Arithmetic approach to p-numerical semigroups

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1 Introduction

Let S be a submonoid of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Namely, S satisfies the conditions:

$$\text{(i) } S \subset \mathbb{N}_0; \quad \text{(ii) } 0 \in S; \quad \text{(iii) if } a, b \in S \text{ then } a + b \in S\,.$$

Then S is called a *numerical semigroup* if and only if $\mathbb{N}_0 \backslash S$ is a finite set, which is equivalent to $1 \in \{x - y | x, y \in S\}$. When S is a numerical semigroup, the maximal element, the cardinality and the sum of the elements of $\mathbb{N}_0 \backslash S$ are called the *Frobenius number*, the *genus* or the *Sylvester number*, and the *Sylvester sum*, respectively, and denoted by $g(S)$, $n(S)$ and $s(S)$, respectively. Several generalized Frobenius numbers have been introduced, but we study the one focusing on the number of representations (nonnegative integer solutions). Let $A := \{a_1, a_2, \dots, a_k\}$ be the set of positive integers with $k \ge 2$. The *denumerant* $d(n) = d(n; a_1, a_2, \dots, a_k)$ is the number of representations to $n = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$ for a given nonnegative integer n. When S is a numerical semigroup and $A \subseteq S$, it is called that S is generated by A and denoted by $S = \langle A \rangle$ if for all $n \in S$, there exist $a_1, a_2, \dots, a_k \in A$ and $x_1, x_2, \dots, x_k \in \mathbb{N}_0$ such that $n = \sum_{j=1}^k a_j x_j$. A is called a *minimal set of generators* of S if $S = \langle A \rangle$ and no proper subset of A has its property. $S = \langle A \rangle$ is called the *canonical form* description of S.

For an nonnegative integer p, let S_p be the set of integers whose nonnegative integral linear combinations of given positive integers $a_1, a_2, \dots, a_k$ are expressed in *more than p ways*. By emphasizing the fact that S is generated by the set A, we also write $S_p(A)$ as S_p. We can see that the set $\mathbb{N}_0 \backslash S_p$ is finite if and only if $\gcd(a_1, a_2, \dots, a_k) = 1$. Then there exists the largest integer $g_p(A) := g(S_p)$ in $\mathbb{N}_0 \backslash S_p$, which is called the *$p$-Frobenius number*. The cardinality of $\mathbb{N}_0 \backslash S_p$ is called the *p-genus* and denoted by $n_p(A) := n(S_p)$. Its sum of elements is called the *p-Sylvester sum* and denoted by $s_p(A) := s(S_p)$.

Several different generalizations have been introduced and studied. However, our generalization is very natural and efficient in terms of the following p-Apéry sets.

2 Preliminaries

Using the elements in the p-Apéry set, we can obtain the p-Frobenius, the p-genus, the p-Sylvester sum and so on very efficiently. Without loss of generality, put $a_1 = \min(A)$. For a nonnegative integer p, the *p-Apéry set* is given by

$$\mathrm{Ap}_p(A) = \mathrm{Ap}_p(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\} . \tag{1}$$

Here, for each $0 \le i \le a_1 - 1$, the positive integer $m_i^{(p)}$ is determined uniquely as

$$\text{(i) } m_i^{(p)} \equiv i \pmod{a_1}, \quad \text{(ii) } m_i^{(p)} \in S_p(A), \quad \text{(iii) } m_i^{(p)} - a_1 \notin S_p(A) .$$

So, the set $\mathrm{Ap}_p(A)$ is a complete residue system modulo a_1, $\{0, 1, \dots, a_1 - 1\}$. In general, the following formula for power sum is given ([13]).

Theorem 1. *Let k, p and μ be integers with $k \ge 2$, $p \ge 0$ and $\mu \ge 0$. Assume that* $\gcd(A) = 1$. *We have*

$$\begin{aligned} s_p^{(\mu)}(A) &:= \sum_{n \in \mathbb{N}_0 \setminus S_p(A)} n^\mu \\ &= \frac{1}{\mu+1} \sum_{\kappa=0}^{\mu} \binom{\mu+1}{\kappa} B_\kappa a_1^{\kappa-1} \sum_{i=0}^{a_1-1} \bigl(m_i^{(p)}\bigr)^{\mu+1-\kappa} + \frac{B_{\mu+1}}{\mu+1}(a_1^{\mu+1} - 1) , \end{aligned}$$

where B_n are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} .$$

When $\mu = 0, 1$ in Theorem 1, together with $g_p(A)$ we have formulas for the p-Frobenius number, the p-Sylvester number and the p-Sylvester sum.

Corollary 1. *Assume that* $\gcd(A) = 1$. *For a nonnegative integer p, we have*

$$\begin{aligned} g_p(A) &= \max_{0 \le i \le a_1 - 1} m_i^{(p)} - a_1 , \\ n_p(A) &= \frac{1}{a_1} \sum_{i=0}^{a_1-1} m_i^{(p)} - \frac{a_1 - 1}{2} , \\ s_p(A) &= \frac{1}{2a_1} \sum_{i=0}^{a_1-1} \bigl(m_i^{(p)}\bigr)^2 - \frac{1}{2} \sum_{i=0}^{a_1-1} m_i^{(p)} + \frac{a_1^2 - 1}{12} . \end{aligned}$$

When $p=0$ in Corollary 1, the formulas are reduced to the classically known ones.

$$g(A)=\left(\max_{1\le i\le a_1-1} m_i\right)-a_1\,,\quad [4]$$

$$n(A)=\frac{1}{a_1}\sum_{i=1}^{a_1-1} m_i-\frac{a_1-1}{2}\,,\quad [22]$$

$$s(A)=\frac{1}{2a_1}\sum_{i=1}^{a_1-1} m_i^2-\frac{1}{2}\sum_{i=1}^{a_1-1} m_i+\frac{a_1^2-1}{12}\,.\quad [24]$$

Notice that $m_0=m_0^{(0)}=0$ is applied when $p=0$. Hence, the sum runs from $i=1$.

