ON ICHINO-IKEDA TYPE FORMULA OF BESSEL PERIODS FOR $(\mathrm{U}(2n), \mathrm{U}(1))$ AND $(\mathrm{GL}(2n), \mathrm{GL}(1))$

KAZUKI MORIMOTO
GRADUATE SCHOOL OF SCIENCE, KOBE UNIVERSITY

Abstract. In this paper, we announce the results given in a joint work [2] with Masaaki Furusawa (Osaka Metropolitan University) on Ichino-Ikeda type formula of Bessel periods for $(\mathrm{U}(2n), \mathrm{U}(1))$.

1. $(\mathrm{U}(2n), \mathrm{U}(1))$-case

Let F be a number field. We denote its ring of adeles by $\mathbb{A} = \mathbb{A}_F$. Let ψ be a non-trivial character of $\mathbb{A}/F$. For a place v of F, we denote by F_v the completion of F at v. Let E be a quadratic extension field of F and $\mathbb{A}_E$ be its ring of adeles. We write $E_v = E \otimes F_v$. Let $E \ni x \mapsto \bar{x}$ denote the non-trivial element of $\mathrm{Gal}(E/F)$. Let $\chi_{E/F}$ denote the quadratic character of $\mathbb{A}^\times/F^\times$ corresponding to E/F. We define a character ψ_E of $\mathbb{A}_E/E$ by $\psi_E(x) = \psi\left(\frac{x+\bar{x}}{2}\right)$. Let Λ be a character of $\mathbb{A}_E^\times/E^\times$ such that the restriction of Λ to $\mathbb{A}^\times$ is trivial.

Let $(V, (\ ,\ )_V)$ be a $2n$-dimensional hermitian spaces over E with a non-degenerate hermitian pairing $(\ ,\ )_V$ such that Witt index of V is at least $n-1$. We may decompose V as a direct sum

$$V = X^+ \oplus L \oplus X^-$$

where L is a 2-dimensional hermitian space over E and $X^\pm$ are totally isotropic $(n-1)$-dimensional subspaces of V which are dual to each other and orthogonal to L. We take a basis $\{e_1, \ldots, e_{n-1}\}$ of X^+ and a basis $\{e_{-1}, \ldots, e_{-n+1}\}$ of X^-, respectively so that

$$(e_i, e_{-j})_V = \delta_{i,j} \tag{1.0.1}$$

for $1 \le i, j \le n-1$, where $\delta_{i,j}$ denotes Kronecker's delta.

Let P' be the maximal parabolic subgroup of $\mathrm{U}(V)$ preserving the isotropic subspace X^-. Let M' and S' denote Levi part and the unipotent part of P', respectively. For an anisotropic vector $e \in L$, we define a character χ_e of $S'(\mathbb{A})$ by

$$\chi_e \begin{pmatrix} 1_{n-1} & A & B \\ & 1_2 & A' \\ & & 1_{n-1} \end{pmatrix} = \psi_E((Ae, e_{n-1})_V).$$

Let U_{n-1} denote the group of upper triangular unipotent matrices in $\mathrm{Res}_{E/F}\mathrm{GL}_{n-1}$. For $u \in U_{n-1}$, we define $\check{u} \in P'$ by

$$\check{u} = \begin{pmatrix} u & & \\ & 1_2 & \\ & & u^* \end{pmatrix}.$$

Here, $u^* = w_{n-1}{}^t\bar{u}^{-1}w_{n-1}$ and w_r denotes the $r\times r$ matrix with ones on the sinister diagonal, zero elsewhere. Then we define an unipotent subgroup S of P' by

$$S = S'S'' \quad \text{where} \quad S'' = \{\check{u} : u \in U_{n-1}\}$$

and we extend χ_e to a character of $S(\mathbb{A})$ by putting

$$\chi_e(\check{u}) = \psi_E(u_{1,2} + \cdots + u_{n-2,n-1}) \quad \text{for} \quad u \in U_{n-1}(\mathbb{A}).$$

