# AN INTEGRALITY OF CRITICAL VALUES OF THE RANKIN–SELBERG $L$-FUNCTION FOR $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$

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ABSTRACT. This article is a survey on the author's preprint [HMN], where the authors study an integrality of critical values of the Rankin–Selberg $L$-function for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ if the base field is a totally imaginary field.

## 1. Introduction

1.1. **Motivations and the main result.** This article is a report of the author's talk at the conference " Analytic and arithmetic aspects of automorphic representations ", which was held at RIMS, Kyoto University during 23th to 27th, January, 2023. The author's talk was based on the joint work with Takashi Hara and Tadashi Miyazaki ([HMN]), where the authors study an integrality of critical values of Rankin-Selberg $L$-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over totally imaginary fields.

Let us briefly recall the history of the study of the critical values of Rankin-Selberg $L$-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$. Manin ([Man72]) and Shimura ([Shi76], [Shi77]) started the study of the rationality of special values of Rankin-Selberg $L$-functions for $\mathrm{GL}_2 \times \mathrm{GL}_1$ over the rational number field. Following their works, Deligne ([Del79]) introduced the notion of critical values and proposed a general conjecture to understand their works as study of a rationality of critical values of $L$-functions attached to pure motives. It is widely believed that there exists a pure motive attached to an irreducible cohomological cuspidal automorphic representation of $\mathrm{GL}_n$. Hence we expect that the critical values of the Rankin-Selberg $L$-functions for them have an algebraic property, although the existence of pure motives attached to it is not yet known. Based on this motivation, Mahnkopf ([Mah05]) and Raghuram ([Rag10], [Rag16]) consider a generalization of Manin and Shimura's works to $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ ($n \geq 2$), by using the generalized modular symbol method due to Kazhdan-Mazur-Schmidt ([KMS00]). However, an unspecific constant still remains in their formula for critical values, which is expected to be non-zero as in [KMS00, page 98, Question]. It is proved that this unspecific constant is non-zoro by Sun ([Sun17]), and hence Sun's result implies that the generalized modular symbol method is non-trivial. Therefore it is natural to study further applications of the generalized modular symbol method, and the one of important applications should be the construction of $p$-adic $L$-functions for Rankin-Selberg $L$-functions beyond the study of the rationality of critical values.

Based on the work of [KMS00], Januszewski ([Jan]) constructed $p$-adic $L$-functions for Rankin-Selberg $L$-functions, but his interpolation formula still contains an unspecific constant. This unspecific constant prevent us from studying congruences between critical values at the different points, since the unspecific constants depend on the critical points. (We say critical points if the evaluation of the $L$-functions at the points is a critical value of the $L$-functions.) This kind of congruences is called Kummer (or Manin) congruences, which is one of the expected properties of $p$-adic $L$-functions. Of course, the existence of Kummer congruences implicitly implies that the critical values must be integral in an appropriate sense. The existence of the unspecific constant also makes difficult to define an integrality of critical values and to formulate Kummer congruences.

In this situation, we decided to calculate explicitly this unspecific constant for the further study of critical values of Rankin-Selberg $L$-functions. We also consider an integrality for these critical values. As a result, if the base field is totally imaginary, we obtain an explicit formula for the unspecific constant (see (4.1)), which was always an obstruction to discuss rationality in the previous works, and we also prove an integrality of critical values with respect to an appropriately normalized period without any unspecific constant. See Theorem 4.1 for the result.

1.2. **Strategy for the study.** The ingredients of our study of the integrality are as follows:

(i) Construction of an appropriate period for automorphic representations;
(ii) Cohomological interpretation of the Rankin-Selberg zeta integrals;
(iii) An explicit formula for Rankin-Selberg zeta integrals.

The first and the most fundamental step is to construct periods for cohomological irreducible cuspidal automorphic representations in an appropriate way for the formulation of the integrality of critical values. According to the philosophy in [Del79], the choice of periods corresponds to the choice of the lattices on cohomology groups. Mahnkopf ([Mah05]) and Raghuram-Shahidi ([RS08]) defined periods attached to cohomological irreducible cuspidal automorphic representations, which are called (Betti-)Whittaker periods, by making choices of Whittaker vectors so that an unspecific constant is non-zero, and rational models for a local systems on locally symmetric spaces. An implicit choice of a Whittaker vector makes impossible to calculate the (namely archimedean) local Rankin-Selberg zeta integral and this becomes one of reasons why an unspecific constant appears. Also we can only formulate an algebraicity of critical values as long as one uses a rational model of local systems. In our work ([HMN]), we found appropriate choices of Whittaker vectors which have a good behavior under the local Rankin-Selberg zeta integrals and also appropriate choices of lattices of local systems. These choices enable us to formulate an integrality of critical values.

The second step is to study the cohomological interpretation of Rankin-Selberg zeta integrals, since the periods are defined in terms of cohomology groups. A general strategy to give such a cohomological interpretation is called the generalized modular symbol method in [KMS00], which we have already mentioned. However the explicit relation between cohomological method and Rankin-Selberg zeta integrals are clarified in few cases: see [Hid94] for the case of $\mathrm{GL}_2 \times \mathrm{GL}_1$ over general number fields and [HN21] for the case of $\mathrm{GL}_3 \times \mathrm{GL}_2$ over the rational number field. In [HMN], we give distinguished cohomology classes, which are called Eichler-Shimura class, and we write down the Rankin-Selberg zeta integrals by using these cohomology classes in an explicit manner. This enables us to discuss the integrality of the Rankin-Selberg zeta integrals.

The third step to give an explicit formula of Rankin-Selberg zeta integrals is basically done due to Ishii-Miyazaki ([IM22]). Hence our main considerations in [HMN] are (i) and (ii).

1.3. **Outline of this article.** As we have already mentioned, the most fundamental problem is to define periods in an appropriate manner. This is done by introducing a lattice for the local systems and by making a choice of a distinguished Whittaker vector. Hence, in this survey article, we concentrate to give a brief discussion about these two subjects. In Section 2, we introduce a model of finite dimensional representations which gives an appropriate description of our local systems. In particular, the notion of Gel'fand-Tsetlin basis will be a fundamental tool, and hence we describe it in some detail. We also recall the definition of local systems and critical values in Section 2. In Section 3, we introduce the notion of Whittaker periods. To state the definition of Whittaker periods precisely, we will explain an explicit description of lattices in cohomolgy groups of local systems, distinguished Whittaker vectors and Eichler-Shimura classes there. The main theorem about an explicit formula for critical values and their integrality is stated in Section 4.

1.4. **Basic notations.** Throughout this article, $F$ always denotes a totally imaginary number field. Let $I_F$ be the set of embeddings of $F$ into $\mathbf{C}$ and $\Sigma_F$ the set of places of $F$. Write the complex conjugate of $* \in \mathbf{C}$ as $\overline{*}$ and denote by $\overline{v}$ the complex conjugate of $v \in \Sigma_{F,\infty}$. Then we identify (non-canonically) $I_F$ with the $\{v, \overline{v} \mid v \in \Sigma_{F,\infty}\}$, and hence regard $\Sigma_{F,\infty}$ as a subset of $I_F$. We also denote by $\Sigma_{F,\infty}$ (resp. $\Sigma_{F,\mathrm{fin}}$) the set of infinite (resp. finite) places of $F$. Define $F_{\mathbf{A}}$ to be the ring of adèles of $F$ and $F_{\mathbf{A},\infty}$ (resp. $F_{\mathbf{A},\mathrm{fin}}$) denotes the ring of infinite (resp. finite) adèles of $F$. Similarly, for an adèlic object $X_{\mathbf{A}}$, the symbol $X_{\mathbf{A},\infty}$ (resp. $X_{\mathbf{A},\mathrm{fin}}, X_v$ $(v \in \Sigma_F)$) denotes the infinite part (resp. finite part, $v$-component) of $X_{\mathbf{A}}$. We also let $X_S = \prod_{v\in S} X_v$ for a subset $S \subset \Sigma_F$.

For cohomological irreducible cuspidal automorphic representations $\pi^{(n)}$ of $\mathrm{GL}_n(F_{\mathbf{A}})$ and $\pi^{(n-1)}$ of $\mathrm{GL}_{n-1}(F_{\mathbf{A}})$, let $L(s, \pi^{(n)} \times \pi^{(n-1)}) = \prod_{v\in\Sigma_F} L_v(s, \pi_v^{(n)} \times \pi_v^{(n-1)})$ denote the (complete) Rankin–Selberg $L$-function of $\pi^{(n)}$ and $\pi^{(n-1)}$.

Let $\Lambda_n = \{(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ and $\mu = (\mu_1, \dots, \mu_{n-1}) \in \Lambda_{n-1}$, we write $\mu \preceq \lambda$ if the inequalities $\lambda_n \leq \mu_{n-1} \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1$ are satisfied. For $\boldsymbol{\lambda} = (\lambda_\sigma)_{\sigma\in I_F} \in \Lambda_n^{I_F}$ and $\boldsymbol{\mu} = (\mu_\sigma)_{\sigma\in I_F} \in \Lambda_{n-1}^{I_F}$, we define $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$ by using the multi-index notation, that is, $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$ holds if and only if $\mu_\sigma \preceq \lambda_\sigma$ holds for each $\sigma \in I_F$. For an integer $m \in \mathbf{Z}$, we abbreviate $(m, m, \dots, m) \in \Lambda_n$ to $m \in \Lambda_n$, since no confusion likely occurs.

## 2. Cohomological representations

In this section, we prepare basic notions on cohomological irreducible cuspidal automorphic representations of $\mathrm{GL}_n(F_{\mathbf{A}})$. The automorphic representation is defined to be cohomological if it appears in a cohomology group of the associated locally symmetric space with coefficients in a certain local systems. The local system is defined by using an irreducible finite dimensional representation, and hence it is useful for the further study of automorphic representations to write down the finite dimensional representation in an explicit way to handle. In [HMN], we adopt Gel'fand-Tsetlin basis to study finite dimensional representations, which is a fundamental tool in our study. Hence we explain it in Section 2.1 in certain details. We also prepare some notions about cohomology groups of local systems in Section 2.2 and their relation to the critical values in Section 2.3.

