.1 of [CZ15]. So, we may regard $\mathcal{S}$ as carrying a commuting action of QCoh$(\mathcal{A}^{(1)})$. I.e., 2-QCoh$(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times))$ is also naturally a module 2-category for 2-QCoh$(\mathcal{A}^{(1)})$.

2.5.6. The desired compatibility then reads as follows.

Conjecture 2.5. The equivalence of 2-categories $\mathbb{L}$ carries a datum of 2-QCoh$(\mathcal{A}^{(1)})$-linearity.

2.5.7. Here is a supplemental remark. One has a forgetful functor

$$\mathrm{Oblv} : \text{D-mod}(G_F)\text{-mod} \to \text{2-QCoh}((\mathfrak{g}^* \otimes \omega_F)^{(1)}/G_F), \tag{2.6}$$

cf. Equation (2.4). As in the global case [BB07], in this way local Langlands in positive characteristic should be closely related to its classical limit, the latter being the subject of ongoing work by Arinkin–Fedorov in characteristic zero. Note, however, the appearance of G_F, rather than its Frobenius twist, in Equation (2.6) above.

More concretely, if one stares at the various categories of intertwining operators in Sections 3 and 4 below, one repeatedly runs into the Azumaya property of differential operators of [BMR08].

2.5.8. Here is a cautionary remark. In the global setting in positive characteristic, and in the local setting of Arinkin–Fedorov in characteristic zero, the conjectures are either known or expected over an open locus in the Hitchin base, but the situation is less clear for general fibers, e.g., the nilpotent cone. The same certainly applies here, and in particular the formulation above is a maximally optimistic formulation of the conjecture.

2.6. **Matching objects.**

2.6.1. We expect certain objects on either side to correspond to one another, essentially exactly as they do in characteristic zero. We mention several of these below.

2.7. **Spherical vectors.**

2.7.1. On the automorphic side, consider the affine Grassmannian Gr_G, i.e., the moduli space of pairs $(\mathcal{P}_G, \tau)$, where $\mathcal{P}$ is a G-bundle on $\mathcal{D}$ and τ a trivialization of $\mathcal{P}_G$ on $\mathcal{D}^\times$.

The natural action of G_F on this by changing the trivialization yields a canonical identification

$$\mathrm{Gr}_G \simeq G_F/G_O,$$

and an action of D-mod(G_F) on D-mod(Gr_G).

2.7.2. On the spectral side, consider the space of $\check{G}$-connections $(\mathcal{P}_{\check{G}}, \nabla)$ on the formal disc $\mathcal{D}$.

Explicitly, this may be presented as the prestack quotient of the affine space of connections on the trivial bundle by gauge equivalence

$$\mathrm{LocSys}_{\check{G}}(\mathcal{D}) := \{d + \check{\mathfrak{g}} \otimes \omega_O\}/\check{G}_O.$$

Via the tautological map

$$\mathrm{LocSys}_{\check{G}}(\mathcal{D}) \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times),$$

we may view $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}))$ as a sheaf of categories over $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$.

We should emphasize that $\mathrm{LocSys}_{\check{G}}(\mathcal{D})$ is no longer $\mathrm{pt}/\check{G}$, as it was in characteristic zero, thanks to the presence of p-curvature.

2.7.3. We then have the following basic compatibility.

Conjecture 2.7. We have $\mathbb{L}(\text{D-mod}(\mathrm{Gr}_G)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}))$.

2.8. **Iwahori vectors.**

2.8.1. On the automorphic side, consider the affine flag variety Fl_G, i.e., the moduli space of triples

$$(\mathcal{P}_G, \tau, \mathcal{F}),$$

where $\mathcal{P}_G$ and τ are as in the definition of Gr_G and $\mathcal{F}$ is a B-reduction of $\mathcal{P}_G$ at x.

Changing the trivialization yields a transitive action of G_F on Fl_G and an identification

$$\mathrm{Fl}_G \simeq G_F/I,$$

where $I \subset G_O$ is the Iwahori subgroup, i.e., the preimage of $B \subset G$ under evaluation at x.

2.8.2. On the spectral side, we will consider a certain moduli space of log-connections, which we denote by $\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}}$.