We define a subgroup D_e of $\mathrm{U}(V)$ by

$$D_e = \left\{ \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix} : h \in \mathrm{U}(L),\, he = e \right\}$$

and let $R_e = D_eS$. Then the elements of $D_e(\mathbb{A})$ stabilize a character χ_e of $R_e(\mathbb{A})$ by conjugation. We may regard Λ as a character of $D_e(\mathbb{A})$ by $d \mapsto \Lambda(\det d)$. Then we define a character $\chi_{e,\Lambda}$ of $R_e(\mathbb{A})$ by

$$\chi_{e,\Lambda}(ts) = \Lambda(t)\chi_e(s) \quad \text{for} \quad t \in D_e(\mathbb{A}), s \in S(\mathbb{A}).$$

For a cusp form φ on $\mathrm{U}(V)(\mathbb{A})$, we define the (e, ψ, Λ)-Bessel period of φ by

$$B_{e,\psi,\Lambda}(\varphi) = \int_{D_e(F)\backslash D_e(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} \chi_{e,\Lambda}^{-1}(ts)\varphi(ts)\, ds\, dt \tag{1.0.2}$$

where we take Tamagawa measures ds and dt on $S(\mathbb{A})$ and $D_e(\mathbb{A})$, respectively. We note that when we take local measure ds_v on $S(F_v)$ at each place v corresponding to the gauge form (with respect to ψ_v), we have $ds = \prod_v ds_v$.

Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $\mathrm{U}(V)(\mathbb{A})$. Let $\langle\ ,\ \rangle$ denote the $\mathrm{U}(V)\,(\mathbb{A})$-invariant Hermitian inner product on V_π given by the Petersson inner product, i.e.

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathrm{U}(V)(F)\backslash \mathrm{U}(V)(\mathbb{A})} \varphi_1\,(g)\ \overline{\varphi_2\,(g)}\, dg \quad \text{for } \varphi_1, \varphi_2 \in V_\pi.$$

Since $\pi = \otimes_v \pi_v$ where π_v is unitary for each place v of F, we may choose a $\mathrm{U}(V)(F_v)$-invariant Hermitian inner product $\langle\ ,\ \rangle_v$ on V_{π_v}, the space of π_v, so that we have

$$\langle \varphi_1, \varphi_2 \rangle = \prod_v \langle \varphi_{1,v}, \varphi_{2,v} \rangle_v \quad \text{for} \quad \varphi_i = \otimes_v \varphi_{i,v} \in V_\pi \quad (i = 1, 2)\,.$$

We choose a local Haar measure dg_v on $\mathrm{U}(V)(F_v)$ for each place v so that $\mathrm{Vol}\,(K_{V,v}, dg_v) = 1$ at almost all v, where $K_{V,v}$ is a maximal compact subgroup of $\mathrm{U}(V)(F_v)$. Let us also choose a local Haar measure dt_v on $D_e\,(F_v)$ at each place v so that $\mathrm{Vol}\,(K_{e,v}, dt_v) = 1$ at almost all v, where $K_{e,v}$ is a maximal compact subgroup of $D_e\,(F_v)$. Then we have

$$dg = C_G \cdot \prod_v dg_v \quad \text{and} \quad dt = C_e \cdot \prod_v dt_v, \tag{1.0.3}$$

for some positive constants C_G and C_e.