### 2.1. Gel'fand-Tsetlin basis.

2.1.1. *Description of the action.* Let $\lambda \in \Lambda_n$. Denote by $(\tau_\lambda, V_\lambda)$ the irreducible holomorphic finite dimensional representation of $\mathrm{GL}_n(\mathbf{C})$ of highest weight $\lambda$. Consider $V_\lambda$ as an hermitian space by fixing a $\mathrm{U}(n)$-invariant hermitian pairing on $V_\lambda$. Here we will describe $(\tau_\lambda, V_\lambda)$ in an explicit manner by using a distinguished basis of $V_\lambda$. Such a basis is called Gel'fand-Tsetlin basis of $V_\lambda$, which is a key ingredient of our study in [HMN].

Consider a finite set $\mathrm{G}(\lambda)$ consisting of the triangle matrices

$$M = (m_{i,j})_{1\leq i\leq j\leq n} = \begin{pmatrix} m_{1,n} \quad m_{2,n} \quad \dots \quad m_{n,n} \\ m_{1,n-1} \quad \cdots \quad m_{n-1,n-1} \\ \cdots \quad \cdots \quad \cdots \\ m_{1,2} \quad m_{2,2} \\ m_{1,1} \end{pmatrix} = \begin{pmatrix} m^{(n)} \\ \vdots \\ m^{(1)} \end{pmatrix}$$

with the following conditions: $m_{i,j} \in \mathbf{Z}$, $m^{(n)} = \lambda$ and $m^{(j)} \preceq m^{(j+1)}$ $(1 \leq j \leq n-1)$. For each $M = (m_{i,j})_{1\leq i<j\leq n} \in \mathrm{G}(\lambda)$, define the weight $\gamma^M = (\gamma_1^M, \dots, \gamma_n^M)$ of $M$ to be

$$\gamma_1^M = m_{1,1}, \qquad \gamma_j^M = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1} \quad (2 \leq j \leq n).$$

Gel'fand and Tsetlin construct a basis of $V_\lambda$ which is indexed by elements in $\mathrm{G}(\lambda)$, which gives an explicit description of the action of $\mathfrak{gl}_n$ on $V_\lambda$ via $\tau_\lambda$:

**Proposition 2.1** ([IM22, Section 2.5]). *There exists an orthonormal basis $\{\zeta_M\}_{M\in\mathrm{G}(\lambda)}$ of $V_\lambda$ with the following formulas on the $\mathfrak{gl}_n$-action*

$$\begin{aligned}
\tau_\lambda(E_{k,k})\zeta_M &= \gamma_k^M\zeta_M && (1\le k\le n),\\
\tau_\lambda(E_{j,j+1})\zeta_M &= \sum_{\substack{1\le i\le j\\ M+\Delta_{i,j}\in\mathrm{G}(\lambda)}} \mathrm{a}_{i,j}(M)\zeta_{M+\Delta_{i,j}} && (1\le j\le n-1),\\
\tau_\lambda(E_{j+1,j})\zeta_M &= \sum_{\substack{1\le i\le j\\ M+\Delta^\vee_{i,j}\in\mathrm{G}(\lambda)}} \mathrm{a}_{i,j}(M^\vee)\zeta_{M+\Delta^\vee_{i,j}} && (1\le j\le n-1).
\end{aligned}$$

*Here $\Delta_{i,j}$ is the integral triangular array of size $n$ with $1$ at the $(i,j)$-th entry and $0$ at the other entries, and $\mathrm{a}_{i,j}(M)$ and $M^\vee=(m^\vee_{i,j})_{1\le i\le j\le n}$ are defined to be $(M=(m_{i,j})_{1\le i\le j\le n})$*

$$\begin{aligned}
\mathrm{a}_{i,j}(M) &:= \left|\frac{\prod_{h=1}^{j+1}(m_{h,j+1}-m_{i,j}-h+i)\prod_{h=1}^{j-1}(m_{h,j-1}-m_{i,j}-h+i-1)}{\prod_{1\le h\le j,\ h\ne i}(m_{h,j}-m_{i,j}-h+i)(m_{h,j}-m_{i,j}-h+i-1)}\right|^{\frac12},\\
m^\vee_{i,j} &:= -m_{j+1-i,j}.
\end{aligned}$$

Let $H(\lambda)=\begin{pmatrix}\lambda_1 & \lambda_2 & & \dots & \lambda_n\\ & \lambda_1 & \dots & \lambda_{n-1} & \\ & & \dots\ \dots & & \\ & & \lambda_1 & & \end{pmatrix}$. Then we note that $\zeta_{H(\lambda)}$ is the highest weight vector.

2.1.2. *An integral structure.* One of our motivations to describe $(\tau_\lambda, V_\lambda)$ in Section 2.1.1 is to give a rational and integral structure on cuspidal cohomology groups, which is one of key ingredients for the definition of periods attached to cohomological irreducible cuspidal automorphic representations. Ishii-Miyazaki ([IM22]) introduced such a rational structure on $V_\lambda$ via Gel'fand-Tsetlin basis aiming for application to the study of the rationality of critical values. Furthermore we can actually consider an integral structure by looking their rational structure carefully, which we describe below.

As in [IM22, Section 2.5], we set

$$\xi_M := \sqrt{\mathrm{r}(M)}\zeta_M \qquad (M=(m_{i,j})_{1\le i\le j\le n}\in\mathrm{G}(\lambda)),$$

where $\mathrm{r}(M)$ is the rational constant defined by

$$\mathrm{r}(M)=\prod_{1\le i\le j<k\le n}\frac{(m_{i,k}-m_{j,k-1}-i+j)!(m_{i,k-1}-m_{j+1,k}-i+j)!}{(m_{i,k-1}-m_{j,k-1}-i+j)!(m_{i,k}-m_{j+1,k}-i+j)!}.$$

By re-writing the action of $\mathfrak{gl}_n$ on $V_\lambda$ via $\xi_M$ ($M\in\mathrm{G}(\lambda)$), Ishii-Miyazaki obtain an explicit formula of the action ([IM22, (2.19), (2.20), (2.21)]). By their explicit description, we can see that it actually gives an integral structure under certain condition as follows:

**Corollary 2.2.** *Let $\mathcal{A}$ be a subring of $\mathbf{C}$ satisfying $\{(\lambda_1-\lambda_n+n-3)!\}^{-1}\in\mathcal{A}$ if $n\ge 3$. Define an $\mathcal{A}$-module $V_\lambda(\mathcal{A})$ to be*

$$V_\lambda(\mathcal{A})=\bigoplus_{M\in\mathrm{G}(\lambda)}\mathcal{A}\,\xi_M.$$

*Then $V_\lambda(\mathcal{A})$ is closed under the action of $\mathrm{GL}_n(\mathcal{A})$ via $\tau_\lambda$.*

**Remark 2.3.** (i) In [HMN], $\mathcal{A}$ is not necessary to be a subring of $\mathbf{C}$. We introduce a $\mathrm{GL}_n(\mathcal{A})$-module $(\tau_\lambda, V_\lambda(\mathcal{A}))$ for an integral domain $\mathcal{A}$ of the characteristic zero by realizing $(\tau_\lambda, V_\lambda(\mathcal{A}))$ as a certain subspace of polynomial functions on $\mathrm{M}_n(\mathcal{A})$ with the regular representation of $\mathrm{GL}_n(\mathcal{A})$, which coincides with $(\tau_\lambda, V_\lambda(\mathcal{A}))$ in Corollary 2.2 if $\mathcal{A}$ is a subring of $\mathbf{C}$. So we will use the notion $(\tau_\lambda, V_\lambda(\mathcal{A}))$ for a general integral domain $\mathcal{A}$ of the characteristic zero satisfying the condition in Corollary 2.2 in the subsequent arguments.

(ii) The idea to introduce an explicit model of the finite dimensional representation for the study of cohomological automorphic representations and their critical values can be found in [Hid94] in the case of $\mathrm{GL}_2$ over general number fields. In this sense, introducing an integral structure as in Corollary 2.2 can be considered as a generalization of Hida's strategy for $\mathrm{GL}_n$ over totaly imaginary fields. We also note that a similar kind of study can be found in [HN21] in the case of $\mathrm{GL}_3$ over the rational number field.

2.1.3. *Branching rule.* Besides an integral structure (Corollary 2.2), we introduce one more distinguished property of Gel'fand-Tsetlin basis of $V_\lambda$. Let $\lambda \in \Lambda_n$, and put $\Xi^+(\lambda) = \{\mu \in \Lambda_{n-1} \mid \mu \preceq \lambda\}$. Consider $\mathrm{GL}_{n-1}$ as a subgroup of $\mathrm{GL}_n$ via the diagonal embedding $\iota_n : \mathrm{GL}_{n-1} \to \mathrm{GL}_n; g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Then the irreducible decomposition of $V_\lambda$ as a representation of $\mathrm{GL}_{n-1}(\mathbf{C})$, which is called the branching rule for $(\mathrm{GL}_n(\mathbf{C}), \mathrm{GL}_{n-1}(\mathbf{C}))$, is given as follows ([GW09, Theorem 8.1.1]):

$$V_\lambda = \bigoplus_{\mu \in \Xi^+(\lambda)} V_{\lambda,\mu}, \qquad V_{\lambda,\mu} \cong V_\mu. \tag{2.1}$$

The Gel'fand-Tsetlin basis describes the above branching rule in an explicit manner. We prepare some notation to introduce such a description according to [IM22, Section 2.5]. For each $M = {}^{\mathrm{t}}\left(m^{(n)}, \dots, m^{(1)}\right) \in \mathrm{G}(\lambda)$, define $\widehat{M}$ to be ${}^{\mathrm{t}}\left(m^{(n-1)}, \dots, m^{(1)}\right)$ and let $\mathrm{G}(\lambda;\mu)$ $(\mu \in \Xi[\lambda])$ be the set consisting of $M \in \mathrm{G}(\lambda)$ such that $\widehat{M} \in \mathrm{G}(\mu)$. Then the following map gives a $\mathrm{GL}_{n-1}(\mathbf{C})$-isomorphism:

$$V_{\lambda,\mu} = \bigoplus_{M \in \mathrm{G}(\lambda;\mu)} \mathbf{C}\xi_M \xrightarrow{\sim} V_\mu; \xi_M \longmapsto \xi_{\widehat{M}}. \tag{2.2}$$

For later use, we prepare a notation for the inverse of (2.2). For $\mu \in \Xi^+(\lambda)$ and $M \in \mathrm{G}(\mu)$, we define $M[\lambda] \in \mathrm{G}(\lambda;\mu)$ to be

$$M[\lambda] = \begin{pmatrix} \lambda \\ M \end{pmatrix}.$$

Then the inverse of (2.2) is given by $V_\mu \xrightarrow{\sim} V_{\lambda,\mu}; \xi_M \to \xi_{M[\lambda]}$ $(M \in \mathrm{G}(\mu))$. An important observation is that these maps are integrally defined with respect to the integral structure defined in Corollary 2.2.