The elements of the p-Apéry set are uniquely determined, but it is not easy to obtain them for the general case. Furthermore, it is even more difficult to find any regularity.

3 Explicit expressions

In the case of two variables, namely, $a_1=a$ and $a_2=b$, by $\{m_i|0\le i\le a-1\}=\{b(pa+i)|0\le i\le a-1\}$, from Corollary 1 we can get the explicit expressions.

Corollary 2. *For a nonnegative integer p, we have*

$$g_p(a,b)=(p+1)ab-a-b\,,$$

$$n_p(a,b)=pab+\frac{(a-1)(b-1)}{2}\,,$$

$$s_p(a,b)=\frac{p^2a^2b^2}{2}+\frac{p(ab-a-b)ab}{2}+\frac{(a-1)(b-1)(2ab-a-b-1)}{12}\,.$$

However, for $k\ge 3$, even though $p=0$, $g(A)$ cannot be given by any set of closed formulas which can be reduced to a finite set of certain polynomials [5]. For $k=3$, there are several useful algorithms to obtain the Frobenius number (e.g., [9, 21, 7]). For the concretely given three positive integers, if the conditions are met, the Frobenius number can be calculated by the method of case-dividing in [25]. Only some special cases, explicit closed formulas have been found, including arithmetic, geometric, Fibonacci, Mersenne, repunits and triangular.

When $p>0$ and $k\ge 3$, the situation is even harder. Though any explicit formula had not been found even for particular triples, recently we have been finally successful to obtain closed formulas for arithmetic [17], triangular [11], repunit [12], Fibonacci [16] and Jacobsthal [**?**].

4 Some relations related to p-Frobenius numbers

In this section, we show some fundamental relations among $g_p(A)$, $n_p(A)$ and $s_p(A)$.

Lemma 1. *For $p \ge 0$, we have*

$$n_p(A) \ge \frac{g_p(A)+1}{2}\,.$$

Proof. For a non-negative integer s, if $s \in S_p$, then $g_p(A) - s \notin S_p$. Hence, by $n_p(A) \ge \#\{s \in S_p | s < g_p(A)\} = g_p(A) + 1 - n_p(A)$, we get the result. □

For emphasis, write $\mathrm{Ap}(S_p, a_1)$ as the p-Apéry set $\mathrm{Ap}_p(A)$ from $S_p(A)$ with a_1 as the least element of the set A.

Proposition 1. *Assume that $S_p(A)$ is minimally generated by $a_1, \dots, a_k$. Set $d = \gcd(a_2, \dots, a_k)$ and $T_p(A) = \{n \in \mathbb{N}_0 | d(n; a_1, a_2/d, \dots, a_k/d) > p\}$ Then we have $\mathrm{Ap}(S_p, a_1) = d\mathrm{Ap}(T_p, a_1)$.*

Proof. From the definition of the Apéry set, $w \in \mathrm{Ap}(S_p, a_1)$ implies that $w - a_1 \notin S_p$. Since $w \in \langle a_2, \dots, a_k \rangle$, we have $w/d \in \langle a_2/d, \dots, a_k/d \rangle$. If $w/d - a_1 \in T_p(A)$, as $w - da_1 \notin S_p$, $w/d - a_1 \notin T_p(A)$. Hence, $w/d \in \mathrm{Ap}(S_p, a_1)$, which implies that $w \in d\mathrm{Ap}(S_p, a_1)$.

On the other hand, if $w \in \mathrm{Ap}(T_p, a_1)$, then $w \in \langle a_2/d, \dots, a_k/d \rangle$, implying that $dw \in \langle a_2, \dots, a_k \rangle \subseteq S_p$. We shall see that $dw - a_1 \notin S_p(A)$, entailing that $dw \in S_p(A)$. Otherwise, for non-negative integers $y_1, \dots, y_k$, $dw - a_1 = a_1 y_1 + \dots + a_k y_k$, implying that $w = a_1(y_1+1)/d + (a_2/d)y_2 + \dots + (a_k/d)y_k$ and $d|(y_1+1)$. But this is impossible because $w - a_1 \notin T_p(A)$. □

By Proposition 1, we can obtain the relations between the p-Frobenius numbers $g_p(A)$ and the p-Sylvester numbers $n_p(A)$. For simplicity, we write $g_p\big(A_d\big) = g\big(T_p(A)\big)$ and $n_p\big(A_d\big) = n\big(T_p(A)\big)$.

Corollary 3. *As the same setting as above, we have*

(i) $g_p(A) = d g_p(A_d) + (d-1)a_1$.

(ii) $n_p(A) = d n_p(A_d) + \dfrac{(d-1)(a_1-1)}{2}$.

(iii) $s_p(A) = d^2 s_p(A_d) + \dfrac{a_1 d(d-1)}{2} n_p(A_d) + \dfrac{(a_1-1)(d-1)(2a_1 d - a_1 - d - 1)}{2}$.

