Problems on Low-dimensional Topology, 2023

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 24–26, 2023.

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1 Hyperbolic L–space knots

(Masakazu Teragaito)

A knot is called an L–space knot if it admits a positive Dehn surgery yielding an L–space. It is not too much to say that L–space knots provide an important class of knots from the perspective of Heegaard Floer theory.

We are interested in concordances from knots to L–space knots or among L–space knots. There are several precedent works on this topic. Zemke [35] gives an obstruction derived from knot Floer complex, and shows that $T(4,5)\#T(4,5)$, $T(4,5)\#T(6,7)$, $T(6,7)\#T(6,7)$, $-T(3,4)\#-T(4,5)\#T(5,6)$ (and others) are not concordant to an L–space knot or the mirror image of an L–space knot. This was extend by Livingston [22] to show that no connected sum of at least two positive torus knots is concordant to an L–space knot. Moreover, Allen [2] showed that if $T(p,q)$ and $T(r,s)$ are positive torus knots and $mT(p,q)\#nT(r,s)$ $(m,n \in \mathbb{Z})$ is concordant to an L–space knot, then $(m,n)=(1,0)$ or $(0,1)$.

Question 1.1 (M. Teragaito)**.** *Do there exist distinct hyperbolic L–space knots which are concordant?*

Dunfield determines 632 hyperbolic L–space knots whose complements consist of at most 9 ideal tetrahedra. The list can be found in [3], and Baker and Kegel [5] give the braid words for these 632 knots. By Krcatovich [20], the Alexander polynomial is a concordance invariant for L–space knots. That is, if two L–space knots K_1 and K_2 are concordant, then they share the same Alexander polynomial. I confirmed that there is no duplication of Alexander polynomials in Dunfield's list of hyperbolic L–space knots. This is the reason why I am skeptical about Question 1.1.

As Motegi informed me, it is known that any L–space knot is tight fibered. Recently, Abe and Tagami [1] show that all tight fibered knots are minimal with respect to ribbon concordance. Hence, Question 1.1 is negative if we replace 'concordant' with 'ribbon concordant'.

Question 1.2 (M. Teragaito)**.** *Does there exist a hyperbolic L–space knot which is concordant to a torus knot?*

Again, I have checked that there is no example in Dunfield's list.

2 Vanising of twisted Alexander polynomials of knots

(Masaaki Suzuki)

Let K be a knot and $G(K)$ the knot group. For a finite group G and for a homomorphism $f: G(K) \to G$, we can consider the twisted Alexander polynomial $\Delta_K^{\rho\circ f}(t)$, where $\rho: G \to \mathrm{GL}(|G|,\mathbb{Z})$ is the regular representation of G. In this situation, Friedl and Vidussi showed the following.

Theorem ([9]) A knot K is non-fibered if and only if there exists a finite group G and a surjective homomorphism $f: G(K) \to G$ such that the twisted Alexander polynomial $\Delta_K^{\rho\circ f}(t)$ is zero.

Then in [26] we define the minimal order $\mathcal{O}(K)$ of a knot K as the smallest order of a finite group G such that there exists a surjective homorphism $f : G(K) \to G$ with $\Delta_K^{\rho\circ f}(t) = 0$. By the above theorem, $\mathcal{O}(K)$ is finite for any non-fibered knot K. On the other hand, we define $\mathcal{O}(K) = +\infty$ for a fibered knot K.

Question 2.1 (T. Morifuji and M. Suzuki)**.** *For a non-fibered knot K, how can we find a finite group such that $\Delta_K^{\rho\circ f}(t) = 0$? Moreover, can we determine $\mathcal{O}(K)$ explicitly for a given knot K? In particular, what is a finite group G such that $\Delta_{7_3}^{\rho\circ f}(t) = 0$? We have an inequality $125 \leq \mathcal{O}(5_2) \leq 2520$, then what is $\mathcal{O}(5_2)$ precisely?*

We see that the twisted Alexander polynomial is not zero for any abelian group. Then it is natural to ask another class of finite groups.