**Remark 2.4.** For the study of the rationality and the integrality of critical values of Rankin-Selberg $L$-functions, we interpret Rankin-Selberg zeta integrals as cup products of distinguished cohomology classes, which we introduce in Section 3.2.3, of locally symmetric spaces. To describe cup products in an explicit way, we will use the branching rule (2.1). Hence a rational (resp. integral) model of branching rule is one of key ingredients for the study of rationality (resp. integrality) of critical values.

The multiplicity one of the branching rule and the Schur's lemma immediately show that the branching rule is defined over a certain number field. This rationality for the branching rule is used in the previous works as in [Mah05], [Rag10] and [Rag16]. However this abstract construction of the rational model of the branching rule becomes one of reasons of an ambiguity of an unspecific constant in the formula for the critical values and this also causes the difficulty of the study of the cohomological interpretation of the Rankin-Selberg zeta integrals.

We also note that, in [Hid94] and [HN21], the branching rule is explicitly described by giving finite dimensional representations explicit models as subspaces of polynomial functions. This description is one of key ingredient for the cohomological interpretation of the Rankin-Selberg zeta integrals in an explicit way in [Hid94] and [HN21]. In [HMN], we consider an analogue of these works by using an explicit description (2.2) of the branching rule.

2.2. **Cohomology of local systems.** For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$, define the contragredient $\lambda^\vee$ of $\lambda$ to be $\lambda^\vee = (-\lambda_n, \dots, -\lambda_1) \in \Lambda_n$. Let $\boldsymbol{\lambda} = (\lambda_\sigma)_{\sigma \in I_F} \in \Lambda_n^{I_F}$ and define $\boldsymbol{\lambda}^\vee$ to be $(\lambda_\sigma^\vee)_{\sigma \in I_F} \in \Lambda_n^{I_F}$. Denote by $F_{\mathrm{nc}}$ the normal closure of $F$ in $\mathbf{C}$ and take a subfield $A$ in $\mathbf{C}$ containing $F_{\mathrm{nc}}$. Let $\widetilde{V}(\boldsymbol{\lambda}^\vee)_A = \bigotimes_{\sigma \in I_F} V_{\lambda_\sigma^\vee}(A)$ and define a representation $(\tau_{\boldsymbol{\lambda}^\vee}, \widetilde{V}(\boldsymbol{\lambda}^\vee)_A)$ of $\mathrm{GL}_n(F)$ by

$$\tau_{\boldsymbol{\lambda}^\vee}(g)\left(\otimes_{\sigma \in I_F} \mathrm{v}_\sigma\right) = \otimes_{\sigma \in I_F} \tau_{\lambda_\sigma^\vee}(\sigma(g))\,(\mathrm{v}_\sigma) \qquad (g \in \mathrm{GL}_n(F), \mathrm{v}_\sigma \in V_{\lambda_\sigma^\vee}).$$

Let $\mathcal{K}$ be an open compact subgroup of $\mathrm{GL}_n(F_{\mathbf{A},\mathrm{fin}})$ and put $\widetilde{K}_n = \prod_{v \in \Sigma_{F,\infty}} \mathbf{C}^\times \mathrm{U}(n)$. Define $Y_{\mathcal{K}}^{(n)}$ to be

$$Y_{\mathcal{K}}^{(n)} = \mathrm{GL}_n(F) \backslash \mathrm{GL}_n(F_{\mathbf{A}}) / \widetilde{K}_n \mathcal{K}.$$

Consider a diagonal left action of $\mathrm{GL}_n(F)$ on the direct product $\mathrm{GL}_n(F_{\mathbf{A}})/\widetilde{K}_n\mathcal{K} \times \widetilde{V}(\boldsymbol{\lambda})_A$. Then let $\widetilde{\mathcal{V}}(\boldsymbol{\lambda})_A$ be the local system on $Y_{\mathcal{K}}^{(n)}$ which is defined to be the sheaf of locally constant sections of the following first projection:

$$\mathrm{GL}_n(F) \backslash \left( \mathrm{GL}_n(F_{\mathbf{A}})/\widetilde{K}_n\mathcal{K} \times \widetilde{V}(\boldsymbol{\lambda})_A \right) \longrightarrow Y_{\mathcal{K}}^{(n)}.$$

Let $\pi^{(n)}$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(F_{\mathbf{A}})$. We recall that $\pi^{(n)}$ is said to be cohomological, if there exists an open compact subgroup $\mathcal{K}$ of $\mathrm{GL}_n(F_{\mathbf{A},\mathrm{fin}})$ and $\boldsymbol{\lambda} \in \Lambda_n^{I_F}$ such that the $\pi_{\mathrm{fin}}^{(n)}$-isotropic part of the cuspidal cohomology group $H^*_{\mathrm{cusp}}(Y_{\mathcal{K}}^{(n)}, \widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$ is non-trivial for some degree $*$. We call $\boldsymbol{\lambda} \in \Lambda_n^{I_F}$ the highest weight associated with $\pi^{(n)}$. If $\pi^{(n)}$ is cohomological, then the range of the degree $*$ with the non-trivial cohomology groups are known to be $\mathrm{b}_{n,F} := \sum_{v \in \Sigma_{F,\infty}} \frac{n(n-1)}{2} \leq * \leq \sum_{v \in \Sigma_{F,\infty}} (\frac{n(n-1)}{2} + n - 1)$. Here we note that the weight $\boldsymbol{\lambda}$ is uniquely determined by the Langlands parameters of the archimedean part $\pi_\infty^{(n)}$ of $\pi^{(n)}$ and it is known that $\boldsymbol{\lambda}$ satisfies the following purity condition:

**(Pur):** there exists an integer $\mathsf{w} \in \mathbf{Z}$ such that $\lambda_\sigma - \lambda_{\overline{\sigma}}^\vee = (\mathsf{w}, \mathsf{w}, \dots, \mathsf{w})$ holds for each $\sigma \in I_F$. Here $\overline{\sigma} \in I_F$ denotes the complex conjugate of $\sigma \in I_F$.

Let us call $\mathsf{w}$ the purity weight of $\boldsymbol{\lambda}$.

2.3. **Critical values.** In this subsection, we prepare some notation and facts about critical values of the Rankin-Selberg $L$-function $L(s, \pi^{(n)} \times \pi^{(n-1)})$. We write the infinite part of $L(s, \pi^{(n)} \times \pi^{(n-1)})$ as $L_\infty(s, \pi_\infty^{(n)} \times \pi_\infty^{(n-1)}) = \prod_{v \in \Sigma_{F,\infty}} L_v(s, \pi_v^{(n)} \times \pi_v^{(n-1)})$. Define a half-integer $\frac{1}{2} + m \in \frac{1}{2} + \mathbf{Z}$ to be a critical point of $L(s, \pi^{(n)} \times \pi^{(n-1)})$ if neither $L_\infty(s, \pi_\infty^{(n)} \times \pi_\infty^{(n-1)})$ nor $L_\infty(1 - s, \pi_\infty^{(n),\vee} \times \pi_\infty^{(n-1),\vee})$ has a pole at $s = \frac{1}{2} + m$. Here $*^\vee$ denotes the contragredient representation of $*$.

Let $\boldsymbol{\mu} \in \Lambda_{n-1}$ be the highest weight associated with $\pi^{(n-1)}$. We always suppose the following condition, which is necessary to apply the generalized modular symbol method due to Kazhdan-Mazar-Schmidt ([KMS00]) for the study of critical values:

- There exists an integer $m_0 \in \mathbf{Z}$ such that $\boldsymbol{\lambda}^\vee \succeq \boldsymbol{\mu} + m_0 \mathbf{1}$.

Here we put $\mathbf{1} = (1)_{\tau \in I_F} \in \Lambda_{n-1}^{I_F}$. Under this condition, we find the following proposition, which is firstly found by Kasten-Schmidt ([KS13, Theorem 2.3]) if the base field is the rational number field:

**Proposition 2.5** ([Rag16, Theorem 2.21])**.** Let $m \in \mathbf{Z}$. Then a half-integer $\frac{1}{2}+m$ is a critical point of $L(s, \pi^{(n)} \times \pi^{(n-1)})$ if and only if $m$ satisfies that $\boldsymbol{\lambda}^\vee \succeq \boldsymbol{\mu} + m\mathbf{1}$.

Proposition 2.5 relates the study of critical values of Rankin-Selberg $L$-functions with the study of the branching rule (2.1), which is explicitly described by the Gel'fand-Tsetlin basis as in (2.2). This is the main reason why we use Gel'fand-Tsetlin basis for the study of critical values, which also describes an integral structure of cuspidal cohomology groups in an explicit way due to Corollary 2.2, and hence we can discuss an integrality of critical values. (See also Remark 3.2.)

## 3. Whittaker periods

In this section, we introduce the notion of Whittaker periods for cohomological irreducible cuspidal automorphic representations of $\mathrm{GL}_n(F_{\mathbf{A}})$. According to the philosophy of Deligne in [Del79], we define the Whittaker period of $\pi^{(n)}$ as a ratio of two algebraic structures associated with $\pi^{(n)}$. Such a definition is introduced in [Mah05], [RS08], and an rationality of critical values $L(\frac{1}{2}+m, \pi^{(n)} \times \pi^{(n-1)})$ with respect to these Whittaker periods is discussed in [Rag10], [Rag16]. In [HMN], we basically follow their formulation, but we also discuss an integral properties of these critical values by using certain integral structure associated with $\pi^{(n)}$ and distinguished Whittaker vectors. We introduce such integral structures on cuspidal cohomology groups in Section 3.1, and we give distinguished elements in Whittaker model of $\pi^{(n)}$ in Section 3.2. By using these data, we define the Whittaker period associated with $\pi^{(n)}$ up to multiplication by $p$-adic units in Secton 3.3, which enables us to discuss an integrality of critical values.

3.1. **Integral structure of cuspidal cohomology groups.** In this subsection, we prepare integral structure on the cuspidal cohomology group of $Y_{\mathcal{K}}^{(n)}$.