By definition, $\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}}$ parametrizes triples $(\mathcal{P}_{\check{G}}, \nabla, \check{\mathcal{F}})$, where

(1) $\mathcal{P}_{\check{G}}$ is a $\check{G}$-bundle on the non-punctured disc $\mathcal{D}$,
(2) $\check{\mathcal{F}} : \mathcal{P}_{\check{B},x} \hookrightarrow \mathcal{P}_{\check{G},x}$ is a $\check{B}$-reduction of $\mathcal{P}_{\check{G}}$ at the central point x, and
(3) ∇ is a connection on $\mathcal{P}_{\check{G}}$ on $\mathcal{D}^\times$ with at most a simple pole and whose residue lies in the subspace

$$\mathcal{P}_{\check{B},x} \overset{\check{B}}{\times} \check{\mathfrak{n}} \hookrightarrow \mathcal{P}_{\check{B},x} \overset{\check{B}}{\times} \check{\mathfrak{g}} \overset{\check{\mathcal{F}}}{\simeq} \mathcal{P}_{\check{G},x} \overset{\check{G}}{\times} \check{\mathfrak{g}}.$$

Explicitly, if we write $\check{I}$ for the Iwahori subgroup of $\check{G}_F$, we may present this as the prestack quotient

$$\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}} \simeq \{d + \check{\mathfrak{g}} \otimes \omega_O + \check{\mathfrak{n}} \otimes \omega_{O(x)}\}/\check{I}, \tag{2.8}$$

where $\omega_{O(x)} \subset \omega_F$ denotes the space of one forms with at most a simple pole. Note that taking the residue of the connection yields a morphism

$$\mathrm{Res}^{\check{\mathcal{F}}} : \mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}} \to \check{\mathfrak{n}}/\check{B},$$

which in characteristic zero is an isomorphism.

2.8.3. We propose the following.

Conjecture 2.9. We have $\mathbb{L}(\text{D-mod}(\text{Fl}_G)) \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{\check{G}})$.

2.8.4. Intermediate between the previous two cases, consider any Langlands dual standard parabolic subgroups

$$P \subset G \quad \text{and} \quad \check{P} \subset \check{G}.$$

On the automorphic side, consider the corresponding partial affine flag variety $\text{Fl}_{G,P}$. On the spectral side introduce $\text{LocSys}^{\text{nilp}}_{\check{G},\check{P}}$ similarly, where one places a $\check{P}$-reduction at the central fiber and asks that the residue again lies in the nilpotent radical. Then we propose the following.

Conjecture 2.10. We have $\mathbb{L}(\text{D-mod}(\text{Fl}_{G,P})) \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{\check{G},\check{P}})$.

2.9. Whittaker model.

2.9.1. On the automorphic side, fix a square root $\omega_O^{1/2}$ of the canonical bundle ω_O on $\mathcal{D}$. Write $2\check{\rho}$ for the sum of the positive coroots of G, and consider the homomorphism

$$2\check{\rho} : \mathbb{G}_m \to T \to G. \tag{2.11}$$

We will be interested in the category of Whittaker equivariant D-modules on the associated G_F-torsor

$$\mathcal{P}_{G_F} := G_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^{1/2,\times}.$$

To spell this out, note the automorphism group of this torsor is given by

$$\mathcal{P}_{G_F^{\text{Ad}}} := G_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^{1/2,\times},$$

where $(\mathbb{G}_m)_F$ now acts on G_F by conjugation via (2.11). In particular, this contains the subgroup

$$\mathcal{P}_{N_F^{\text{Ad}}} \simeq N_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^{1/2,\times},$$

which has a canonical additive character induced by our pinning, namely

$$N_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^{1/2,\times} \xrightarrow{\psi_F} \mathbb{A}^1_F \overset{(\mathbb{G}_m)_F}{\times} \omega_F^{1/2,\times} \simeq \omega_F \xrightarrow{\text{Res}} \mathbb{A}^1.$$

We still denote this, along with the pullback of the exponential character D-module on $\mathbb{A}^1$, by ψ.

The desired Whittaker model is then the category of equivariant D-modules

$$\text{D-mod}(\mathcal{P}_{G_F}/\mathcal{P}_{N_F^{\text{Ad}}}, \psi),$$

which carries a $\text{D-mod}(G_F)$-action via left convolution.