Question 2.2 (T. Morifuji and M. Suzuki)**.** *Can we characterize finite groups such that $\Delta_K^{\rho\circ f}(t) \neq 0$ for any K and f?*

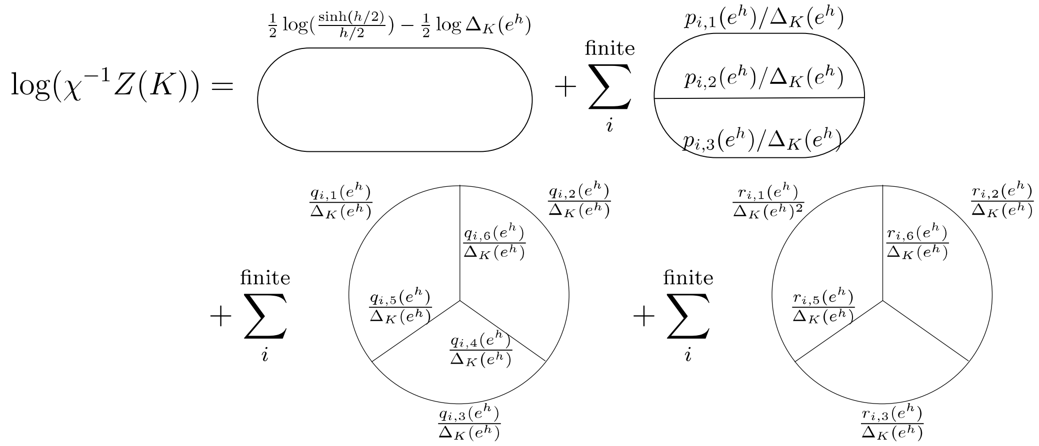
We are not sure the minimal order $\mathcal{O}(K)$ is unbounded or not.

Question 2.3 (T. Morifuji, Y. Nozaki, and M. Suzuki)**.** *Is the minimal order $\mathcal{O}(K)$ unbounded?*

3 Calculation of the 3-loop invariant of knots

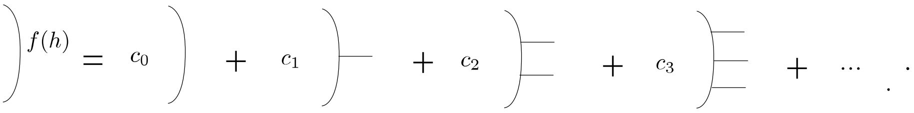
(Kouki Yamaguchi)

The Kontsevich invariant $\chi^{-1}Z(K)$ of knots is a powerful invariant which is universal among all quantum invariants of knots and Vassiliev invariants of knots, and it is expected that the Kontsevich invariant would classify knots. The Kontsevich invariant of a (0-framed) knot K can be presented in the following form, which is called the loop expansion,



$+$ (terms of (>3)-loop part),

where $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^h), q_{i,j}(e^h), r_{i,j}(e^h)$ are polynomials in $e^{\pm h}$. Here, a labeling of $f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$ implies that ,



For details, see [11, 21, 27]. The 1-loop part is presented by the Alexander polynomial. The 2-loop part is presented by the 2-loop polynomial $\Theta_K(t_1, t_2, t_3)$, which is given by

$$\Theta_K(t_1,t_2,t_3) = \sum_{\substack{m \\ \epsilon=\pm1 \\ \{i,j,k\}=\{1,2,3\}}} p_{m,1}(t_i^\epsilon) p_{m,2}(t_j^\epsilon) p_{m,3}(t_k^\epsilon) \in \mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}]/(\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}, t_1 t_2 t_3 = 1).$$

Further, the 3-loop part is presented by the 3-loop invariant (or, the 3-loop polynomial) $\Lambda_K(t_1, t_2, t_3, t_4)$, which is given by

$$\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4) \\
&= \sum_{\substack{i \\ \tau\in\mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(3)}^{-\mathrm{sgn}\tau}) q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1 t_4^{-1}) \Delta_K(t_2 t_4^{-1}) \Delta_K(t_3 t_4^{-1}) \Delta_K(t_2 t_3^{-1}) \Delta_K(t_3 t_1^{-1}) \Delta_K(t_1 t_2^{-1})} \\
&+ \sum_{\substack{i \\ \tau\in\mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)} t_{\tau(4)}^{-1})^2 \Delta_K(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_K(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_K(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_K(t_{\tau(1)} t_{\tau(2)}^{-1})} \\
&\in \frac{1}{\hat{\Delta}^2} \cdot \mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}]/(\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1),
\end{aligned}$$

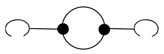
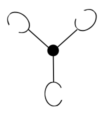
where we put

$$\hat{\Delta} = \Delta_K(t_1 t_4^{-1}) \Delta_K(t_2 t_4^{-1}) \Delta_K(t_3 t_4^{-1}) \Delta_K(t_2 t_3^{-1}) \Delta_K(t_3 t_1^{-1}) \Delta_K(t_1 t_2^{-1}).$$

In particular, if $\Delta_K(t) = 1$, then $\Lambda_K(t_1, t_2, t_3, t_4)$ is a polynomial, so in this case, we call it the *3-loop polynomial*. For details, see [34]. In general, for an arbitrarily given knot K, it is not easy to calculate the 2-loop polynomial and the 3-loop invariant of K concretely.