Let $p$ be a prime number and fix an isomorphism $\boldsymbol{i} : \mathbf{C} \to \mathbf{C}_p$ as fields. Recall that $F_{\mathrm{nc}}$ is the normal closure of $F$ in $\mathbf{C}$, and let $\mathfrak{r}_{F_{\mathrm{nc}}}$ be the ring of integer of $F_{\mathrm{nc}}$. Define $\mathcal{O}_{\mathrm{nc}}$ to be the closure of $\iota_p(\mathfrak{r}_{F_{\mathrm{nc}}})$ in $\mathbf{C}_p$, where $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ is the embedding induced by the fixed isomorphism $\boldsymbol{i} : \mathbf{C} \to \mathbf{C}_p$. Let $\mathcal{A}$ be a subring of $\mathbf{C}_p$ containing $\mathcal{O}_{\mathrm{nc}}$.

Let $\boldsymbol{\lambda} = (\lambda_\sigma)_{\sigma \in I_F} \in \Lambda_n^{I_F}$. If $n \geq 2$, suppose that $p$ satisfies

$$p > \max\{\lambda_{\sigma,1} - \lambda_{\sigma,n} + n - 2 \mid \sigma \in I_F\}. \tag{3.1}$$

Let $(\tau_{\lambda_\sigma}, V_{\lambda_\sigma}(\mathcal{A}))$ $(\sigma \in I_F)$ be the finite rank representation of $\mathrm{GL}_n(\mathcal{A})$ as in Corollary 2.2. Consider $\widetilde{V}(\boldsymbol{\lambda})_{\mathcal{A}}^{(p)} = \bigotimes_{\sigma \in I_F} V_{\lambda_\sigma}(\mathcal{A})$, and then we define an action $\tau_{\boldsymbol{\lambda}}^{(p)}$ of $\mathrm{GL}_n(\mathfrak{r}_F)$ on $\widetilde{V}(\boldsymbol{\lambda})_{\mathcal{A}}^{(p)}$ as follows:

$$\tau_{\boldsymbol{\lambda}}^{(p)}(g)\left(\bigotimes_{\sigma \in I_F} \mathrm{v}_\sigma\right) := \bigotimes_{\sigma \in I_F} \left(\tau_{\lambda_\sigma}(\boldsymbol{i} \circ \sigma(g))\mathrm{v}_\sigma\right) \qquad \text{for } g \in \mathrm{GL}_n(\mathfrak{r}_F).$$

We extend this action to the action of $\mathrm{GL}_n(\mathfrak{r}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)$ as follows. Define $I_{F,v}$ to be

$$I_{F,v} := \{\sigma \in I_F \mid v \text{ is induced by } \boldsymbol{i} \circ \sigma \colon F \hookrightarrow \mathbf{C}_p\}.$$

For $\sigma \in I_{F,v}$, let $\sigma_v$ denote the automorphism of $F_v$ induced by $\boldsymbol{i} \circ \sigma$. Then define an action of $\mathrm{GL}_n(\mathfrak{r}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)$ on $\widetilde{V}(\boldsymbol{\lambda})_{\mathcal{A}}^{(p)}$ by

$$\tau_{\boldsymbol{\lambda}}^{(p)}\big((g_v)_{v|(p)}\big)\left(\bigotimes_{v|(p)} \bigotimes_{\sigma \in I_{F,v}} \mathrm{v}_\sigma\right) = \bigotimes_{v|(p)} \bigotimes_{\sigma \in I_{F,v}} \tau_{\lambda_\sigma}(\sigma_v(g_v))\mathrm{v}_\sigma \quad \text{for } (g_v)_{v|(p)} \in \mathrm{GL}_n(\mathfrak{r}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p).$$

If $\mathcal{A}$ is a field, then we also define the action of $\mathrm{GL}_n(F\otimes_{\mathbf{Q}}\mathbf{Q}_p)$ on $\widetilde{V}(\boldsymbol{\lambda})^{(p)}_{\mathcal{A}}$ in the same way.

Let $\mathcal{K}$ be an open compact subgroup of $\mathrm{GL}_n(F_{\mathbf{A},\mathrm{fin}})$ so that $\mathcal{K}_p$ is a subgroup of $\mathrm{GL}_n(\mathfrak{r}_F\otimes_{\mathbf{Z}}\mathbf{Z}_p)$. Define a right action of $\mathcal{K}$ on the direct product $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(F_{\mathbf{A}})/\widetilde{K}_n\times\widetilde{V}(\boldsymbol{\lambda})^{(p)}_{\mathcal{A}}$ to be $([g],\mathbf{v}_{[g]})\cdot u=\left([gu],\tau^{(p)}_{\boldsymbol{\lambda}}(u^{-1})\mathbf{v}_{[g]}\right)$ ($[g]\in\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(F_{\mathbf{A}})/\widetilde{K}_n,\mathbf{v}_{[g]}\in\widetilde{V}(\boldsymbol{\lambda})^{(p)}_{\mathcal{A}}$). Then define the local system $\widetilde{\mathcal{V}}(\boldsymbol{\lambda})^{(p)}_{\mathcal{A}}$ on $Y^{(n)}_{\mathcal{K}}$ to be the sheaf of locally constant sections of the following first projection:

$$\left(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(F_{\mathbf{A}})/\widetilde{K}_n\times\widetilde{V}(\boldsymbol{\lambda})^{(p)}_{\mathcal{A}}\right)/\mathcal{K}\longrightarrow Y^{(n)}_{\mathcal{K}}.$$

**Lemma 3.1.** *Retain the notations. Let $\Phi$ be a morphism of local systems on $Y^{(n)}_{\mathcal{K}}$ defined as*

$$\Phi:\widetilde{\mathcal{V}}(\boldsymbol{\lambda})_{\mathbf{C}}\longrightarrow\widetilde{\mathcal{V}}(\boldsymbol{\lambda})^{(p)}_{\mathbf{C}_p};([g],\mathbf{v}_{[g]})\longmapsto\left([g],\tau^{(p)}_{\boldsymbol{\lambda}}(g_p^{-1})\boldsymbol{i}(\mathbf{v}_{[g]})\right),$$

*where $g_p=(g_v)_{v|p}\in\mathrm{GL}_n(F_{\mathbf{A},p})$ is the $p$-component of $g\in\mathrm{GL}_n(F_{\mathbf{A}})$, and we consider $\boldsymbol{i}(\mathbf{v}_{[g]})$ as an element in $\widetilde{V}(\boldsymbol{\lambda})^{(p)}_{\mathbf{C}_p}$ via the fixed embedding $\boldsymbol{i}:\mathbf{C}\to\mathbf{C}_p$. Then $\Phi$ gives an isomorphism of local systems.*

Let $E\subset\mathbf{C}$ be a finite extension of $F_{\mathrm{nc}}$, $\mathcal{E}$ the $p$-adic closure of its image via the fixed isomorphism $\boldsymbol{i}:\mathbf{C}\to\mathbf{C}_p$, $\mathcal{O}$ the ring of integers of $\mathcal{E}$, and $\mathfrak{P}$ the prime ideal of the ring of integers $\mathfrak{r}_E$ of $E$ which is induced by $\boldsymbol{i}$. For a local system $\widetilde{\mathcal{V}}$ on $Y^{(n)}_{\mathcal{K}}$, denote by $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}})$ the cohomology group of local system $\widetilde{\mathcal{V}}$ for $?=\emptyset$ and the cohomology group with compact supports for $?=\mathrm{c}$. Let $\mathfrak{r}_{E,(\mathfrak{P})}$ be the localization of $\mathfrak{r}_E$ at $\mathfrak{P}$. Then the natural inclusion $E\hookrightarrow\mathbf{C}$ induces a morphism from $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_E)$ to $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$, which gives an isomorphism after taking scalar extension to $\mathbf{C}$. We also have the natural inclusion $\mathcal{O}\hookrightarrow\mathbf{C}_p$ and hence this induces a morphism from $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathcal{O}})$ to $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathbf{C}_p})$ which also gives an isomorphism after taking scalar extension to $\mathbf{C}_p$. Write the image of $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathcal{O}})$ as $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathcal{O}})'$. Furthermore, we can identify $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$ with $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathbf{C}_p})$ via $\Phi$ due to Lemma 3.1. Hence we define $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathfrak{r}_{E,(\mathfrak{P})}})$ to be $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_E)\cap H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)^{(p)}_{\mathcal{O}})'$ by taking the intersection in $H^*_?(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$. We also define $H^*_{\mathrm{cusp}}(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathfrak{r}_{E,(\mathfrak{P})}})$ to be

$$H^*_{\mathrm{cusp}}(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathfrak{r}_{E,(\mathfrak{P})}})=H^*_{\mathrm{c}}(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathfrak{r}_{E,(\mathfrak{P})}})\cap H^*_{\mathrm{cusp}}(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}}),$$

which gives the integral structure of $H^*_{\mathrm{cusp}}(Y^{(n)}_{\mathcal{K}},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$ in [HMN].

**Remark 3.2.** Januszewski ([Jan]) chooses a lattice of $\widetilde{V}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}}$ to be the $\mathrm{GL}_n(\mathfrak{r}_{E,(\mathfrak{P}_0)})$-submodule generated by a highest weight vector. In his formulation, it becomes difficult to describe the branching rule in integral coefficients and hence this is one of reasons why an unspecific constant appears in his formula for the critical values. On the other hand, we immediately find an integral model of branching rule from (2.2) and Corollary 2.2 according to the formulation in [HMN].

3.2. **Choice of Whittaker vectors.** Let $\pi^{(n)}$ be an irreducible cohomological cuspidal automorphic representation of $\mathrm{GL}_n(F_{\mathbf{A}})$, which appears in the cuspidal cohomology group of a local system $\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)$. We introduce distinguished Whittaker vectors in the Whittaker model of $\pi^{(n)}$.