Proof. We shall prove (iii). By Corollary 1,

$$
\begin{aligned}
s_p(A) &= \frac{1}{2a_1}\sum_{w\in\mathrm{Ap}_p(S;a_1)} w^2 - \frac{1}{2}\sum_{w\in\mathrm{Ap}_p(S;a_1)} w + \frac{a_1^2-1}{12}\\
&= \frac{d^2}{2a_1}\sum_{w\in\mathrm{Ap}_p(T;a_1)} w^2 - \frac{d}{2}\sum_{w\in\mathrm{Ap}_p(T;a_1)} w + \frac{a_1^2-1}{12}\\
&= d^2\left(\frac{1}{2a_1}\sum_{w\in\mathrm{Ap}_p(T;a_1)} w^2 - \frac{1}{2}\sum_{w\in\mathrm{Ap}_p(T;a_1)} w + \frac{a_1^2-1}{12}\right)\\
&\quad - \frac{d}{2}\sum_{w\in\mathrm{Ap}_p(T;a_1)} w + \frac{a_1^2-1}{12}\\
&= d^2 s_p(A_d) + \frac{a_1 d(d-1)}{2} n_p(A_d) + \frac{(a_1-1)(d-1)(2a_1 d - a_1 - d - 1)}{2}\,.
\end{aligned}
$$

□

Example 1. Let $S = \langle 20, 30, 17\rangle$ and $T = \langle 2, 3, 17\rangle = \langle 2, 3\rangle$ with $d = 10$. Then, for $p = 3$, by $s_3(A_{10}) = 136$ and $n_3(A_{10}) = 17$, we get $s_3(A) = 10^2 s_3(A_{10}) + 17\cdot 10\cdot 9/2 n_3(A_{10}) + 16\cdot 9(2\cdot 17\cdot 10 - 17 - 10 - 1)/12 = 30349$.

Example 2. When $S = \langle a, b\rangle$, by putting $d = b$, we get

$$
g_p(A_d) = ap - 1,\quad n_p(A_d) = ap \quad\text{and}\quad g_p(A_d) = \frac{a^2p^2 - ap}{2}\,.
$$

Therefore, Corollary 2 is reduced again.

4.1 p-Hilbert series

For a non-negative integer p, the *p-Hilbert series* of $S_p(A)$ is defined by

$$
H_p(A; x) := H(S_p; x) = \sum_{s\in S_p(A)} x^s\,.
$$

When $p = 0$, the 0-Hilbert series is the original Hilbert series. In addition, the p-gaps generating function is defined by

$$
\Psi_p(A; x) = \sum_{s\in\mathbb{N}_0\setminus S_p(A)} x^s\,,
$$

satisfying $H_p(A; x) + \Psi_p(A; x) = 1/(1-x)$ $(|x| < 1)$. By using p-Apéry set, we see that $S_p(A) = \mathrm{Ap}_p(A; a) + a\mathbb{N}_0$, with $a = \min(A)$. Hence,

$$
H_p(A; x) = \frac{1}{1-x^a}\sum_{m\in\mathrm{Ap}_p(A;a)} x^m\,. \tag{2}
$$

For three or more variables, it is not easy to obtain an explicit form of the p-Hilbert series. However, the p-Hilbert series may be explicitly given when the structure of the p-Apéry set is known. We give one of the simplest cases, though the expression of the p-Hilbert series often becomes very complicated.

For example, let $A := \{a, a+1, \dots, 2a-1\}$ for an integer with $a \ge 3$. Then its p-Apéry set is given as follows.

Lemma 2. *Let $a \ge 3$. For $p = 0$, we have*

$$\mathrm{Ap}_0(a, a+1, \dots, 2a-1) = \{0, a+1, a+2, \dots, 2a-1\}$$

and $1 \le p \le (a-1)/2$, we have

$$\mathrm{Ap}_p(a, a+1, \dots, 2a-1) = \{3a, \dots, 3a+2p-1, 2a+2p, \dots, 3a-1\}\,.$$

Therefore, by (2), when $p = 0$, we obtain

$$\begin{aligned} H_0(a, a+1, \dots, 2a-1; x) &= \frac{1}{1-x^a}(x^{a+1} + x^{a+2} + \cdots + x^{2a-1}) \\ &= \frac{x^{a+1}(1-x^{a-1})}{(1-x)(1-x^a)}\,. \end{aligned}$$

When $p > 0$, we obtain

$$\begin{aligned} H_p(a, a+1, \dots, 2a-1; x) &= \frac{1}{1-x^a}(x^{2a+2p} + x^{2a+2p+1} + \cdots + x^{3a+2p-1}) \\ &= \frac{x^{2a+2p}}{1-x}\,. \end{aligned}$$

This result looks simple, but the expression of the p-Hilbert series usually becomes very complicated because its structure of the corresponding p-Apéry set is uncertain or complicated. For example, concerning the sequence of consecutive odd integers $A := \{2a+1, 2a+3, \dots, 4a+3\}$ $(a \ge 1)$, no exact explicit form of the p-Apéry set has been found for general p.

5 p-symmetric semigroup

By arranging the elements $m_i^{(p)}$ $(0 \le i \le a_1 - 1)$ of the Apéry set in (1) in ascending order, let $\ell_0(p) < \ell_1(p) < \cdots < \ell_{a_1-1}(p)$. That is, the sequence $\ell_0(p)$,$\ell_1(p)$,$\dots$,$\ell_{a_1-1}(p)$ is the ascending permutation of $m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}$.

The p-numerical semigroup $S_p = S_p(A)$ is called *p-symmetric* if for all $x \in \mathbb{Z}\backslash S_p$, $\ell_0(p) + g_p(A) - x \in S_p$, where $\ell_0(p)$ is the least element of S_p, that is the *p-multiplicity* of S_p if $p \geq 1$; $\ell_0(p) = 0$ if $p = 0$. When $p = 0$, "0-symmetric" is "symmetric". If a p-symmetric numerical semigroup S_p further satisfies $\ell_0(p) = g_p(A) + 1 := c_p(A)$, which is called p-conductor, then S_p is called *p-completely-symmetric*.

From the definition, the following is obvious.