For the 2-loop polynomial, some clasper surgery formulas are concretely presented in [28], which describe the changes of the 2-loop polynomial under clasper surgeries. Such formulas are useful to calculate the 2-loop polynomial for some classes of knots. However, such clasper surgery formulas have not been presented concretely yet for the 3-loop invariant so far.

Problem 3.1 (K. Yamaguchi). *Present clasper surgery formulas concretely for the 3-loop invariant for clasper surgeries along claspers such as*

 *and*  .

From the viewpoint of the classification problem of knots, it is an important problem to determine the image of the Kontsevich invariant. When we restrict this problem to the (≤ 3)-loop part, it is a problem to determine the set of possible values of the triple $\big(\Delta_K(t), \Theta_K(t_1,t_2,t_3), \Lambda_K(t_1,t_2,t_3,t_4)\big)$. We consider a further simplified case of this problem in the following problem.

Problem 3.2 (K. Yamaguchi)**.** *For a (0-framed) knot K with $\Delta_K(t) = 1$ and $\Theta_K(t_1,t_2,t_3) = 0$, determine the set of possible values of the 3-loop polynomial $\Lambda_K(t_1,t_2,t_3,t_4)$.*

4 An embedding of the Kauffman bracket skein algebra of a surface into a localized quantum torus

(Ramanujan Santharoubane)

For Σ a compact connected oriented surface with genus at least one without boundary component, we denote by $S(\Sigma)$ the skein algebra of $\Sigma \times [0,1]$, we see it as a $\mathbb{Z}[A^{\pm 1}]$-algebra. For a given pants decomposition $\mathcal{P}$ of Σ, a localized quantum torus $\mathcal{A}(\mathcal{P})$ is defined in [7]. Recall that we need first to consider the quantum torus $\mathcal{T}(\mathcal{P})$ over $\mathbb{Z}[A^{\pm 1}]$ with variables $\{E_e, Q_e \,|\, e \in \mathcal{P}\}$ where all variables commute except Q_e and E_e (for all $e \in \mathcal{P}$) that satisfy $Q_e E_e = A E_e Q_e$. Viewed as a ring, the quantum torus $\mathcal{T}(\mathcal{P})$ is an integral domain so we can define $\mathcal{A}(\mathcal{P})$ to be a $\mathbb{Z}[A^{\pm 1}]$-algebra containing $\mathcal{T}(\mathcal{P})$ where $A^k Q_e^2 - A^{-k} Q_e^{-2}$ is invertible for all $e \in \mathcal{P}$ and $k \in \mathbb{Z}$. One of the main result in [7] was to build an embedding

$$\sigma_{\mathcal{P}} : S(\Sigma) \to \mathcal{A}(\mathcal{P}).$$

The first natural question is to know what happens when we change the pants decomposition. It is known that we can go from one pant decomposition to another via a finite number of elementary moves.

Question 4.1 (R. Santharoubane)**.** *Given two pants decompositions $\mathcal{P}, \mathcal{P}'$ of Σ, can we find a $\mathbb{Z}[A^{\pm 1}]$-algebra homomorphism $\varphi_{\mathcal{P},\mathcal{P}'} : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{P}')$ such that $\sigma_{\mathcal{P}'} = \varphi_{\mathcal{P},\mathcal{P}'} \circ \sigma_{\mathcal{P}}$?*

A positive answer to this question would allow us to define an universal localized quantum torus in which $S(\Sigma)$ would embed.

Another problem concerns $\mathrm{Aut}(\mathcal{A}(\mathcal{P}), S(\Sigma))$ which is the group of automorphisms of $\mathcal{A}(\mathcal{P})$ being the identity on $\sigma_{\mathcal{P}}(S(\Sigma))$.

Question 4.2 (R. Santharoubane)**.** *What is $\mathrm{Aut}(\mathcal{A}(\mathcal{P}), S(\Sigma))$?*

In [10], Frohman and Gelca built an embedding of the skein algebra of the torus into a quantum torus (not localized) and they proved that the image of the skein algebra of the torus (by the embedding) is exactly the invariant set of certain automorphisms of the quantum torus. This allowed them to get a beautiful formula for the product of simple curves in the skein algebra of the torus. Hence we can raise the following vague question.