We should mention that the question of taking invariants or coinvariants here with respect to $(\mathcal{P}_{N_F^{\text{Ad}}}, \psi)$, and the relation between them, is a basic technical issue, cf. [Ras21].

2.9.2. This Whittaker category should be exchanged with the categorical structure sheaf, namely we propose the following.

Conjecture 2.12. We have $\mathbb{L}(\text{D-mod}(\mathcal{P}_{G_F}/\mathcal{P}_{N_F^{\text{Ad}}}, \psi)) \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(\mathcal{D}^\times))$.

We recall that, as in characteristic zero, this will not entirely characterize $\mathbb{L}$, e.g., due to the appearance of ind-coherent sheaves of categories on the spectral side.

2.9.3. For ease of reading, in what follows we will use a chosen coordinate t to trivialize ω_O, and in particular view it as its own square root in the tautological way. With this, we may identify

$$\text{D-mod}(\mathcal{P}_{G_F}/\mathcal{P}_{N_F^{\mathrm{Ad}}}, \psi) \simeq \text{D-mod}(G_F/N_F, \psi),$$

where ψ is now a generic character of N_F of conductor zero.

2.10. **Kac–Moody representations.**

2.10.1. Consider the affine Lie algebra associated to $\mathfrak{g}$ and minus one half times the Killing form, i.e., the critical central extension

$$0 \to k \cdot \mathbf{1} \to \widehat{\mathfrak{g}}_{\kappa_c} \to \mathfrak{g}_F \to 0.$$

We denote its category of smooth modules on which $\mathbf{1}$ acts by the identity by

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod};$$

see [FG09a] for the appropriate renormalization in characteristic zero.

2.10.2. Let us denote the moduli space of $\check{G}$-opers on the punctured disc by

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times).$$

Passing to the underlying connection of an oper yields a map

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times).$$

2.10.3. Following a beautiful idea of Frenkel–Gaitsgory in characteristic zero [FG06b], we propose the following.

Conjecture 2.13. We have $\mathbb{L}(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}) \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}})$.

2.11. The equivalence should also be compatible with parabolic induction and Jacquet functors, and presumably with Frobenius twists. Together with the above matchings, this should largely pin it down.

3. Intertwining operators

3.1. In this section, we spell out some concrete consequences of the predictions of the previous section, namely by matching categories of homomorphisms between objects on opposite sides of the equivalence. We emphasize that in all the conjectures which follow, we expect one needs to renormalize appropriately, e.g., pass to various categories of ind-coherent D-modules and ind-coherent sheaves, for them to hold.

3.2. An orienting comment we should make is that, compared to characteristic zero, many of the basic objects in what follows are double affine, rather than affine. Informally speaking, one circle comes from the loop variable t, and the second from the Frobenius.

3.3. **Double affine Hecke categories.**

3.3.1. First, by taking endomorphisms of the spherical modules on either side of Conjecture 2.3, cf. Section 2.7, we obtain the following Satake isomorphism, due in characteristic zero to Lusztig, Drinfeld, Ginzburg, Mirković–Vilonen, and Bezrukavnikov–Finkelberg [Lus83], [Gin90], [MV07], [BF08].

Conjecture 3.1. There is an equivalence of monoidal factorization categories

$$\text{D-mod}(G_O\backslash G_F/G_O) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LocSys}_{\check{G}}(\mathcal{D})). \tag{3.2}$$

Let us make some basic remarks about this conjecture.

First, the analog of the abelian Satake category, i.e., the 'un-derived' Hecke operators of the global correspondence, is now played by the monoidal functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D})) \xrightarrow{\Delta_*} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LocSys}_{\check{G}}(\mathcal{D})) \simeq \text{D-mod}(G_O\backslash G_F/G_O),$$

where the first arrow is pushforward along the diagonal.

Second, unlike in characteristic zero, the inclusion of local systems with regular singularities into all local systems is not formally étale, due to tangent directions coming from p-curvature. So, fiber products, here and below, really need to be taken over all local systems.