Let $\mathrm{B}_n$ be the subgroup of $\mathrm{GL}_n$ consisting of upper triangle matrices. Define $\mathrm{N}_n$ (resp. $\mathrm{T}_n$) to be the subgroup of $\mathrm{B}_n$ consisting of the unipotent (resp. diagonal) matrices. We fix an additive character $\psi_{\varepsilon,\mathrm{N}_n}$ of $\mathrm{N}_n(F_{\mathbf{A}})$ to introduce the notion of Whittaker models of $\pi^{(n)}$. Let $\varepsilon\in\{\pm\}$ and $\psi_\varepsilon:\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}\to\mathbf{C}^\times$ be the additive character which is characterized by the following properties: $\psi_{\varepsilon,\infty}(x) = \exp(\varepsilon 2\pi\sqrt{-1}x)$ for $x \in \mathbf{R}$, and $\psi_{\varepsilon,p}$ is trivial on $\mathbf{Z}_p$ and non-trivial on $p^{-1}\mathbf{Z}_p$ for each prime $p$. Let $\psi_{\varepsilon,\mathrm{N}_n}$ be a character of $\mathrm{N}_n(F_\mathbf{A})$ defined by

$$\psi_{\varepsilon,\mathrm{N}_n}(x) := \psi_\varepsilon\left(\mathrm{Tr}_{F/\mathbf{Q}}(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n})\right) \qquad (x = (x_{i,j}) \in \mathrm{N}_n(F_\mathbf{A})). \tag{3.2}$$

When $n = 1$, we understand that $\psi_{\varepsilon,\mathrm{N}_1}$ is the trivial character of $\mathrm{N}_1(F_\mathbf{A}) = \{1\}$.

Let $\mathcal{W}(\pi^{(n)}, \psi_\varepsilon) \cong \otimes'_v \mathcal{W}(\pi_v^{(n)}, \psi_{\varepsilon,v})$ be the Whittaker model associated with $\pi^{(n)}$ and the additive character $\psi_{\varepsilon,\mathrm{N}_n}$. The purpose of this subsection is to give distinguished vectors in $\mathcal{W}(\pi_v^{(n)}, \psi_{\varepsilon,v})$ for each place $v$ of $F$. In particular, we describe the Whittaker vectors at infinite places, since the argument at inifinte places is the essential part of [HMN].

3.2.1. *At infinite places.* We introduce distinguished Whittaker vectors in the Whittaker models at infinite places $v$ of $F$. Here we follow a formulation given in [IM22, Section 2.4]. See references therein for the basic facts on the Whittaker model. It is known that $\pi_v^{(n)}$ is isomorphic to an irreducible principal series representation $\pi_{\mathrm{B}_n,d_v,\nu_v}$ of $\mathrm{GL}_n(F_v) = \mathrm{GL}_n(\mathbf{C})$. So here we recall some notion of the principal series representation $\pi_{\mathrm{B}_n,d_v,\nu_v}$ and the Whittaker model associated with $\pi_{\mathrm{B}_n,d_v,\nu_v}$.

Let $d_v = (d_{v,1}, d_{v,2}, \ldots, d_{v,n}) \in \mathbf{Z}^n$ and $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n}) \in \mathbf{C}^n$. Define a character $\chi_{d_v,\nu_v}$ of $\mathrm{T}_n(\mathbf{C})$ to be

$$\chi_{d_v,\nu_v}(a) := \prod_{i=1}^n \left(\frac{a_i}{|a_i|}\right)^{d_{v,i}} |a_i|^{2\nu_{v,i}} \qquad (a = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in \mathrm{T}_n(\mathbf{C})). \tag{3.3}$$

Let $\rho_n = (\rho_{n,1}, \rho_{n,2}, \ldots, \rho_{n,n}) \in \mathbf{Q}^n$ with $\rho_{n,i} := \frac{n+1}{2} - i$ $(1 \le i \le n)$. Denote the space of $C^\infty$-functions on $\mathrm{GL}_n(\mathbf{C})$ by $C^\infty(\mathrm{GL}_n(\mathbf{C}))$. We define a (smooth) principal series representation $(\pi_{\mathrm{B}_n,d_v,\nu_v}, I^\infty_{\mathrm{B}_n}(d_v,\nu_v))$ of $\mathrm{GL}_n(\mathbf{C})$ by

$$I^\infty_{\mathrm{B}_n}(d_v,\nu_v) := \left\{ f \in C^\infty(\mathrm{GL}_n(\mathbf{C})) \;\middle|\; \begin{array}{l} f(xag) = \chi_{d_v,\nu_v+\rho_n}(a) f(g) \\ (x \in \mathrm{N}_n(\mathbf{C}),\ a \in \mathrm{T}_n(\mathbf{C}),\ g \in \mathrm{GL}_n(\mathbf{C})) \end{array} \right\} \tag{3.4}$$

and $(\pi_{\mathrm{B}_n,d_v,\nu_v}(g)f)(h) = f(hg)$ $(g, h \in \mathrm{GL}_n(\mathbf{C}),\ f \in I^\infty_{\mathrm{B}_n}(d_v,\nu_v))$. Let $d_v^{\mathrm{dom}}$ be the unique element in $\Lambda_n \cap \{\sigma d_v \mid \sigma \in \mathfrak{S}_n\}$. Suppose that $\pi_{\mathrm{B}_n,d_v,\nu_v}$ is irreducible. Then $V_{d_v^{\mathrm{dom}}}$ gives the minimal $\mathrm{U}(n)$-type of $\pi_{\mathrm{B}_n,d_v,\nu_v}$, and, in particular, $\mathrm{Hom}_{\mathrm{U}(n)}(V_{d_v^{\mathrm{dom}}}, I^\infty_{\mathrm{B}_n}(d_v,\nu_v))$ is one dimensional. Let $\mathrm{f}_{\mathrm{B}_n,d_v,\nu_v} : V_{d_v^{\mathrm{dom}}} \to I^\infty_{\mathrm{B}_n}(d_v,\nu_v)$ be the $\mathrm{U}(n)$-embedding which is characterized by $\mathrm{f}_{\mathrm{B}_n,d_v,\nu_v}(\xi_{H(d_v^{\mathrm{dom}})})(1_n) = 1$.

If $\mathrm{Re}(\nu_{v,1}) > \mathrm{Re}(\nu_{v,2}) > \cdots > \mathrm{Re}(\nu_{v,n})$, we define the Jacquet integral $\mathcal{J}_\varepsilon : I^\infty_{\mathrm{B}_n}(d_v,\nu_v) \to \mathbf{C}$ to be

$$\mathcal{J}_\varepsilon(f) := \int_{\mathrm{N}_n(\mathbf{C})} f(w_n x)\psi_{-\varepsilon,\mathrm{N}_n,v}(x)\mathrm{d}x$$

for $f \in I^\infty_{\mathrm{B}_n}(d_v,\nu_v)$. Here $w_n$ is an anti-diagonal matrix of size $n$ whose all anti-diagonal entries are 1. The Jacquet integral $\mathcal{J}_\varepsilon(f)$ is absolutely convergent, and it is holomorphically continued to whole $\nu \in \mathbf{C}^n$. Define $\mathrm{W}_\varepsilon(f)(g)$ $(f \in I^\infty_{\mathrm{B}_n}(d_v,\nu_v), g \in \mathrm{GL}_n(\mathbf{C}))$ to be $\mathcal{J}_\varepsilon(\pi_{\mathrm{B}_n,d_v,\nu_v}(g)f)$ and set

$$\mathcal{W}(\pi_{\mathrm{B}_n,d_v,\nu_v}, \psi_{\varepsilon,v}) = \left\{ \mathrm{W}_\varepsilon(f) \;\middle|\; f \in I^\infty_{\mathrm{B}_n}(d_v,\nu_v) \right\}.$$

Then the right-translation by $\mathrm{GL}_n(\mathbf{C})$ on $\mathcal{W}(\pi_{\mathrm{B}_n,d_v,\nu_v}, \psi_{\varepsilon,v})$ gives the Whittaker model of $\pi_{\mathrm{B}_n,d_v,\nu_v}$.

For each $\mathrm{v} \in V_{d^{\mathrm{dom}}}$, we define the normalized Whittaker function $\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v})$ to be

$$\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v}) = (-1)^{\sum_{i=1}^n (i-1)d^{\mathrm{dom}}_{v,i}} (\varepsilon\sqrt{-1})^{\sum_{i=1}^n (i-1)d_{v,i}} \Gamma_n(\nu_v; d_v) \mathrm{W}_\varepsilon(\mathrm{f}_{\mathrm{B}_n d_v,\nu_v}(\mathrm{v})),$$

where we put $\Gamma_n(\nu_v; d_v) = \prod_{1 \le i < j \le n} \Gamma(\nu_{v,i} - \nu_{v,j} + 1 + \frac{|d_{v,i} - d_{v,j}|}{2})$. Since it is well-known that $I^\infty_{\mathrm{B}_n}(d_v,\nu_v) \cong I^\infty_{\mathrm{B}_n}(\sigma d_v, \sigma\nu_v)$ as $\mathrm{GL}_n(\mathbf{C})$-modules for each $\sigma \in \mathfrak{S}_n$, we have $\mathcal{W}(\pi_{\mathrm{B}_n,d_v,\nu_v}, \psi_{\varepsilon,v}) =$

$\mathcal{W}(\pi_{\mathrm{B}_n,\sigma d_v,\sigma\nu_v},\psi_{\varepsilon,v})$. This implies that there exists a constant $c_\sigma \in \mathbf{C}^\times$ such that $\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v}) = c_\sigma \mathbf{W}^{(\varepsilon)}_{\sigma d_v,\sigma\nu_v}(\mathrm{v})$ for each $\mathrm{v} \in V_{d^{\mathrm{dom}}}$. Based on an inductive argument for the Whittaker functions due to [Jac09] and [IM22], we obtain the following proposition:

**Proposition 3.3.** *For each $\sigma \in \mathfrak{S}_n$, we have $c_\sigma = 1$. In other words, we have $\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v}) = \mathbf{W}^{(\varepsilon)}_{\sigma d_v,\sigma\nu_v}(\mathrm{v})$ for each $\mathrm{v} \in V_{d_v^{\mathrm{dom}}}$.*

Proposition 3.3 shows that $\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v})$ has a distinguished property in $\mathcal{W}(\pi_{\mathrm{B}_n,d_v,\nu_v},\psi_{\varepsilon,v})$. Concerning the Rankin-Selberg zeta integrals, we also find a good property on $\mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\mathrm{v})$ as follows. Let $d_v \in \mathbf{Z}^n, \nu_v \in \mathbf{C}^n$, $d'_v \in \mathbf{Z}^{n-1}$ and $\nu'_v \in \mathbf{C}^{n-1}$. Then for each $W \in \mathcal{W}(\pi_{\mathrm{B}_n,d_v,\nu_v},\psi_{\varepsilon,v})$, $W' \in \mathcal{W}(\pi_{\mathrm{B}_{n-1},d'_v,\nu'_v},\psi_{-\varepsilon,v})$ and $s \in \mathbf{C}$ with sufficiently large $\mathrm{Re}(s)$, we consider the following archimedean local Rankin-Selberg zeta integral $Z(s,W,W')$:

$$Z(s,W,W') = \int_{\mathrm{N}_{n-1}(\mathbf{C})\backslash \mathrm{GL}_{n-1}(\mathbf{C})} W(\iota_n(g))W'(g)|\det g|^{2s-1}\mathrm{d}g.$$