Question 4.3 (R. Santharoubane). *Can we get from the knowledge of* $\sigma_{\mathcal{P}}$ *and* $\mathrm{Aut}(\mathcal{A}(\mathcal{P}), S(\Sigma))$*, a formula à la Frohman-Gelca for the product of two curves in* $S(\Sigma)$*?*

5 The distance on Teichmüller space via renormalized volume

(Hidetoshi Masai)[2]

Let S be a closed surface of genus $g \geq 2$, and $\mathcal{T}(S)$ the Teichmüller space of S. The space of quasi-Fuchsian manifolds is parameterized by $\mathcal{T}(S) \times \mathcal{T}(S)$. Let $\mathrm{qf}(X,Y)$ denote the quasi-Fuchsian manifold corresponding to $(X,Y) \in \mathcal{T}(S) \times \mathcal{T}(S)$. We denote by $V_R(X,Y)$ the renormalized volume of the quasi-Fuchsian manifold $\mathrm{qf}(X,Y)$. The map $V_R : \mathcal{T}(S) \times \mathcal{T}(S)$ is not a distance. In fact, it is known ([23, Theorem 7.2]) that the function $V_R : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}$ does NOT satisfy the triangle inequality.

In [23], we define a distance d_R on $\mathcal{T}(S)$ via the renormalized volume, and demonstrate that the distance d_R is natural to the volume of hyperbolic 3-manifolds. Given $X, Y \in \mathcal{T}(S)$, let

$$d_R(X,Y) := \sup_{Z \in \mathcal{T}(S)} V_R(X,Z) - V_R(Y,Z).$$

One important feature of d_R is the following.

Theorem ([23]) *Let* $\psi \in \mathrm{MCG}(S)$ *be a pseudo-Anosov mapping class and* $M(\psi) = S \times I/(x,1) \sim (\psi(x),0)$ *denote the mapping torus of* ψ*. Then the translation distance of* ψ *with respect to* d_R *is equal to the hyperbolic volume of* $M(\psi)$*, that is, for any* $X \in \mathcal{T}(S)$*, we have*

$$\lim_{n\to\infty} \frac{1}{n} d_R(X, \psi^n X) = \mathrm{vol}(M(\psi)).$$

In the proof, we utilize some ergodic theory, which is inspired by Karlsson-Ledrappier [14, Proof of Theorem 1.1], see [23, Theorem 7.10] for the proofs.

The distance d_R is still very mysterious. Similarly to the case of Weil-Petersson (WP) metric, as $(\mathcal{T}(S), d_R)$ is not complete, we may not use Hopf-Rinow Theorem to find geodesics.

Question 5.1 (H. Masai). *Is* $(\mathcal{T}(S), d_R)$ *a geodesic space?*

As $d_R(\cdot,\cdot) \leq 3\sqrt{\pi(g-1)}\, d_{\mathrm{wp}}(\cdot,\cdot)$, one easily sees that $\hat{\mathcal{T}}(S)$ is contained in the completion of $(\mathcal{T}(S), d_R)$.

Question 5.2 (H. Masai). *What is the metric completion of* $(\mathcal{T}(S), d_R)$*?*

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Also it is interesting to understand the action of pseudo-Anosov maps. Let $\psi \in \mathrm{MCG}(S)$ be pseudo-Anosov. Then the axis of ψ should be a geodesic of d_R invariant under ψ.

Question 5.3 (H. Masai)**.** *Does every pseudo-Anosov map have a (unique?) geodesic axis?*

The distance d_R is quasi-isometric to d_{wp} [23], and $(\mathcal{T}(S), d_{\mathrm{wp}})$ is CAT(0) [33]. Although CAT(0)-ness is not invariant under quasi-isometry, we might expect:

Question 5.4 (H. Masai)**.** *Is $(\mathcal{T}(S), d_R)$ a CAT(0) space?*

The horoboundaries may be used to identify isometry groups (see e.g. [31]). In [31], except for some sporadic cases, Walsh identified the isometry group of the Thurston metric with the so-called extended mapping class groups (see [31] for the definition).

Question 5.5 (H. Masai)**.** *Is $\mathrm{Isom}(\mathcal{T}(S), d_R)$ equal to the extended mapping class group? What about self-maps on $\mathcal{T}(S)$ preserving V_R?*

Since we are taking supremum in the definition of d_R, several properties of V_R (say, smoothness) is not a priori inherited to d_R. Let us finish the paper with the following question.