Finally, we expect that the left-hand side of (3.2), which *a priori* is Azumaya in the appropriate sense over the Frobenius twist of its cotangent bundle, may have its gerbe trivialized. Combined with the map from the right-hand side to the spaces of p-curvatures, this should largely reduce to an earlier conjecture of Bezrukavnikov–Finkelberg–Mirković [BFM05].

3.3.2. Passing to Iwahori level, we similarly obtain the following, which is a version of Bezrukavnikov's equivalence [Bez16] for double affine Hecke categories.

Conjecture 3.3. There is an equivalence of monoidal categories

$$\text{D-mod}(I\backslash G_F/I) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}} \underset{\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}}).$$

Let us now spell out for the convenience of the reader how ones sees this should be a double affine Hecke category, at the level of counting parameters. The strata, i.e., double cosets, are indexed by the affine Weyl group, i.e., the semi-direct product of the Weyl group and the cocharacter lattice. On the other hand, each stratum is of the form pt/S_w, where S_w is a prosolvable group with reductive quotient T.

As in characteristic zero, the category of D-modules on such a stratum identifies with comodules for the de Rham cohomology of the group, but now this takes the form

$$\Gamma_{\mathrm{dR}}(S_w, \mathrm{Fun}(S_w)) \simeq \mathrm{Sym}_{\mathrm{Fun}(S_w^{(1)})}(T^* S_w^{(1)}[-1]).$$

In particular, the abelian category of D-modules is the category of representations of the Frobenius twist $S_w^{(1)}$, which contributes a copy of the character lattice to the parameter count, as desired.

Phrased differently, one can again expect the gerbe to trivialize, so that we may identify this with

$$\mathrm{QCoh}(T^*(I\backslash G_F/I)^{(1)}),$$

i.e., coherent sheaves on the affine Steinberg variety. The parameter count for it is identical, and the relationship with the double affine Hecke category has been well studied by Garland–Grojnowski, Bezrukavnikov–Finkelberg–Mirković, Varagnolo–Vasserot, Cautis–Williams, and many others; see [GG95], [BFM05], [VV10], [VV09], [CW19], and references therein for a partial indication.

3.3.3. Intermediate between the two prior conjectures, we have the following version of Arkhipov–Bezrukavnikov–Ginzburg [ABG04].

Conjecture 3.4. There is an equivalence

$$\text{D-mod}(I\backslash G_F/G_O) \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{\check{G},\check{B}} \underset{\text{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \text{LocSys}_{\check{G}}(\mathcal{D})).$$

This equivalence, as with many below, should be compatible with the actions of the Hecke categories on both sides.

3.3.4. We also have the following versions of the geometric Casselman–Shalika formula, due to Frenkel–Gaitsgory–Vilonen [FGV01], and its Iwahori level version, due to Arkhipov–Bezrukavnikov [AB09].

Conjecture 3.5. There is an equivalence

$$\text{D-mod}(G_O\backslash G_F/N_F, \psi) \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(\mathcal{D})).$$

Conjecture 3.6. There is an equivalence

$$\text{D-mod}(I\backslash G_F/N_F, \psi) \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{\check{G},\check{B}}).$$

One may replace I and G_O in the above conjectures by any standard parahoric via Conjecture 2.10.

3.4. **Kac–Moody representations - spectral realization.**

3.4.1. In characteristic zero, Frenkel–Gaitsgory applied their local Langlands correspondence to obtain spectral realizations of certain categories of Kac–Moody representations, and also to obtain localization theorems, i.e., purely automorphic statements [FG06a], [FG06b], [FG09a], [FG09b], [FG09c].

Unlike in characteristic zero, in the present setting both types of theorems should give coherent realizations as in [BMR08].

Let us spell out what form some of these take, beginning in this subsection with the spectral statements.

3.4.2. By pairing Kac–Moody with the other objects encountered so far, we obtain the following.

Conjecture 3.7. There is an equivalence of factorization categories

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{G_O} \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(\mathcal{D}) \underset{\text{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \text{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Here the left-hand side denotes the Kazhdan–Lusztig category, i.e., modules for the Harish-Chandra pair $(\widehat{\mathfrak{g}}_{\kappa_c}, G_O)$.

Conjecture 3.8. There is an equivalence

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I} \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{\check{G},\check{B}} \underset{\text{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \text{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Pairing with Whittaker, we obtain the following, which is closely to the Feigin–Frenkel isomorphism $\mathcal{W}$-algebras in the critical limit [FF91].