Proposition 2. *For a p-semigroup S_p ($p \geq 0$), the following conditions are equivalent.*

(i) *S_p is p-symmetric.*

(ii) $\#S_p \cap \{\ell_0(p), \dots, g_p(A)\} = \#(\mathbb{N}_0 \backslash S_p) \cap \{\ell_0(p), \dots, g_p(A)\} = \dfrac{g_p(A) - \ell_0(p) + 1}{2}$.

(iii) *If $x + y = \ell_0(p) + g_p(A)$, then exactly one of non-negative integers x and y belongs to S_p and another to $\mathbb{N}_0 \backslash S_p$.*

Example

When $A = \{4, 5, 6\}$, we get that

$$S_8 = \{36, 38, 40, 41, \mapsto\},$$
$$\mathbb{N}_0 \backslash S_8 = \{0, 1, \dots, 35, 37, 39\}.$$

Then we know that

$$g_8(A) = 39 \quad \text{and} \quad \mathrm{Ap}_8(A) = \{36 = m_0^{(8)}, 38 = m_2^{(8)}, 41 = m_1^{(8)}, 43 = m_3^{(8)}\}.$$

Hence, we see that

$$36 + 39 = 38 + 37 = 40 + 35 = 41 + 34 = \cdots = 75 + 0 = 76 + (-1) = \cdots .$$

Therefore, $S_8(A)$, where $A = \{4, 5, 6\}$, is 8-symmetric. In fact, among the elements in $\mathrm{Ap}_8(A)$, we can obtain

$$36 + 43 = 38 + 41 .$$

This fact is explained in the next lemma, which is a generalization of the result by Apéry [1].

Lemma 3. *For a non-negative integer p, $S_p = S_p(A)$ is p-symmetric if and only if $\ell_i(p) + \ell_{a-i-1}(p) = g_p(A) + \ell_0(p) + a$ ($i = 1, 2, \dots, \lfloor a/2 \rfloor$).*

If one element $m_i^{(p)}$ in $\mathrm{Ap}_p(A; a)$ with $a = \min(A)$ can extend such that $m_i^{(p)} \equiv i \pmod{a}$ for any i, Lemma 3 can be restated as follows. For simplicity, put $g = g_p(A)$ and $\ell = \ell_0(p)$.

Lemma 4. *For a non-negative integer p, S_p is p-symmetric if and only if $m_{(g+\ell+1)/2+j}(p) + m_{(g+\ell-1)/2+j}(p) = g_p + \ell + a$ ($j \in \mathbb{Z}$).*

From Lemma 3 or Lemma 4, we have a relation between p-Frobenius number $g_p(A)$ and p-Sylvester number $n_p(A)$.

Proposition 3. *For a non-negative integer p, $S_p = S_p(A)$ is p-symmetric if and only if*

$$n_p(A) = \frac{g_p(A) + \ell_0(p) + 1}{2}\,.$$

Let us consider the two variables' case. For any integer $n \in S_p(A)$ for $A = \{a, b\}$ with $\gcd(a, b) = 1$ and $a < b$, let x_0 be the largest integer x satisfying $n = ax + by$ $(y \ge 0)$. Then there exists the least non-negative integer y_0 such that $n = ax_0 + by_0$, which is called the *standard form* of the representation of n. Since $S_p(A) \subseteq \mathbb{N}_0 \subseteq \mathbb{Z}$ and $\mathbb{Z}$ is Euclidean domain, the standard form is unique.

Lemma 5. *Let $n = ax_0 + by_0$ be the standard form of n. Then*

(i) $0 \le y_0 \le a - 1$.

(ii) *For any integer $n \in S(A) = S_0(A)$, $n \in S_p(A)$ if and only if $x_0 \ge pb$.*

By Proposition 3 and Lemma 5, together with the formulas in Corollary 2, we can show the p-symmetric property for two variables.

Theorem 2. *For any non-negative integer p, $S_p(a, b)$ with $\gcd(a, b) = 1$ is p-symmetric.*

Proof. When $A = \{a, b\}$ with $\gcd(a, b) = 1$, by Lemma 5, the least integer whose number of representations in terms of a and b is more than p is pab. That is, the non-negative integral solutions of $ax + by = pab$ are $(x, y) = \bigl(jb, (p - j)a\bigr)$ $(j = 0, 1, \dots, p)$. Since $\ell_0(p) = pab$, by Proposition 3 together with the formulas in Corollary 2, we have

$$\begin{aligned}\frac{g_p(A) + \ell_0(p) + 1}{2} &= \frac{(p+1)ab - a - b + pab + 1}{2}\\ &= pab + \frac{(a-1)(b-1)}{2} = n_p(a, b)\,.\end{aligned}$$

□

At the end of this section, we consider a p-symmetric property in terms of the valuation. Let $R_0 := \mathbb{K}[[t^s | s \in S_p^{(0)}(A)]]$, $\bar{R}_0$ be the integral closure of R_0, f be the algebraic conductor from $t^{\ell_0(p)}R_0$ to $\bar{R}_0$, $c_p = g_p + 1$ (p-conductor). Since R_0 is the ring associated to a numerical semigroup $S_p^{(0)}(A)$, it is a discrete valuation ring with the valuation v.

Lemma 6. $f = \{x \in \bar{R}_0 | v(x) \ge c_p + \ell_0(p)\}$.

Proof. For any $x \in f$ and $r \in \bar{R}_0$, we have $rx \in t^{\ell_0(p)}R_0$. So $v(rx) = v(x) + v(r) \in v(t^{\ell_0(p)}R_0)$. For $x = t^{\ell_0(p)}x'$ we get $v(r) + v(t^{\ell_0(p)}) + v(x') \in v(t^{\ell_0(p)}R_0) = v(R_0) + v(t^{\ell_0(p)})$. By the arbitrariness of r and $v(r) \geq 0$, we have $v(x) \geq c_p + \ell_0(p)$, that is, $f \subseteq \{x \in \bar{R}_0 | v(x) \geq c_p + \ell_0(p)\}$.