Question 5.6 (H. Masai)**.** *Is there a Riemannian or a Finsler metric on $\mathcal{T}(S)$ which defines d_R?*

6 Invariants of 3-manifolds obtained from the Heisenberg doubles of Hopf algebras

(Sakie Suzuki)

The Heisenberg double of a finite-dimensional Hopf algebra H has a canonical element that satisfies a pentagon relation [17]. By associating the canonical element with ideal tetrahedra, we constructed in [24, 25] an invariant $Z(M, f; H)$ for framed closed 3-manifolds (M, f) with vanishing first Betti number $b_1(M)$.

Let us take $H = u_q(\mathfrak{sl}_2^+)$ the small quantum Borel subalgebra with q the n-th primitive root of unity. In this case we have

$$Z(S^3, f; u_q(\mathfrak{sl}_2^+)) = q^{-1},$$

where f is the framing extending the combing induced by the Hopf fibering, and we have

$$Z(L(2,1), f; u_q(\mathfrak{sl}_2^+)) = 2q^{-1}\frac{1 - q^{-\lfloor \frac{n+1}{2} \rfloor}}{1 - q^{-1}},$$

where $L(2,1)$ is the lens space and f is the framing extending the canonical combing induced by its Seifert fibered structure. When q is a primitive root of unity of odd order N, the above values match, up to multiplication by q, the SO(3) Witten-Reshetikhin-Turaev (WRT) invariant $\tau_N^{\mathrm{SO}(3)}(M)$ [19, 29, 32] times the cardinality $|H_1(M)|$ of the first homology group.

Question 6.1 (S. M. Mihalache, S. Suzuki, Y. Terashima [25]). *Let M be a closed oriented framed 3-manifold with $b_1(M) = 0$. For a primitive root of unity q of odd order N, is it true that*

$$Z(M, f; u_q(\mathfrak{sl}_2^+)) = q^k \cdot |H_1(M)| \cdot \tau_N^{SO(3)}(M)$$

for some integer k?

Recall that the WRT invariant is an invariant of 2-framed 3-manifold, where one usually chooses canonical 2-framing to compute it. Since framing f induces a 2-framing v_2, it is natural to ask the following question.

Question 6.2 (S. M. Mihalache, S. Suzuki, Y. Terashima [25]). *Under the same assumption in Question 6.1, does the following equation hold?*

$$Z(M, f; u_q(\mathfrak{sl}_2^+)) = |H_1(M)| \cdot \tau_N^{SO(3)}(M, v_2).$$

The Turaev-Viro invariant [6, 30] is defined as a state sum invariant using triangulations and $6j$-symbols. In particular, we can obtain the invariant $TV_r(M)$ associated with the quantum group $U_q(\mathfrak{sl}_2)$, where $q^{1/2}$ is a $4r$-th primitive root of unity. It is known that the Turaev-Viro invariant is equal to the absolute square of the WRT invariant.

Problem 6.3 (S. M. Mihalache, S. Suzuki, Y. Terashima). *Reconstruct $TV_r(M)$ from $Z(M, f; u_q(\mathfrak{sl}_2^+))$ by establishing a relation between the canonical element and the $6j$-symbol.*

Problem 6.4 (S. M. Mihalache, S. Suzuki, Y. Terashima). *More generally, construct $6j$-symbols from the canonical elements of the Heisenberg doubles.*

For the quantum Borel subalgebra $U_\hbar(\mathfrak{sl}_2^+)$ ($\hbar$-adic ver.) of $\mathfrak{sl}_2$, the pentagon equation of the canonical element of Heisenberg double turns out to be essentially the Fadeev-Kashaev's pentagon identity for the quantum dilogarithm [8, 17]. The pentagon identity for the quantum dilogarithm was the crucial result sitting behind a sequence of important works [4, 12, 13, 15, 16, 18] related to the Kashaev invariant of links, the volume conjecture, and investigation of 3-manifolds invariants and TQFT using the quantum dilogarithm and quantum Teichmüller Theory.

Problem 6.5 (S. M. Mihalache, S. Suzuki, Y. Terashima). *Define $Z(M, f; H)$ for the infinite dimensional Hopf algebra $H = U_\hbar(\mathfrak{sl}_2)$, which realizes an invariant associating the quantum dilogarithm with ideal tetrahedra. Generalize it to TQFT and establish a relation with the quantum Teichmüller Theory.*

Problem 6.6 (S. M. Mihalache, S. Suzuki, Y. Terashima). *Formulate a volume conjecture using the anticipated invariant $Z(M, f; U_h(sl_2))$ in Problem 6.5 (or using the family $Z(M, f; u_q(\mathfrak{sl}_2^+))$ for roots of unity) by specifying a relation between the canonical element of the Heisenberg double and the volume of ideal tetrahedra.*

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