Conjecture 3.9. There is an equivalence

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{N_F,\psi} \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Finally, we obtain a statement about affine Harish–Chandra bimodules.

Conjecture 3.10. There is an equivalence of monoidal factorization categories

$$\widehat{\mathfrak{g}}_{\kappa_c} \oplus \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{G_F} \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \underset{\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

3.5. **Kac–Moody representations - localization.**

3.5.1. To orient the reader for the statements of the localization theorems, it may be helpful to recall the following formal observation.

Consider a map of spaces over the moduli of local systems

$$X \to Y \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times).$$

We may use this to express the Langlands transform of X via that of Y via the identity

$$\mathbb{L}^{-1}(\mathrm{QCoh}(X)) = \mathbb{L}^{-1}(\mathrm{QCoh}(Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(X)) \simeq \mathbb{L}^{-1}(\mathrm{QCoh}(Y)) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(X).$$

Consider now a commutative diagram of the form

$$\begin{CD} X @>>> Y \\ @VVV @VVV \\ Z @>>> \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times). \end{CD}$$

If we apply the previous observation to either circuit of the diagram, we obtain an equivalence

$$\mathbb{L}^{-1}(Z) \underset{\mathrm{QCoh}(Z)}{\otimes} \mathrm{QCoh}(X) \simeq \mathbb{L}^{-1}(Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(X).$$

3.5.2. If we consider the space of $\check{G}$-opers on the non-punctured disc, this fits into a commutative diagram

$$\begin{CD} \mathrm{Op}_{\check{G}}(\mathcal{D}) @>>> \mathrm{LocSys}_{\check{G}}(\mathcal{D}) \\ @VVV @VVV \\ \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) @>>> \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times). \end{CD}$$

We propose the following, as in [FG06b].

Conjecture 3.11.

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} \underset{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}{\otimes} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D})) \simeq \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}))}{\otimes} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D})).$$

The functor should again be given by convolution with the vacuum algebra; in particular, the conjecture encodes a modular birth of opers, due in characteristic zero to Beilinson–Drinfeld [BD].

3.5.3. Let us now consider the space $\mathrm{Op}^{\mathrm{nilp}}_{\check{G}}$ of nilpotent $\check{G}$-opers, as introduced by Frenkel–Gaitsgory [FG06b].

We recall a convenient description from *loc. cit.* of this moduli space. To do so, write

$$\check{\mathfrak{g}}_{-\alpha_\iota} \subset \check{\mathfrak{g}}, \quad \iota \in I,$$

for the negative simple root spaces of $\check{\mathfrak{g}}$ afforded by our pinning. In addition, for a line bundle $\mathcal{L}$ on $\mathcal{D}$, write

$$\mathcal{L}^{\times}_{O} \subset \Gamma(\mathcal{D}, \mathcal{L})$$

for the open locus of sections which do not vanish in the central fiber $\mathcal{L}_x$. With this, we have

$$\mathrm{Op}^{\mathrm{nilp}}_{\check{G}} \simeq \{d + \sum_{\iota} (\check{\mathfrak{g}}_{-\alpha_\iota} \otimes \omega_O)^{\times} + \check{\mathfrak{b}} \otimes \omega_O + \check{\mathfrak{n}} \otimes \omega_{O(x)}\}/\check{B}_O.$$

From this presentation, we see it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Op}^{\mathrm{nilp}}_{\check{G}} & \longrightarrow & \mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}} \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}) & \longrightarrow & \mathrm{LocSys}_{\check{G}}(\mathcal{D}^{\times}). \end{array}$$

We propose the following, again as in [FG06b].

Conjecture 3.12.

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} \underset{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}))}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{nilp}}_{\check{G}}) \simeq \mathrm{D\text{-}mod}(\mathrm{Fl}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{nilp}}_{\check{G},\check{B}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{nilp}}_{\check{G}}).$$

3.6. **Global aspects.**

3.6.1. Let us briefly remark on some expectable interactions with global phenomena.

3.6.2. Let X be a smooth projective curve, $\mathcal{D}$ the formal disc around a closed point $x \in X$, and

$$\mathrm{Bun}_G(X) \quad \text{and} \quad \mathrm{LocSys}_{\check{G}}(X)$$

the moduli spaces of G-bundles and $\check{G}$-connections on X, respectively.