For any $x \in \bar{R}_0$ and $v(x) \geq c_p + \ell_0(p)$, we have $v(x) = v(r)$ for some $r \in t^{\ell_0(p)}R \subseteq R$. Then for any $r' \in \bar{R}_0$, we have $v(xr') = v(x) + v(r') = v(r) + v(r') \geq c_p + \ell_0(p)$. By the definition of c_p, we have $xr' \in t^{\ell_0(p)}R_0$. So, $f \supseteq \{x \in \bar{R}_0 | v(x) \geq c_p + \ell_0(p)\}$. □

For simplicity, let d_1 and d_2 be the lengths of ideal of R_0/f and of R_0-submodule of $\bar{R}_0/f$, respectively, and d_3 be the number of elements in $S_p(A) \cap \{1, 2, \ldots c_p + \ell_0(p) - 1\}$.

Theorem 3. *$S_p(A)$ is p-symmetric if and only if $d_1 = \frac{d_2}{2}$.*

Proof. By Proposition 2 together with the facts that all the elements in $\{1, \ldots i_p - 1\}$ are in $\mathbb{N}_0 \backslash S_p(A)$ and $\{g_p + 1, \ldots \ell_0(p) + g_p - 1\}$ are all in $S_p(A)$, $S_p(A)$ is p-symmetric if and only if $d_3 = \frac{\ell_0(p) + g_p - 1}{2}$.

Consider the ideal chain $R_0 \supset R_1 \supset R_2 \cdots \supset R_{d_3} \supset f$, where $R_i = \{r \in R_0 | v(r) \geq v_i\}$ and $v_1 < v_2 < \cdots < v_{d_3}$ are the elements in $S_p(A) \cap \{1, 2, \ldots c_p + \ell_0(p) - 1\}$ arranged in ascending order. This sequence is maximal because if we adjoin an element $r \in R_0$ of value v_{i-1} to R_i, we get all of R_{i-1}. So, $d_1 = d_3 + 1$.

Similarly consider the maximal R_0-submodule chain of $\bar{R}_0/f$: $\bar{R}_0 = b_0 \supset b_1 \supset b_2 \cdots \supset b_{\ell_0(p)+g_p+1} = f$ where $b_i = \{r \in \bar{R}_0 | v(r) \geq i\}$. So we have $d_2 = \ell_0(p) + g_p + 1$. Hence, $S_p(A)$ is p-symmetric if and only if $d_1 - 1 = \frac{d_2 - 2}{2}$. □

5.1 p-pseudo-symmetric semigroup

For a non-negative integer p, let $S_p(A)$ be a p-numerical semigroup. $x \in \mathbb{Z}$ is called a *p-pseudo-Frobenius number* if $x \notin S_p(A)$ and $x + s - \ell_0(p) \in S_p(A)$ for all $s \in S_p(A) \backslash \{\ell_0(p)\}$, where $\ell_0(p)$ is the least element of $S_p(A)$, so is of $\mathrm{Ap}_p(A; a)$ with $a = \min(A)$. The set of p-pseudo-Frobenius numbers is denoted by $\mathrm{PF}_p(A)$. The *p-type* is denoted by $t_p(A) := \#\bigl(\mathrm{PF}_p(A)\bigr)$. Notice that the p-Frobenius number is given by $g_p(A) = \max\bigl(\mathrm{PF}_p(A)\bigr)$.

For $p \geq 0$, the p-numerical semigroup $S_p = S_p(A)$ is called *p-pseudo-symmetric* if for all $x \in \mathbb{Z} \backslash S_p$ with $x \neq \bigl(\ell_0(p) + g_p(A)\bigr)/2 \in \mathbb{Z}$, $\ell_0(p) + g_p(A) - x \in S_p$, where $\ell_0(p)$ is the least element of S_p. When $p = 0$, "0-pseudo-symmetry" is "pseudo-symmetry".

For simplicity, put the p-Frobenius number as $g := g_p(A)$ and the p-multiplicity as $\ell := \ell_0(p)$ ($p \geq 1$) with $\ell_0(0) = 0$. Denote the p-Apéry set by $\mathrm{Ap}_p(A; a)$ with $a = \min(A)$.

Theorem 4. *For a non-negative integer p, the following conditions are equivalent:*

(i) *$S_p = S_p(A)$ is p-pseudo-symmetric*

(ii)

$$m^{(p)}_{(g+\ell)/2+j}+m^{(p)}_{(g+\ell)/2-j}=g+\ell+\begin{cases}2a & \textit{if } j=0 \textit{ and } (g+\ell)/2\in\mathbb{N}_0\backslash S_p(A);\\ 0 & \textit{if } j=0 \textit{ and } (g+\ell)/2\in S_p(A);\\ a & \textit{if } j>0.\end{cases}$$

(iii) $n_p(A)=\dfrac{g+\ell}{2}+\begin{cases}1 & \textit{if } (g+\ell)/2\in\mathbb{N}_0\backslash S_p(A);\\ 0 & \textit{if } (g+\ell)/2\in S_p(A).\end{cases}$

Corollary 4. *Let $S_p(A)$ be a p-numerical semigroup. The following conditions are equivalent.*

(i) S_p *is p-symmetric.*

(ii) $\mathrm{PF}_p(A)=\{g_p(A)\}$ *with* $g_p(A)\not\equiv\ell_0(p)\pmod 2$.

(iii) $t_p(A)=1$ *with* $g_p(A)\not\equiv\ell_0(p)\pmod 2$.