In [HMN], we prove the following theorem, which gives an explicit formula for the archimedean local Rankin-Selberg zeta integrals:

**Theorem 3.4.** *Suppose that $d'^{\mathrm{dom}} \preceq d^{\mathrm{dom}\vee}$.*

(i) *$(V_{d^{\mathrm{dom}}} \otimes V_{d'^{\mathrm{dom}}})^{\mathrm{U}(n-1)}$ is one dimensional and it is spanned by the following element:*

$$\sum_{M \in \mathrm{G}(d'^{\mathrm{dom}\vee})} \frac{(-1)^{\mathrm{q}(M)}}{\mathrm{r}(M)} \xi_{M[d^{\mathrm{dom}}]} \otimes \xi_{M^\vee}.$$

*Here $\mathrm{q}(M) = \sum_{1\le i\le j\le n-1} m_{i,j}$.*

(ii) *We have*

$$\sum_{M \in \mathrm{G}(d'^{\mathrm{dom}\vee})} \frac{(-1)^{\mathrm{q}(M)}}{\mathrm{r}(M)} Z(s, \mathbf{W}^{(\varepsilon)}_{d_v,\nu_v}(\xi_{M[d^{\mathrm{dom}}]}), \mathbf{W}^{(-\varepsilon)}_{d'_v,\nu'_v}(\xi_{M^\vee}))$$
$$= (-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1} d'_{v,i}} L(s, \pi_{\mathrm{B}_n,d_v,\nu_v} \times \pi_{\mathrm{B}_{n-1},d'_v,\nu'_v}).$$

If $(d_{v,n},\ldots,d_{v,1}) = d_v^{\mathrm{dom}}$ and $d'_v = d'^{\mathrm{dom}}_v$, Theorem 3.4 is proved in [IM22, Corollary 2.10], and in fact, our proof of Theorem 3.4 is reduced to the case.

3.2.2. *At finite places.* As well as the archimedean case, we also need explicit formulas for the non-archimedean local Rankin-Selberg zeta integrals. For this purpose, we have to impose some assumptions on $\pi^{(n)}$ and $\pi^{(n-1)}$.

To begin with, we prepare some notation. Let $\mathfrak{r}_F$ be the ring of integers of $F$ and put $\widehat{\mathfrak{r}}_F = \mathfrak{r}_F \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$. Denote by $D_F$ the discriminant of $F$ and let $\delta = \prod_{v\in\Sigma_{F,\mathrm{fin}}} \delta_v \in \widehat{\mathfrak{r}}_F$ be a generator of different ideal of $\widehat{\mathfrak{r}}_F$. Put $\boldsymbol{\delta}^{(n)}$ to be the diagonal matrix $\mathrm{diag}(\delta^{n-1},\ldots,\delta,1) \in \mathrm{GL}_n(F_{\mathbf{A},\mathrm{fin}})$. For each ideal $\mathfrak{N}$ of $\widehat{\mathfrak{r}}_F$, define the mirahoric group $\mathcal{K}_{n,1}(\mathfrak{N})$ of level $\mathfrak{N}$ to be

$$\mathcal{K}_{n,1}(\mathfrak{N}) = \left\{ k = (k_{ij})_{1\le i,j\le n} \in \mathrm{GL}_n(\widehat{\mathfrak{r}}_F) \;\middle|\; \begin{array}{ll} k_{nj} \equiv 0 & (\mathrm{mod}\ \mathfrak{N}) \text{ for } 1 \le j \le n-1, \\ k_{nn} \equiv 1 & (\mathrm{mod}\ \mathfrak{N}) \end{array} \right\}.$$

Then we impose the following conditions on $\pi^{(n)}$ and $\pi^{(n-1)}$:

- $\pi^{(n)}_{\mathrm{fin}}$ has a $\mathcal{K}_{n,1}(\mathfrak{N})$-fixed vector for some ideal $\mathfrak{N}$ of $\widehat{\mathfrak{r}}_F$; suppose that $\mathfrak{N}$ is maximum among such ideals;
- $\mathfrak{N}$ is prime to $D_F$.
- $\pi^{(n-1)}_{\mathrm{fin}}$ is spherical, that is, $\pi^{(n-1)}_{\mathrm{fin}}$ has a $\mathrm{GL}_n(\widehat{\mathfrak{r}}_F)$-fixed vector.

If $\pi_v^{(n)}$ is spherical, we normalize the spherical vector $w_v^{\mathrm{sph}}(\pi_v^{(n)})$ in $\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})$ so that $w_v^{\mathrm{sph}}(\pi_v^{(n)})((\boldsymbol{\delta}_v^{(n)})^{-1})=1$. Define the non-archimedean local zeta integral $Z_v(s,W,W')$ ($W\in\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v}), W'\in\mathcal{W}(\pi_v^{(n-1)},\psi_{-\varepsilon,v})$) to be

$$Z_v(s,W,W')=\int_{\mathrm{N}_{n-1}(F_v)\backslash\mathrm{GL}_{n-1}(F_v)}W(\iota_n(g))W'(g)|\det g|_v^{s-\frac{1}{2}}\mathrm{d}g.$$

Then we have the following lemma due to [JPSS81]:

**Lemma 3.5.** *The $\mathcal{K}_{n,1}(\mathfrak{N})$-fixed subspace of $\mathcal{W}(\pi_{\mathrm{fin}}^{(n)},\psi_{\varepsilon,\mathrm{fin}})$ is one dimensional. Moreover, for each finite place $v$ of $F$, there exists a unique $\mathcal{K}_{n,1}(\mathfrak{N})_v$-fixed vector $w_v^{\mathrm{ess}}(\pi_v^{(n)})\in\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})$ such that, if $\pi_v^{(n-1)}$ is spherical,*

$$Z_v(s,w_v^{\mathrm{ess}}(\pi_v^{(n)}),w_v^{\mathrm{sph}}(\pi_v^{(n-1)}))=\omega_{\pi_v^{(n-1)}}(\delta_v)^{-1}|\delta_v|_v^{-\frac{1}{2}n(n-1)(s-\frac{1}{2})}L_v(s,\pi_v^{(n)}\times\pi_v^{(n-1)}).$$

*Here $\omega_{\pi_v^{(n-1)}}$ is the central character of $\pi_v^{(n-1)}$ and $w_v^{\mathrm{sph}}(\pi_v^{(n-1)})$ is the fixed spherical vector of $\mathcal{W}(\pi_v^{(n-1)},\psi_{-\varepsilon,v})$. In particular, if $v$ does not divide $\mathfrak{N}$, $w_v^{\mathrm{ess}}(\pi_v^{(n)})$ is given by $w_v^{\mathrm{sph}}(\pi_v^{(n)})$.*

The vector $w_v^{\mathrm{ess}}(\pi^{(n)})$ is called the essential vector in $\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})$. We define distinguished vectors $w^{\mathrm{ess}}(\pi^{(n)})\in\mathcal{W}(\pi_{\mathrm{fin}}^{(n)},\psi_{\varepsilon,\mathrm{fin}})$ and $w^{\mathrm{sph}}(\pi^{(n-1)})\in\mathcal{W}(\pi_{\mathrm{fin}}^{(n-1)},\psi_{-\varepsilon,\mathrm{fin}})$ to be

$$w^{\mathrm{ess}}(\pi^{(n)})=\otimes_{v\in\Sigma_{F,\mathrm{fin}}}w_v^{\mathrm{ess}}(\pi_v^{(n)}),\quad w^{\mathrm{sph}}(\pi^{(n-1)})=\otimes_{v\in\Sigma_{F,\mathrm{fin}}}w_v^{\mathrm{sph}}(\pi_v^{(n-1)}).$$

3.2.3. *Eichler-Shimura classes.* Here we introduce an explicit construction of a non-trivial element in $H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$, which enables us to reduce the study of critical values of Rankin-Selberg $L$-functions to a study of local zeta integrals and cohomology classes. In the case of $\mathrm{GL}_2$, such explicit cohomology classes are called Eichler-Shimura classes, which can be found in [Shi71, Section 8] (see also [Hid93, Section 7]) if the base field is the rational number field and [Hid94, Section 3] if the base field is general. We consider the generalization of these constructions in the case of $\mathrm{GL}_n$ over totally imaginary fields.

To begin with, recall that the $[\pi^{(n)}]$-isotropic part of $H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$ is given by the image of the following natural map:

$$H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty},\widetilde{K}_n;\pi^{(n)}\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})^{\mathcal{K}_{n,1}(\mathfrak{N})}\longrightarrow H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)},\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}}),$$

where $\mathfrak{gl}_n$ is the Lie algebra of $\mathrm{GL}_n$, $\mathfrak{gl}_{n\mathbf{C}}=\mathfrak{gl}_n\otimes_{\mathbf{R}}\mathbf{C}$, $\mathfrak{gl}_{n,\infty}=\prod_{v\in\Sigma_{F,\infty}}\mathfrak{gl}_{n\mathbf{C}}$, and $\mathcal{K}=\mathcal{K}_{n,1}(\mathfrak{N})$. We realize $\pi^{(n)}$ as the Whittaker model $\mathcal{W}(\pi^{(n)},\psi_\varepsilon)$, and hence the definition of the $(\mathfrak{g},K)$-cohomology immediately shows that

$$\begin{aligned}&H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty},\widetilde{K}_n;\pi^{(n)}\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})^{\mathcal{K}_{1,n}(\mathfrak{N})}\\&\qquad\cong H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty},\widetilde{K}_n;\mathcal{W}(\pi_\infty^{(n)},\psi_{\varepsilon,\infty})\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})\otimes\mathcal{W}(\pi_{\mathrm{fin}}^{(n)},\psi_{\varepsilon,\mathrm{fin}})^{\mathcal{K}_{1,n}(\mathfrak{N})}.\end{aligned}$$