3.6.3. As pointed out by Bezrukavnikov, it should be possible to use Kac–Moody localization and birth of opers as in Conjecture 3.11 to construct Hecke eigensheaves.

This was done in characteristic zero by Beilinson–Drinfeld for local systems admitting an oper structure [BD], and work of Frenkel–Gaitsgory yields a similar construction for singular oper structures, as recorded by Faergeman–Raskin [FR22].

3.6.4. Very relatedly, let us write

$$\mathrm{Bun}_{G,\infty\cdot x}(X) \quad \text{and} \quad \mathrm{LocSys}_{\check{G}}(X \setminus x)$$

for the moduli spaces of G-bundles with full level structure at x and $\check{G}$-connections on $X \setminus x$, respectively. Recall there is a natural Hecke modification action of G_F on $\mathrm{Bun}_{G,\infty,x}(X)$, and a restriction map

$$\mathrm{LocSys}_{\check{G}}(X \setminus x) \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^{\times}).$$

Exactly as in [Gai16], we may expect that

$$\mathbb{L}(\mathrm{D\text{-}mod}(\mathrm{Bun}_{G,\infty,x})) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X \setminus x)).$$

3.6.5. Consider the 'un-derived' Hecke category, i.e., $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}))$, at each point of the curve $x \in X$, acting on $\text{D-mod}(\mathrm{Bun}_G)$. One should be able to formulate a vanishing conjecture to the effect that these actions induce a single action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on $\text{D-mod}(\mathrm{Bun}_G(X))$, cf. [Gai10].

3.6.6. Relatedly, many of the basic results for chiral algebras and their homology should extend to positive characteristic.

As a relevant example, the space of dormant opers on a curve X, i.e., opers with vanishing p-curvature, should be the chiral homology of the corresponding quotient of the critical $\mathcal{W}$-algebra $\mathcal{W}_{\kappa_c}$. This may give a useful perspective on the very interesting work of Joshi, Mochizuki, Pauly, and Wakabayashi, among others, on counting dormant opers; see [Moc96], [Jos17], [Wak17], [Wak22] and references therein.

We highlight in particular Section 10 of [Jos17], wherein a relationship between the dimension of the space of dormant opers and the nonabelian theta functions of level $p \cdot \kappa_b + \kappa_c$, where κ_b denotes the basic form, is indicated. It would be interesting to understand this via the map from $\mathcal{W}_{\kappa_c}$ to the universal vertex algebra for G_F, i.e., the dual Weyl G_F-module of highest weight zero, of level $p \cdot \kappa_b + \kappa_c$.

4. Local quantum geometric Langlands

4.1. In this final section we would like to indicate the analogues of the previous two sections at noncritical level, i.e., what form local quantum Langlands may take in positive characteristic.

4.2. **Statement.**

4.2.1. Let us now write $\ell_{\mathfrak{g},c}$ and $\ell_{\check{\mathfrak{g}},c}$ for the critical levels for G and $\check{G}$, respectively, and for simplicity assume p is odd.

We recall that, as with ρ-shifts elsewhere in representation theory, these critical levels behave as the central points in the spaces of levels, rather than the zero forms.

With this in mind, let κ be an Ad-invariant bilinear form on $\mathfrak{g}$ such that $\kappa - \ell_{\mathfrak{g},c}$ is nondegenerate. We write $\check{\kappa}$ for the Ad-invariant bilinear form on $\check{\mathfrak{g}}$ such that

$$\kappa - \ell_{c,\mathfrak{g}} \quad \text{and} \quad \check{\kappa} - \ell_{c,\check{\mathfrak{g}}},$$

after restriction to $\mathfrak{t}$ and $\check{\mathfrak{t}}$, respectively, are dual bilinear forms. Similarly, by an abuse of notation we will denote the level $-\check{\kappa} + 2 \cdot \ell_{\check{\mathfrak{g}},c}$ simply by $-\check{\kappa}$.