Corollary 5. *Let $S_p(A)$ be a p-numerical semigroup. The following conditions are equivalent.*

(i) S_p *is p-pseudo-symmetric.*

(ii) $\mathrm{PF}_p(A)=\begin{cases}\{g_p(A),\big(g_p(A)+\ell_0(p)\big)/2\} & \textit{if } \big(g_p(A)+\ell_0(p)\big)/2\in\mathbb{N}_0\backslash S_p(A);\\ \{g_p(A)\} & \textit{if } \big(g_p(A)+\ell_0(p)\big)/2\in S_p(A).\end{cases}$

(iii) $t_p(A)=\begin{cases}2 & \textit{if } \big(g_p(A)+\ell_0(p)\big)/2\in\mathbb{N}_0\backslash S_p(A);\\ 1 & \textit{if } \big(g_p(A)+\ell_0(p)\big)/2\in S_p(A).\end{cases}$

For $a,b\in\mathbb{Z}$, define a partial order relation $a\le_{S_p}b$ (or $a\le_S b$ for short) as $b-a\in S_p$. The set of p-pseudo-Frobenius numbers $\mathrm{PF}_p(A)$ can be determined with this order relation in terms of the p-maximal gaps.

Proposition 4. *For a p-numerical semigroup $S_p=S_p(A)$, we have*

$$\mathrm{PF}_p(A)=\mathrm{Maximals}_{\le_S}(\mathbb{N}_0\backslash S_p)\,.$$

The set of p-pseudo-Frobenius numbers $\mathrm{PF}_p(A)$ can be also determined in terms of the p-Apéry set.

Proposition 5. *Let $S_p = S_p(A)$ be a p-numerical semigroup with $a = \min(A)$. Then for $n \in S_p$ we have*

$$\mathrm{PF}_p(A) = \{w - a | w \in \mathrm{Maximals}_{\le S} \mathrm{Ap}_p(A; a)\} .$$

At the end of this subsection, we mention a partially corresponding result to Theorem 3.

Theorem 5. *If $S_p(A)$ is p-pseudo-symmetric, then $2d_1 + 1 = d_2$.*

Proof. If $S_p(A)$ is p-pseudo-symmetric, then we have $2d_3 = \ell_0(p) + g_p - 2$.
Again, consider the maximal ideal chain $R_0 \supset R_1 \supset R_2 \cdots \supset R_{d_3} \supset f$ as in the proof of Theorem 3. Thus, we get $d_1 = d_3 + 1$. And consider the R_0-submodule chain of $\bar{R}_0/f$: $\bar{R}_0 = b_0 \supset b_1 \supset b_2 \cdots \supset b_{\ell_0(p)+g_p+1} = f$. We have $d_2 = \ell_0(p) + g_p + 1$. Hence, if $S_p(A)$ is p-pseudo-symmetric, then $2(d_1 - 1) = d_2 - 3$. $\square$

When is a p-numerical semigroup p-symmetric, and when p-pseudo-symmetric?
Let $A = \{6, 7, 17, 28\}$. S_0 is pseudo-symmetric. In addition,

S_1	=	{	24	30	31	34	35	36	37	38	40	$\mapsto$	}
G_1	=	{	39	33	32	29	28	27	26	25	23	$\mapsto$	}
S_2	=	{	41	42			47	48	49	51	$\mapsto$	}	
G_2	=	{	50		46	45	44	43		40	$\mapsto$	}	
S_3	=	{	48				54	55		58	$\mapsto$	}	
G_3	=	{	57	56	53	52	51	50	49	47	$\mapsto$	}	
S_4	=	{	65	66	68	$\mapsto$	}						
G_4	=	{	67		64	$\mapsto$	}						

Hence, S_1 is symmetric, S_2 and S_3 are not symmetric, and S_4 is pseudo-symmetric. By continuing, S_p is p-symmetric for $p = 1, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, \underline{11}, \underline{12}, \underline{13}, 15, \underline{17}, \underline{18}, \underline{21}, \underline{22}, \underline{24}, \dots$. (For underlined p's, they are completely-symmetric.) S_p is p-pseudo-symmetric for $p = 0, 4, 5, 19, 20, 23, 25, \dots$. S_p is neither p-symmetric nor p-pseudo-symmetric for $p = 2, 3, 14, 16, \dots$.

Conjecture 1. *If $\gcd(a_i, a_j) = 1$ $(i \neq j)$ for $A := \{a_1, a_2, \dots, a_k\}$, then S_p is p-completely-symmetric for enough large p.*

5.2 p-irreducible numerical semigroup

A numerical semigroup S is irreducible if it cannot be expressed as the intersection of two proper oversemigroups. A p-numerical semigroup $S_p = S_p(A)$ is called *p-irreducible* if it is either p-symmetric or p-pseudo-symmetric. It is known that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups.

By Theorem 2, we have the p-irreducible property for two variables.

Corollary 6. *For any non-negative integer p, $S_p(a,b)$ with $\gcd(a,b)=1$ is p-irreducible.*

Every p-numerical semigroup can be also expressed as a finite intersection of irreducible numerical semigroups ([2]).