The Künneth formula yields that

$$\begin{aligned}&H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty},\widetilde{K}_n;\mathcal{W}(\pi_\infty^{(n)},\psi_{\varepsilon,\infty})\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})\\&\qquad\cong\bigotimes_{v\in\Sigma_{F,\infty}}H^{\mathrm{b}_n}(\mathfrak{gl}_{n,\mathbf{C}},\mathbf{C}^\times\mathrm{U}(n);\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}_v^\vee)),\end{aligned}$$

where $\mathrm{b}_n:=\frac{n(n-1)}{2}$ and $\widetilde{V}(\boldsymbol{\lambda}_v^\vee)=V_{\lambda_\sigma^\vee}(\mathbf{C})\otimes V_{\lambda_{\overline{\sigma}}^\vee}(\mathbf{C})$ for the embeddings $\sigma,\overline{\sigma}:F\to\mathbf{C}$ corresponding to $v$. Furthermore, as in [BW80, Section I.5], we find that

$$\begin{aligned}&H^{\mathrm{b}_n}(\mathfrak{gl}_{n,\mathbf{C}},\mathbf{C}^\times\mathrm{U}(n);\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}_v^\vee))\\&\qquad=\Big(\mathcal{W}(\pi_v^{(n)},\psi_{\varepsilon,v})\otimes\wedge^{\mathrm{b}_n}\mathfrak{p}_{n\mathbf{C}}^0\otimes_{\mathbf{C}}\widetilde{V}(\boldsymbol{\lambda}_v^\vee)\Big)^{\mathrm{U}(n)}.\end{aligned}\tag{3.5}$$

Here write the Cartan decomposition of $\mathfrak{gl}_n$ as $\mathfrak{gl}_n = \mathfrak{u}(n) \oplus \mathfrak{p}$ where $\mathfrak{u}(n)$ is the Lie algebra of $\mathrm{U}(n)$. Let $\mathfrak{p}_n^0$ be the subspace of $\mathfrak{p}_n$ consisting of matrices whose traces are zero and put $\mathfrak{p}_{n\mathbf{C}}^0 = \mathfrak{p}_n^0 \otimes_{\mathbf{R}} \mathbf{C}$. We have already chosen a vector $w^{\mathrm{ess}}(\pi^{(n)}) \in \mathcal{W}(\pi_{\mathrm{fin}}^{(n)}, \psi_{\varepsilon,\mathrm{fin}})^{\mathcal{K}_{n,1}(\mathfrak{N})}$ in Section 3.2.2, and hence it suffices to construct an element in the right-hand side of (3.5) for the construction of an element in $H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)}, \widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$.

Note that, considering $\wedge^{\mathrm{b}_n}\mathfrak{p}_{n\mathbf{C}}^0$ as a representation space of $\mathrm{U}(n)$ via the adjoint action, $\wedge^{\mathrm{b}_n}\mathfrak{p}_{n\mathbf{C}}^0$ contains $V_{2\rho_n}$ with multiplicity one. Hence we have a $\mathrm{U}(n)$-equivariant homomorphism $V_{2\rho_n} \to \wedge^{\mathrm{b}_n}\mathfrak{p}_{n\mathbf{C}}^0$, which is normalized by giving a specific element $\mathbf{E}_n \in \wedge^{\mathrm{b}_n}\mathfrak{p}_{n\mathbf{C}}^0$ as the image of $\xi_{H(2\rho_n)}$. We omit the detail of the construction of $\mathbf{E}_n$ here, since it needs more notation although it is easy. Recall that $\pi_v^{(n)}$ is isomorphic to some principal series representation $\pi_{\mathrm{B}_n,d_v,\nu_v}$ of $\mathrm{GL}_n(\mathbf{C})$. Due to Proposition 3.3, we may suppose that $d_v$ is dominant. Since $\pi^{(n)}$ appears in $H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)}, \widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee))$, $d_v$ and $\nu_v$ are given by the following formulas:

$$d_v = 2\boldsymbol{\lambda}_v + 2\rho_n - \mathsf{w}, \qquad \nu_v = (\mathsf{w}/2, \dots, \mathsf{w}/2),$$

where we recall that $\mathsf{w}$ is the purity weight of $\boldsymbol{\lambda}$ (see **(Pur)**). Since the minimal $\mathrm{U}(n)$-type of $\pi_{\mathrm{B}_n,d_v,\nu_v}$ is given by $V_{d_v}$, we can define a $\mathrm{U}(n)$-equivariant homomorphism $V_{d_v} \to \mathcal{W}(\pi_v^{(n)}, \psi_{\varepsilon,v})$ to be $\mathbf{W}_{d_v,\nu_v}^{(\varepsilon)}$ as in Section 3.2.1. Hence, to construct an element in the right-hand side of (3.5), it suffices to construct a $\mathrm{U}(n)$-invariant element in $\left(V_{d_v} \otimes_{\mathbf{C}} V_{2\rho_n} \otimes_{\mathbf{C}} \widetilde{V}(\boldsymbol{\lambda}_v^\vee)\right)$. For this purpose, we need some claims from the finite dimensional representation theory by using Gel'fand-Tsetlin basis as follows:

**Lemma 3.6.** *Let $\lambda, \lambda' \in \Lambda_n$.*

(i) *We have a unique injective $\mathrm{GL}_n(\mathbf{C})$-homomorphism $I_{\lambda+\lambda'}^{\lambda,\lambda'} : V_{\lambda+\lambda'} \to V_\lambda \otimes_{\mathbf{C}} V_{\lambda'}$, which is characterized by $I_{\lambda+\lambda'}^{\lambda,\lambda'}(\xi_{H(\lambda+\lambda')}) = \xi_{H(\lambda)} \otimes \xi_{H(\lambda')}$.*

(ii) *Let $(\tau_\lambda^{\mathrm{conj}}, V_\lambda^{\mathrm{conj}})$ be the complex conjugate representation of $(\tau_\lambda, V_\lambda)$, that is,*

$$V_\lambda^{\mathrm{conj}} = V_\lambda, \qquad \tau_\lambda^{\mathrm{conj}}(g) = \tau_\lambda(\overline{g}) \quad (g \in \mathrm{GL}_n(\mathbf{C})).$$

*Then a $\mathbf{C}$-linear map $I_\lambda^{\mathrm{conj}} : V_\lambda \to V_{\lambda^\vee}^{\mathrm{conj}}; \xi_M \mapsto (-1)^{\mathrm{q}(M)}\xi_{M^\vee}$ is a $\mathrm{U}(n)$-isomorphism.*

(iii) *The following element gives a $\mathrm{U}(n)$-invariant element in $V_\lambda \otimes V_{\lambda^\vee}$:*

$$[\mathrm{id}_{V_\lambda}] := \sum_{M \in \mathrm{G}(\lambda)} \frac{(-1)^{\mathrm{q}(M)}}{\mathrm{r}(M)} \xi_M \otimes \xi_{M^\vee}.$$

Due to Lemma 3.6, we define a $\mathrm{U}(n)$-invariant element in $V_{d_v} \otimes_{\mathbf{C}} V_{2\rho_n} \otimes_{\mathbf{C}} \widetilde{V}(\boldsymbol{\lambda}_v^\vee)$ to be the image of $[\mathrm{id}_{V_d}]$ via the following composite maps:

$$\begin{aligned} V_d \otimes_{\mathbf{C}} V_{d^\vee} &\xrightarrow{\mathrm{id}\otimes \mathrm{I}_{d^\vee}^{2\rho_n, 2\lambda^\vee+\mathsf{w}}} V_d \otimes_{\mathbf{C}} V_{2\rho_n} \otimes_{\mathbf{C}} V_{2\lambda^\vee+\mathsf{w}} \\ &\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes \mathrm{I}_{2\lambda^\vee+\mathsf{w}}^{\lambda^\vee, \lambda^\vee+\mathsf{w}}} V_d \otimes_{\mathbf{C}} V_{2\rho_n} \otimes_{\mathbf{C}} V_{\lambda^\vee} \otimes_{\mathbf{C}} V_{\lambda^\vee+\mathsf{w}} \\ &\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes \mathrm{I}_{\lambda^\vee+\mathsf{w}}^{\mathrm{conj}}} V_{d_v} \otimes_{\mathbf{C}} V_{2\rho_n} \otimes_{\mathbf{C}} \widetilde{V}(\boldsymbol{\lambda}_v^\vee). \end{aligned}$$

Finally we obtain a cohomology class in $H_{\mathrm{cusp}}^{\mathrm{b}_{n,F}}(Y_{\mathcal{K}}^{(n)}, \widetilde{\mathcal{V}}(\boldsymbol{\lambda}^\vee)_{\mathbf{C}})$, which we denote by $\delta(\pi^{(n)})$ and call Eichler-Shimura class of $\pi^{(n)}$.

3.3. **Definition of Whittaker periods.** For each $\boldsymbol{\lambda} = (\lambda_\sigma)_{\sigma \in I_F} \in \Lambda_n^{I_F}$ and $\alpha \in \mathrm{Aut}(\mathbf{C})$, define ${}^\alpha\boldsymbol{\lambda} \in \Lambda_n^{I_F}$ by ${}^\alpha\lambda_\sigma = \lambda_{\alpha^{-1}\circ\sigma}$ $(\sigma \in I_F)$. Put $\mathbf{Q}(\boldsymbol{\lambda}) = \mathbf{C}^{\{\alpha \in \mathrm{Aut}(\mathbf{C}) \mid {}^\alpha\boldsymbol{\lambda} = \boldsymbol{\lambda}\}}$. Define also $\alpha$-twists ${}^\alpha\pi_?^{(n)}$ of $\pi_?^{(n)}$ to be $\pi_?^{(n)} \otimes_{\mathbf{C},\alpha} \mathbf{C}$ for $? = \mathrm{fin}, \infty$. Note that if the highest weight associated with $\pi_\infty^{(n)}$ is $\boldsymbol{\lambda}$, then the highest weight associated with ${}^\alpha\pi_\infty^{(n)}$ is given by ${}^\alpha\boldsymbol{\lambda}$. Then [Clo90, Théorème 3.13] yields that there exists a unique cohomological irreducible cuspidal automorphic representation ${}^{\alpha}\pi^{(n)}$ of $\mathrm{GL}_n(F_{\mathbf{A}})$ such that the finite (resp. infinite) part of ${}^{\alpha}\pi^{(n)}$ is given by ${}^{\alpha}\pi^{(n)}_{\mathrm{fin}}$ (resp. ${}^{\alpha}\pi^{(n)}_{\infty}$) and also that $\mathbf{Q}(\pi^{(n)}_{\mathrm{fin}}) := \mathbf{C}^{\{\alpha\in\mathrm{Aut}(\mathbf{C})\ \mid\ {}^{\alpha}\pi^{(n)}_{\mathrm{fin}}=\pi^{(n)}_{\mathrm{fin}}\}}$ is a number field. Define the field of rationality of $\pi^{(n)}$ to be $\mathbf{Q}(\pi^{(n)}) := \mathbf{Q}(\pi^{(n)}_{\mathrm{fin}})\mathbf{Q}(\boldsymbol{\lambda})$. Let $\mathfrak{r}(\pi^{(n)})$ be the ring of integers of $\mathbf{Q}(\pi^{(n)})$ and $\mathfrak{P}$ the prime ideal of $\mathfrak{r}(\pi^{(n)})$ which is induced by the fixed isomorphism $\mathbf{Q}(\pi^{(n)}) \subset \mathbf{C} \xrightarrow{\boldsymbol{i}} \mathbf{C}_p$. Then, since the $\mathfrak{r}(\pi^{(n)})_{(\mathfrak{P})}$-module