4.2.2. To κ, we may assign the corresponding monoidal category of κ-twisted D-modules on G_F, which we denote by

$$\text{D-mod}(G_F)_\kappa.$$

Exactly as in characteristic zero, these arise as modules for the twisted differential operators corresponding to a power of the determinant line bundle on G_F, i.e., the Kac–Moody central extension.

4.2.3. A naive formulation of the desired correspondence, i.e., without appropriate renormalization, reads as follows.

Conjecture 4.1. There is an equivalence of 2-categories

$$\mathbb{L}_\kappa : \text{D-mod}(G_F)_\kappa\text{-mod} \simeq \text{D-mod}(\check{G}_F)_{-\check{\kappa}}\text{-mod}\,.$$

As in characteristic zero, in the limit $\kappa \to \kappa_c$ this should tend to Conjecture 2.3, and in the limit $\kappa \to \infty$ to Conjecture 2.3 with the roles of G and $\check{G}$ reversed.

The corresponding global statement was studied by Travkin in the work [Tra16].

4.3. **Matching objects.**

4.3.1. Exactly as in characteristic zero [Gai16], we propose the following.

Conjecture 4.2. Spherical vectors are exchanged under the equivalence, i.e.,

$$\mathbb{L}_\kappa(\text{D-mod}(\text{Gr}_G)_\kappa) \simeq \text{D-mod}(\text{Gr}_{\check{G}})_{-\check{\kappa}}.$$

More generally, for dual standard parabolic subgroups $P \subset G$ and $\check{P} \subset \check{G}$, the corresponding parahoric fixed vectors are exchanged, i.e.,

$$\mathbb{L}_\kappa(\text{D-mod}(\text{Fl}_{G,P})_\kappa \simeq \text{D-mod}(\text{Fl}_{\check{G},\check{P}})_{-\check{\kappa}}.$$

Conjecture 4.3. Kac–Moody modules and Whittaker models are exchanged by duality, i.e., we have

$$\mathbb{L}_\kappa(\widehat{\mathfrak{g}}_\kappa\text{-mod}) \simeq \text{D-mod}(\check{G}_F/\check{N}_F, \check{\psi})_{-\check{\kappa}}.$$

$$\mathbb{L}_\kappa(\text{D-mod}(G_F/N_F, \psi)) \simeq \widehat{\check{\mathfrak{g}}}_{-\check{\kappa}}\text{-mod}.$$

4.4. **Intertwining operators.**

4.4.1. Let us briefly mention a few of the consequent predictions for intertwining operators.

Conjecture 4.4. There is an equivalence of monoidal factorization categories

$$\text{D-mod}(G_O\backslash G_F/G_O)_\kappa \simeq \text{D-mod}(\check{G}_O\backslash\check{G}_F/\check{G}_O)_{\check{\kappa}}.$$

Conjecture 4.5. There is an equivalence of monoidal categories

$$\text{D-mod}(I\backslash G_F/I)_\kappa \simeq \text{D-mod}(\check{I}\backslash\check{G}_F/\check{I})_{\check{\kappa}}.$$

The following should again follow from a Feigin–Frenkel type isomorphism [FF91] for modular $\mathcal{W}$-algebras.

Conjecture 4.6. There is an equivalence of factorization categories

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{N_F,\psi} \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{N}_F,\check{\psi}}.$$

Finally, we have the following versions of the Fundamental Local Equivalence, as introduced in characteristic zero by Gaitsgory and Lurie [Gai08].

Conjecture 4.7. There is an equivalence of factorization categories

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{G_O} \simeq \text{D-mod}(\check{N}_F, \check{\psi}\backslash\check{G}_F/\check{G}_O)_{\check{\kappa}}.$$

Conjecture 4.8. There is an equivalence

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{I} \simeq \text{D-mod}(\check{N}_F, \check{\psi}\backslash\check{G}_F/\check{I})_{\check{\kappa}}.$$

4.4.2. We would like to mention one consequence of the Fundamental Local Equivalences.

For ease of notation, let us take G to be almost simple. In this case, levels form a line, which one compactifies to a copy of $\mathbb{P}^1$ by adding the point ∞ dual to $\ell_{\check{\mathfrak{g}},c}$.