Proposition 6. *For a non-negative integer p, let S_p be a p-numerical semigroup. Then, there exist finitely many irreducible numerical semigroups $\mathcal{S}_1,\dots,\mathcal{S}_r$ such that $S_p=\mathcal{S}_1\cup\cdots\cup\mathcal{S}_r$.*

Remark. It has not been known that for any fixed non-negative integer p, a p-numerical semigroup can be expressed as an intersection of all p-irreducible numerical semigroups.
Example. For $A=\{5,9,16\}$, we see that $S_2(A)=\{41,45,46,48,50,\mapsto\}$, which is neither (2-)symmetric nor (2-)pseudo-symmetric. But it can be expressed as an intersection of two (0−)numerical semigroups: $S_2(A)=S(A_1)\cup\mathcal{S}(A_2)$ with $A_1=\{41,43,45,46,48,50,\mapsto\}$ and $A_2=\{41,45,46,47,48,50,\mapsto\}$. Here both $S(A_1)$ and $S(A_2)$ are (0−)pseudo-symmetric. In addition, these 0-numerical semigroups are given by canonical forms:

$$S(A_1)=\langle 41,43,45,46,48,\underbrace{50,\dots,81},83,85\rangle\,,$$
$$S(A_2)=\langle 41,45,46,47,48,\underbrace{50,\dots,81},83,84,85\rangle\,.$$

6 Lipman semigroup and dual

For simplicity, set $S_p^\circ=S_p\cup\{0\}$. Then the *p-dual* of $S_p(A)$ is defined to be

$$B(S_p):=(S_p^\circ-S_p)=(S_p-S_p)\,.$$

Note that for ideals I and J, $I+J=\{i+j|i\in I,\,j\in J\}$, and $lI=\underbrace{I+\cdots+I}_{l}$.

The *p-Lipman semigroup* is defined to be $L_p(S)=L(S_p^\circ):=\cup_{h\ge1}(hS_p-hS_p)$. Then two kinds of chains of semigroups are obtained by duals and blow-ups, respectively:

$$S_p=:B_0(S_p)\subseteq B\big(B_0(S_p)\big)=:B_1(S_p)\subseteq\cdots\subseteq B\big(B_h(S_p)\big)=:B_{h+1}(S_p)\subseteq\cdots$$
$$S_p=:L_0(S_p)\subseteq L\big(L_0(S_p)\big)=:L_1(S_p)\subseteq\cdots\subseteq L\big(L_h(S_p)\big)=:L_{h+1}(S_p)\subseteq\cdots$$

If these sequences coincide, the semigroup S is called the *p-Arf numerical semigroup*. $\beta_p(S)=\beta(S_p)$ and $\lambda_p(S)=\lambda(S_p)$ denote the least integers such that $B_{\beta_p(S)}=L_{\lambda_p(S)}=\mathbb{N}_0$. Two chains play a role to characterize classes of certain local Noetherian domains.

The following is a generalization of the result in [3].

Proposition 7. *For a nonnegative integer p, let $S_p(A)$ be a p-numerical semigroup with canonical form $\langle A\rangle$. Then*

(i) $g_p\big(B_p(A)\big) = g_p\big(S_p(A)\big) - \ell_0(p)$*, where $\ell_0(p)$ is the least non-zero element of $S_p(A)$.*

(ii) $L_p(A) = \langle \ell_1(p) - \ell_0(p), \ell_2(p) - \ell_0(p), \dots, \ell_{a_1-1}(p) - \ell_0(p)\rangle$.

Example. For $A = \{5, 9, 16\}$, we see that

$$S_0(A) = \{0, 5, 9, 10, 14, 15, 16, 18, 19, 20, 21, 23, \mapsto\}\,,$$
$$\mathbb{N}_0\backslash S_0(A) = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 17, 22\}\,.$$

So, we see that $B_0(A) = (M - M) = \{0, 5, 9, 10, 11, 14, 15, 16, 18, \mapsto\}$. Hence, $g_0\big(S_0(A)\big) - \ell_0(0) = 22 - 5 = 17 = g_0\big(B_0(A)\big)$.

Since $2M = \{10, 14, 15, 18, 19, 20, 21, 23, \mapsto\}$, we get $(2M - 2M) = \{0, 5, 9, 10, 11, 13, \mapsto\}$. Since $3M = \{15, 19, 20, 23, \mapsto\}$, we get $(3M - 3M) = \{0, 4, 5, 8, \mapsto\}$, which is also equal to $(4M - 4M)$ because $4M = \{20, 24, 25, 28, \mapsto\}$. Thus, $L(S) = \langle 5, 9 - 5, 16 - 5\rangle = \langle 4, 5, 11\rangle = \{0, 4, 5, 8, \mapsto\} = \cup_{h=1}^{3}(hM - hM)$.

Since

$$G_1(A) = \{0, \dots, 24, 26, 27, 28, 29, 31, 33, 38\}\,,$$
$$S_1(A) = \{25, 30, 32, 34, 35, 36, 37, 39, \mapsto\}\,,$$

we see that $B_1(A) = \big(S_1(A) - S_1(A)\big) = \{0, 5, 7, 9, 10, 11, 12, 14, \mapsto\}$. Hence, $g_1\big(S_1(A)\big) - \ell_0(1) = 38 - 25 = 13 = g_1\big(B_1(A)\big)$. In addition, $B_1^{(2)} = \{0, 5, 7, 9, \mapsto\}$, $B_1^{(3)} = \{0, 2, 4, \mapsto\}$, $B_1^{(4)} = \{0, 2, \mapsto\}$ and $B_1^{(5)} = \mathbb{N}$. Since $2S_1(A) = \{50, 55, 57, 59, 60, 61, 62, 64, \mapsto\}$, we get $\big(2S_1(A) - 2S_1(A)\big) = \{0, 5, 7, 9, 10, 11, 12, 14, \mapsto\}$, which is the same for $\big(hS_1(A) - S_1(A)\big)$ when $h \ge 3$. Hence, $L_1(S) = B_1(S) = \{30-25, 32-25, 34-25, 35-25, 36-25, 37-25, 39-25, \mapsto\}$. Similarly, $L_1^{(2)}(S) = \{0, 2, 4, 5, 7, 9, \mapsto\}$, $L_1^{(3)}(S) = \{0, 2, \mapsto\}$ and $L_1^{(4)}(S) = \mathbb{N}_0$.