$$H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty}, \widetilde{K}_n; \pi^{(n)} \otimes_{\mathbf{C}} \widetilde{V}(\boldsymbol{\lambda}^{\vee})_{\mathbf{C}})^{\mathcal{K}_{n,1}(\mathfrak{N})} \cap H^{\mathrm{b}_{n,F}}_{\mathrm{cusp}}(Y^{(n)}_{\mathcal{K}}, \widetilde{\mathcal{V}}(\boldsymbol{\lambda}^{\vee})_{\mathfrak{r}_{E,(\mathfrak{P})}})$$

is free of rank one over $\mathfrak{r}(\pi^{(n)})_{(\mathfrak{P})}$, let us choose a generator $\eta(\pi^{(n)})$ of this module. Since $H^{\mathrm{b}_{n,F}}(\mathfrak{gl}_{n,\infty}, \widetilde{K}_n; \pi^{(n)} \otimes_{\mathbf{C}} \widetilde{V}(\boldsymbol{\lambda}^{\vee})_{\mathbf{C}})^{\mathcal{K}_{n,1}(\mathfrak{N})}$ is one dimensional over $\mathbf{C}$, and it is spanned by the Eichler-Shimura class $\delta(\pi^{(n)})$, there exists a constant $p^{\mathrm{b}}(\pi^{(n)}) \in \mathbf{C}^{\times}$ such that

$$\delta(\pi^{(n)}) = p^{\mathrm{b}}(\pi^{(n)})\eta(\pi^{(n)}).$$

We call $p^{\mathrm{b}}(\pi^{(n)})$ the (Betti-)Whittaker period of $\pi^{(n)}$. Note that $p^{\mathrm{b}}(\pi^{(n)})$ is determined up to a multiplication by an element in $\mathfrak{r}(\pi^{(n)})^{\times}_{(\mathfrak{P})}$, and hence we can discuss the integrality with respect to this period $p^{\mathrm{b}}(\pi^{(n)})$.

**Remark 3.7.** To describe the behavior of critical values under the action of $\alpha \in \mathrm{Aut}(\mathbf{C})$, we need to choose $p^{\mathrm{b}}({}^{\alpha}\pi^{(n)})$ in a compatible way under the $\alpha$-twist of cohomology groups and $\pi^{(n)}$. In this article, we omit this point of view for the sake of simplicity. See [RS08, Definition/Proposition 3.3] for the details of $\alpha$-twists and the choice of $p^{\mathrm{b}}({}^{\alpha}\pi^{(n)})$. Note that Raghuram-Shahidi define their periods up to multiplication by rational constants, but we can discuss in a similar way by using our integral models.

## 4. Main Theorem

In this section, we introduce the main theorem in [HMN]. Let us summarize assumptions on $\pi^{(n)}$ and $\pi^{(n-1)}$, which we made in the previous sections:

- $\pi^{(n)}$ (resp. $\pi^{(n-1)}$) is cohomological, which appears in the cohomology group of a local system $\widetilde{\mathcal{V}}(\boldsymbol{\lambda}^{\vee})_{\mathbf{C}}$ (resp. $\widetilde{\mathcal{V}}(\boldsymbol{\mu}^{\vee})_{\mathbf{C}}$);
- There exists an integer $m_0 \in \mathbf{Z}$ such that $\boldsymbol{\lambda}^{\vee} \succeq \boldsymbol{\mu} + m_0\mathbf{1}$;
- $\pi^{(n)}_{\mathrm{fin}}$ has a $\mathcal{K}_{n,1}(\mathfrak{N})$-fixed vector and $\mathfrak{N}$ is maximal among such ideals; $\pi^{(n-1)}_{\mathrm{fin}}$ has a $\mathrm{GL}_{n-1}(\mathfrak{r}_F)$-fixed vector.
- The discriminant $D_F$ of $F$ is prime to $\mathfrak{N}$;
- $p$ is a prime number which is coprime to $D_F\mathfrak{N}$ and satisfies (3.1).

Define $\mathfrak{r}(\pi^{(n)}, \pi^{(n-1)})$ to be the ring of integers of the composite $\mathbf{Q}(\pi^{(n)})\mathbf{Q}(\pi^{(n-1)})$ of the fields of rationality of $\pi^{(n)}$ and $\pi^{(n-1)}$. Denote by $\mathfrak{P}_0$ the prime ideal of $\mathfrak{r}(\pi^{(n)}, \pi^{(n-1)})$ which is induced by the fixed isomorphism $\mathbf{Q}(\pi^{(n)})\mathbf{Q}(\pi^{(n-1)}) \subset \mathbf{C} \xrightarrow{\boldsymbol{i}} \mathbf{C}_p$. Define a constant $\mathcal{C}(m, \pi^{(n)} \times \pi^{(n-1)})$ ($m \in \mathbf{Z}$) to be

$$\begin{aligned}\mathcal{C}(m, \pi^{(n)} \times \pi^{(n-1)}) =& \omega_{\pi^{(n-1)}}(\delta)^{-1} D_F^{\frac{1}{2}n(n-1)m} \\ & \times \prod_{v\in\Sigma_{F,\infty}} \left(2^{-n(n-1)}(\sqrt{-1})^{-\mathrm{b}_{n-1}}(\varepsilon\sqrt{-1})^{\mathrm{b}_n(\mathsf{w}-\mathsf{w}')}(-1)^{(m+1)\mathrm{b}_n}\right).\end{aligned} \tag{4.1}$$

Here we recall that $\varepsilon \in \{\pm 1\}$, $\mathrm{b}_n = \frac{n(n-1)}{2}$, and $\mathsf{w}$ (resp. $\mathsf{w}'$) is the purity weight of $\boldsymbol{\lambda}$ (resp. $\boldsymbol{\mu}$) as in **(Pur)**. Note that, if the prime number $p$ satisfying the above condition is odd, then $\mathcal{C}(m, \pi^{(n)} \times \pi^{(n-1)})$ is a unit in $\mathfrak{r}(\pi^{(n)}, \pi^{(n-1)})_{(\mathfrak{P}_0)}$, and hence $\mathcal{C}(m, \pi^{(n)} \times \pi^{(n-1)})$ can be understood as a harmless constant when we discuss the integrality. However, we consider $\mathcal{C}(m, \pi^{(n)} \times \pi^{(n-1)})$ as an important constant to discuss the Kummer congruences for critical values, since the information $m$ of critical points appears in its definition.

The following statement is the main theorem in [HMN]:

**Theorem 4.1.** *For each critical point $m+\frac{1}{2}$ of $L(s,\pi^{(n)}\times\pi^{(n-1)})$, the value*

$$\mathcal{C}(m,\pi^{(n)}\times\pi^{(n-1)})\frac{L(m+\frac{1}{2},\pi^{(n)}\times\pi^{(n-1)})}{p^{\mathrm{b}}(\pi^{(n)})p^{\mathrm{b}}(\pi^{(n-1)})}$$

*is indeed contained in* $\mathfrak{r}(\pi^{(n)},\pi^{(n-1)})_{(\mathfrak{P}_0)}$.

**Remark 4.2.** (i) Theorem 4.1 yields that the unspecific constant appearing in the critical value formulas in the previous researches, which we mentioned in Section 1.1, is given by the product of the constant $\mathcal{C}(m,\pi^{(n)}\times\pi^{(n-1)})$ and $L_\infty(m+\frac{1}{2},\pi_\infty^{(n)}\times\pi_\infty^{(n-1)})$.

Li-Liu-Sun ([LLS]) proved that the unspecific constant is a product of an easy constant depending on the critical points, which is similar to $\mathcal{C}(m,\pi^{(n)}\times\pi^{(n-1)})$, $L_\infty(m+\frac{1}{2},\pi_\infty^{(n)}\times\pi_\infty^{(n-1)})$ and an unspecific non-zero constant which does not depend on the critical points in the case that the base field is general. This implies that they also get a formula similar to Theorem 4.1 after suitably normalizing Whittaker vectors. However their normalization of Whittaker vectors on $\mathrm{GL}_n$ a priori depends on the normalization of those of $\mathrm{GL}_N$ $(1\leq N\leq n-1)$, and hence the period $p^{\mathrm{b}}(\pi^{(n)})$ also a priori depends on information of representations $\mathrm{GL}_N$. Also it seems to be difficult to discuss the integrality according to their formulation, since we need an integral branching rule as mentioned in Remark 3.2.

(ii) The formula for critical values in Theorem 4.1 implies that the product of Whittaker periods plays a role of Deligne's period for the tensor product of the expected pure motives attached to $\pi^{(n)}$ and $\pi^{(n-1)}$. So it is natural to ask a motivic background of the single Whittaker period $p^{\mathrm{b}}(\pi^{(n)})$ in terms of the expected pure motive attached to $\pi^{(n)}$. The study in this direction can be found in [HN].

**Remark 4.3.** The key ingredient of the proof of Theorem 4.1 is the cohomological interpretation of Rankin-Selberg zeta integrals, as we have already mentioned in Section 1.2. Following the generalized modular symbol method due to [KMS00], we write down all cohomological manipulations in an explicit way and we reduce the calculation to the explicit formula for archimedean Rankin-Selberg zeta integrals in Theorem 3.4 (ii), which is essentially due to [IM22]. We note that a similar strategy can be found in [HN21] in the case of $\mathrm{GL}_3\times\mathrm{GL}_2$ over the rational number field.

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