Within this $\mathbb{P}^1$ one has the integral levels, i.e., those arising from a multiplicative line bundle on G_F, and not merely a k-multiple of one. Provided p is not too small relative to G and $\check{G}$, these are simply the forms extended by scalars from the split form of $\mathfrak{g}$ over $\mathbb{F}_p$. In particular, after compactifying, integral forms for G form a copy of $\mathbb{P}^1(\mathbb{F}_p)$ and are dual to the $\mathbb{P}^1(\mathbb{F}_p)$ of integral forms for $\check{G}$.

This phenomenon is markedly different from characteristic zero. Indeed, duality acts on levels essentially by the involution

$$S : z \mapsto -\frac{1}{z},$$

which does not preserve $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{C})$, but does preserve $\mathbb{P}^1(\mathbb{F}_p) \subset \mathbb{P}^1(k)$. In particular, the phenomena of admissible vs. dual admissible rational levels, metaplectic dual groups, etc., from characteristic zero, are less present, as rational is now the same as integral.

We are ready to state to desired consequence. Notice that any integral translation of the level, which we can think of as essentially

$$T : z \mapsto z + 1,$$

does not affect the right hand sides of Conjectures 4.7 and 4.8, and permutes $\mathbb{P}^1(\mathbb{F}_q) \setminus \infty$ transitively. Applying S, we permute $\mathbb{P}^1(\mathbb{F}_q) \setminus 0$ transitively, and hence can bring any point to ∞ to obtain the following.

Conjecture 4.9. Suppose κ is a noncritical integral level. Then there are equivalences

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{G_O} \simeq \text{QCoh}(\text{LocSys}_G(\mathcal{D})) \quad \text{and} \quad \widehat{\mathfrak{g}}_\kappa\text{-mod}^I \simeq \text{QCoh}(\text{LocSys}^{\text{nilp}}_{G,B}).$$

4.4.3. The consequence of Conjecture 4.9 that $\widehat{\mathfrak{g}}_\kappa$-mod^I and $\widehat{\mathfrak{g}}_{\kappa'}$-mod^I would be equivalent for all noncritical integral levels κ and κ' may be surprising, given how different it looks from the situation in characteristic zero. Let us therefore indicate a plausibility check.

Suppose we conjecture that the linkage principle for $\widehat{\mathfrak{g}}_\kappa$-mod^I at an integral level κ is again controlled by the action of the double affine Hecke category

$$\text{D-mod}(I \backslash G_F / I),^3$$

and in particular at the level of highest weights is given by the level κ dot action of the affine Weyl group along with translations by p times the root lattice, i.e., the double affine Weyl group. One can check that the resulting group of affine transformations indeed only depends on whether κ is critical or non-critical.

4.5. **Final remarks.**

4.5.1. We finish by recording a few thoughts.

[3] Strictly speaking, really we should replace G_F by the neutral component of the loop group of the adjoint form.

4.5.2. In characteristic zero, a striking conjecture of Gaiotto equates the highest weight representations of basic classical affine Lie superalgebras $\widehat{\mathfrak{s}}$ with certain equivariant D-modules on affine flag varieties, generalizing the Fundamental Local Equivalence. More generally, it predicts where local quantum Langlands sends $\widehat{\mathfrak{s}}$-mod, viewed as a categorical representation of its even part. This has been studied and partially confirmed by Braverman, Finkelberg, Ginzburg, Travkin, and Yang in an interesting series of papers [BFGT21], [BFT21], [BFT22a], [BFT22b], [TY22]. One may conjecture the analogous statements hold in positive characteristic.

4.5.3. In characteristic zero, it is expected by physicists that the S and T transformations generate an $SL(2,\mathbb{Z})$ of quantum geometric Langlands equivalences in the appropriate sense. This includes crucially the $STS = TST$ relation, which controls various phenomena including the coset realizations of $\mathcal{W}$-algebras, as was explained to us by Gaiotto. It seems reasonable to expect an action of $SL(2,\mathbb{F}_p)$ in the present case.

4.5.4. Finally, various experts have sensed connections between higher Teichmüller theory and its quantizations and geometric Langlands in characteristic zero. It is natural to wonder about the connection in positive characteristic to the work of Mochizuki [Moc96], and presumably this is related to the analogies at the end of [BTCZ16].

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