Since $S_2(A) = \{41, 45, 46, 48, 50, \mapsto\}$, we see that $B_2(A) = \{0, 5, 7, 9, \mapsto\}$, $B_2^{(2)}(A) = \{0, 2, 4, \mapsto\}$, $B_2(A)^{(3)} = \{0, 2, \mapsto\}$ and $B_2^{(4)}(A) = \mathbb{N}_0$. We have $g_2\big(S_2(A)\big) - \ell_0(2) = 49 - 41 = 8 = g_1\big(B_2(A)\big)$. In addition, $L_2(S) = \{0, 4, 5, 7, \mapsto\} = \langle 45 - 41, 46 - 41, 48 - 41, 50 - 41\rangle$ and $L_2^{(2)}(S) = \mathbb{N}_0$.

7 p-Arf numerical semigroup

A numerical semigroup S is called an *Arf numerical semigroup* if for every $x, y, z \in S$ such that $x \ge y \ge z$, then $x + y - z \in S$. Arf semigroups help to characterize Arf rings, an important class of rings in commutative algebra and algebraic geometry.

Proposition 8. *If $S(A)$ for $A = \{a, b\}$ with $\gcd(a, b) = 1$ is an Arf numerical semigroup, then $S_p(A)$ is also an Arf numerical semigroup.*

Proof. Assume that for every $x, y, x \in S_p$ such that $x \geq y \geq z$. We write x, y and z in the standard form as $x = ak_1 + bh_1$, $y = ak_2 + bh_2$ and $z = ak_3 + bh_3$. Then by Lemma 5, $k_i \geq pb$ $(i = 1, 2, 3)$. Put $x' = x - pb = a(k_1 - pb) + bh_1$, $y' = y - pb = a(k_y - pb) + bh_y$ and $z' = z - pb = a(k_3 - pb) + bh_3$. Since $k_1 - pb \geq 0$ and $h_i \geq 0$ $(i = 1, 2, 3)$, we get $x', y', z' \in S$ with $x' \geq y' \geq z$. As S is an Arf, we have $x' + y' - z' \in S$. Hence, $x' + y' - z'$ has the standard form $x' + y' - z' = ak_0 + bh_0$ with $k_0, h_0 \geq 0$. Then by $x + y - z = x' + y' - z' + pab = a(pb + k_0) + bh_0$ and Lemma 5, we have $x + y - z \in S_p$, so S_p is also an Arf. □

Proposition 9. *Let $S = S(A)$ be an Arf numerical semigroup with $a = \min(A)$. For a nonnegative integer p, let p-conductor be c_p, that is, $c_p = g_p(A) + 1$. $\overline{c_p}$ denotes the residue modulo a, that is $c_p \equiv \overline{c_p} \pmod{a}$ with $0 \leq \overline{c_p} < a$. Then, we have*

(i) $m_1^{(p)} = \begin{cases} c_p + 1 & \text{if } c_p \equiv 0 \pmod{a} \\ c_p - \overline{c_p} + a + 1 & \text{otherwise.} \end{cases}$

(ii) $m_{a-1}^{(p)} = c_p - \overline{c_p} + a - 1.$

Proof. As $a \nmid g_p(A)$, we see that $c_p \not\equiv 1 \pmod{a}$. Let $c_p \equiv 0 \pmod{a}$. Since $ah + 1 \notin S_p$ and $ah + a - 1 \notin S_p$ for $h < c_p/a$, we have $m_1^{(p)} = a(c_p/a) + 1 = c_p + 1$ and $m_{a-1}^{(p)} = a(c_p/a) + a - 1 = c_p + a - 1$.

Let $c_p \not\equiv 0 \pmod{a}$. Since $ah + 1 \notin S_p$ and $ah + a - 1 \notin S_p$ for $h < (c_p - \overline{c_p})/a$, we have $m_{a-1}^{(p)} = a\big((c_p - \overline{c_p})/a + 1\big) + 1 = c_p - \overline{c_p} + a + 1$ and $m_{a-1}^{(p)} = a\big((c_p - \overline{c_p})/a\big) + a - 1 = c_p - \overline{c_p} + a - 1$. □

For a nonnegative integer p and every $i \in \{0, 1, \dots\}$, there is a positive integer $k_i^{(p)}$ such that $m_i^{(p)} = k_i^{(p)} a + i$. Then $(k_0^{(p)}, k_1^{(p)}, \dots, k_{a-1}^{(p)})$ are called *p-Kunz coordinates* of S_p.

Proposition 10. *Let $S_p(A)$ be an Arf numerical semigroup with $a = \min(A)$, p-conductor c_p and p-Kunz coordinates $(k_0^{(p)}, k_1^{(p)}, \dots, k_{a-1}^{(p)})$. Then,*

$$k_1^{(p)} = \left\lceil \frac{c_p}{a} \right\rceil \quad \text{and} \quad k_{a-1}^{(p)} = \left\lfloor \frac{c_p}{a} \right\rfloor .$$

Proof. When $c_p \equiv 0 \pmod{a}$, by Lemma 9, we have $m_1^{(p)} = k_1^{(p)} a + 1 = c_p + 1$ and $m_{a-1}^{(p)} = k_{a-1}^{(p)} a + a - 1 = c_p + a - 1$. Hence, $k_1^{(p)} = k_{a-1}^{(p)} = c_p/a$. When $c_p \not\equiv 0 \pmod{a}$, by Lemma 9, we have $m_1^{(p)} = k_1^{(p)} a + 1 = c_p - \overline{c_p} + a + 1$ and $m_{a-1}^{(p)} = k_{a-1}^{(p)} a + a - 1 = c_p - \overline{c_p} + a - 1$. Hence, $k_1^{(p)} = (c_p - \overline{c_p})/a + 1$ and $k_{a-1}^{(p)} = (c_p - \overline{c_p})/a$. □

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