At each place v, Liu [8] defined local (e, ψ_v, Λ_v)-Bessel period using stable integral at finite places and Fourier transform at archimedean places. We write (e, ψ_v, Λ_v)-Bessel period by $\alpha_v\,(\varphi_v, \varphi_v')$ for φ_v, $\varphi_v' \in V_{\pi_v}$ when we use $\langle -, - \rangle_v$ and local measures given above in the definition. Then Liu's theorem [8, Theorem 2.2] implies that at almost all finite places v, we have

$$\alpha_v(\varphi_v, \varphi_v') = \frac{L(1/2, \pi_v \times \Lambda_v) \prod_{j=1}^{2n} L\left(j, \chi_{E,v}^j\right)}{L(1, \pi_v, \mathrm{Ad}) L(1, \chi_{E,v})}$$

Moreover, we define the normalized local Bessel period $\alpha_v^{\natural}(\varphi_v, \varphi_v')$ at each place v of F by

$$\alpha_v^{\natural}(\varphi_v, \varphi_v') := \frac{L(1, \pi_v, \mathrm{Ad}) L(1, \chi_{E,v})}{L(1/2, \pi_v \times \Lambda_v) \prod_{j=1}^{2n} L\left(j, \chi_{E,v}^j\right)} \cdot \alpha_v(\varphi_v, \varphi_v'). \tag{1.0.4}$$

Then for non-zero $\varphi = \otimes \varphi_v \in V_\pi$, we have $\alpha_v^{\natural}(\varphi_v, \varphi_v) = 1$ at almost all finite places v. Here, local L-factors are defined as follows. For each place v of F, let ϕ_{π_v} denote the local Langlands parameter of π_v given by [4, 9]. Then we define

$$L(s, \pi_v, \mathrm{Ad}) := L(s, \phi_{\pi_v}, \mathrm{Ad}).$$

Since

$$L(s, \phi_{\pi_v}, \mathrm{Ad}) = L(s, \mathrm{BC}(\phi_{\pi_v}), \mathrm{As}),$$

we have

$$L(s, \pi, \mathrm{Ad}) = L(s, \Pi, \mathrm{As})$$

where Π denotes the base change lift of π to $\mathrm{GL}_{2n}(\mathbb{A}_E)$ and As stands for the Asai L-function. The existence of Π follows from [4] and [9]. We know that $L(s, \pi, \mathrm{Ad})$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$ and is holomorphic and non-zero at $s = 1$ by [9, Corollary 2.5.9] and [13, Theorem 5.1]. Meanwhile we may define $L(s, \pi_v \times \Lambda_v)$ by the γ-factors defined by the doubling method as in Lapid and Rallis [7]. By the uniqueness of γ-factors, we have

$$L(s, \pi_v \times \Lambda_v) = L(s, \phi_{\pi_v} \times \Lambda_v),$$

which is holomorphic for $\mathrm{Re}(s) > 0$ by Yamana [15] since π_v is tempered.

Let us define a quasi-split unitary group $\mathbb{G}_n$ by

$$\mathbb{G}_n(F) = \left\{ g \in \mathrm{GL}_{2n}(E) : {}^t\bar{g} J_n g = J_n \right\} \quad \text{where} \quad J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}.$$

We shall consider the theta lift of π to $\mathbb{G}_n$. Recall that we may define the Weil representation $\omega_{\psi,\Lambda}$ of $\mathbb{G}_n \times \mathrm{U}(V)$ associated to $\chi_\Lambda = (\Lambda, \Lambda)$ and ψ using the splitting by Kudla [5]. We realize $\omega_{\psi,\Lambda}$ by the space $\mathcal{S}(V^n(\mathbb{A}))$ of Schwartz functions on $V^n(\mathbb{A})$. Then we define the theta function by

$$\theta_{\psi,\Lambda}^{\phi}(g, h) = \sum_{x \in V^n(F)} \omega_{\psi,\Lambda}(g, h)\phi(x)$$

for $\phi \in \mathcal{S}(V^n(\mathbb{A}))$. Then for $\phi \in \mathcal{S}(V^n(\mathbb{A}))$ and a cusp form f on $\mathrm{U}(V)(\mathbb{A})$, we define the theta lift of f to $\mathbb{G}_n(\mathbb{A})$ by

$$\theta_{\psi,\Lambda}(f; \phi)(g) = \int_{\mathrm{U}(V)(F) \backslash \mathrm{U}(V)(\mathbb{A})} \theta_{\psi,\Lambda}^{\phi}(g, h) f(h)\, dh.$$

Further, we define the theta lift of π by

$$\theta(\pi, \psi, \Lambda) = \langle \theta_{\psi,\Lambda}(f; \phi) : f \in V_\pi,\ \phi \in \mathcal{S}(V^n(\mathbb{A})) \rangle.$$

Let N_n be the group of upper triangular unipotent matrices in $\mathbb{G}_n$. For $\lambda \in F^\times$, let $\psi_{N_n,\lambda}$ be a non-degenerate character of $N_n(\mathbb{A})$ defined by

$$\psi_{N_n,\lambda}(u) = \psi_E\left(\sum_{i=1}^{n-1} u_{i,i+1} + \frac{\lambda}{2}u_{n,2n}\right).$$

For an automorphic form f on $\mathbb{G}_n(\mathbb{A})$, we define $\psi_{N_n,\lambda}$-Whittaker period of f by

$$W_{\psi,\lambda}(f;g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} f(u)\psi^{-1}_{N_n,\lambda}(u)\,du.$$

Then we note that $\psi_{N_n,\lambda}$-Whittaker period is related to Bessel period as follows. For $\phi \in \mathcal{S}(V^n(\mathbb{A}))$ and $\varphi \in \pi$, we have

$$(1.0.5)\quad W_{\psi,(e,e)_V}(\theta^{\phi}_{\psi,\Lambda}(\varphi);1) = \int_{R'_e(\mathbb{A})\backslash G(\mathbb{A})} \omega_{\psi,\Lambda}(g,1)\phi(e_{-1},\dots,e_{-n+1},e)B_{e,\psi,\Lambda}(\pi(g)\varphi)\,dg$$

where

$$R'_e = \{g \in G : ge_{-1} = e_{-1},\dots,ge_{-n+1} = e_{-n+1}, ge = e\}.$$

(see [10, Proposition 1,1]). Our main result in the case of unitary groups are stated as follows.

Theorem 1.1 ([2])**.** *Let (π, V_π) be as above. Suppose that Ichino-Ikeda type formula of $\psi_{N_n,\lambda}$-Whittaker period holds for $\theta_{\psi,\Lambda}(\pi)$. Then for any non-zero decomposable vector $\varphi = \otimes\varphi_v \in V_\pi$, we have*

$$(1.0.6)\quad \frac{|B_{e,\Lambda,\psi}(\varphi)|^2}{\langle\varphi,\varphi\rangle} = \frac{C_e}{2\cdot|S(\Psi(\pi))|}\left(\prod_{j=1}^{2n} L(j,\chi_E^j)\right) \times \frac{L(\frac{1}{2},\pi\times\Lambda)}{L\,(1,\pi,\mathrm{Ad})\,L(1,\chi_{E/F})}\prod_v \frac{\alpha_v^{\natural}(\varphi_v,\varphi_v)}{\langle\varphi_v,\varphi_v\rangle}$$

where $\Psi(\pi)$ denote the A-parameter of π and $S(\Psi(\pi))$ its S-group.

In recent preprint [11, 12], the author proved Ichino-Ikeda type formula of Whittaker periods under certain conditions. As a consequence, we have the following theorem.

Theorem 1.2. *Let (π, V_π) be as above. Assume that the following conditions hold*

- *E/F is CM extension*
- *at each archimedean place, π_v is discrete series representation.*

Then the formula (1.1) *holds for any non-zero decomposable vectors of V_π.*

Our idea of the proof of Theorem 1.1 is similar as the proof of [1, Theorem 1]. Indeed, this is proved combining the following ingredients.

(1) Rallis inner product formula for the theta lift $\theta_{\psi,\Lambda}(\pi)$ (see Yamana [15])
(2) Pull-back formula of Whittaker periods for the theta lift $\theta_{\psi,\Lambda}(\pi)$ (see (1.0.5))
(3) Ichino-Ikeda type formula of Whittaker periods for $\theta_{\psi,\Lambda}(\pi)$
(4) Local analogue of pull-back computation of Whittaker periods for the theta lift (cf. [1, Proposition 4])

2. $(\mathrm{GL}(2n), \mathrm{GL}(1))$-case

Similar argument as the previous section can be applied to $(\mathrm{GL}(2n), \mathrm{GL}(1))$-case. Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ and η a character of $\mathbb{A}^\times / F^\times$. Then as in (1.0.2), we may define (ψ, η)-Bessel period $B_{\psi,\eta}(\varphi)$ of $\varphi \in V_\pi$. Here, we take Tamagawa measure dt on $\mathrm{GL}_1(\mathbb{A})$ with the decomposition $dt = C_1 \prod_v dt_v$. We define the Petersson inner product on V_π by

$$\langle \varphi_1, \varphi_2 \rangle = \int_{Z_k(\mathbb{A})\mathrm{GL}_k(F)\backslash \mathrm{GL}_k(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\, dg, \quad \varphi_i \in V_\pi$$

with the Tamagawa measure dg. Then we decompose the Petersson inner product into a product of local Hermitian pairing as $\langle\, ,\, \rangle = \prod_v \langle\, ,\, \rangle_v$ of V_π. Our main result in the general linear group case is stated as follows.

Theorem 2.1. *Let π and η be as above. Then for any non-zero decomposable vector $\varphi = \otimes \varphi_v \in V_\pi$, we have*

$$\text{(2.0.1)} \quad \frac{|B_{\psi,\eta}(\varphi)|^2}{\langle \varphi, \varphi \rangle} = C_1 \left(\prod_{j=2}^{n} \zeta_F(j) \right) \times \frac{L\left(\frac{1}{2}, \pi \times \eta\right) L\left(\frac{1}{2}, \pi^\vee \times \eta^{-1}\right)}{L(1, \pi, \mathrm{Ad})\, \mathrm{Res}_{s=1}\zeta_F(s)} \prod_v \frac{\alpha_v^\natural(\varphi_v, \varphi_v)}{\langle \varphi_v, \varphi_v \rangle_v}$$

where $\alpha_v^\natural(\varphi_v, \varphi_v)$ denotes the normalized local Bessel period.

The idea of our proof of this theorem is same as that of Theorem 1.1. Indeed, we may compute the pull-back of Whittaker period of the theta lift from GL_{2n} to GL_{2n} as in the unitary group case. See [2] for the detail. Moreover, Lapid and Mao [6] showed Ichino-Ikeda type formula of Whittaker periods for GL_m in general. Since local Bessel period in this case is same as that of the unitary group case at split place, the local pull-back computation in this case is given in the proof of Theorem 1.1. Hence, our remaining task is to prove an analogue of Rallis inner product formula for our theta lifts. Indeed, this is proved by a direct computation with an explicit description of theta lifts given by Watanabe [14, Lemma 3]. For the convenience to the reader, we only state our inner product formula, and see [2] for the detail.

We denote $\mathcal{S}(\mathrm{Mat}_{\ell,m}(\mathbb{A}))$, the space of Schwartz functions on $\mathrm{Mat}_{\ell,m}(\mathbb{A})$, by $\mathcal{S}_{\ell,m}(\mathbb{A})$. We also write $\mathcal{S}(\mathrm{Mat}_{\ell,m}(F_v))$ by $\mathcal{S}_{\ell,m}(F_v)$ for any place v of F. Then the Weil representation $\omega_{\ell,m}$ of $\mathrm{GL}_\ell(\mathbb{A}) \times \mathrm{GL}_m(\mathbb{A})$ on $\mathcal{S}_{\ell,m}(\mathbb{A})$ is given by

$$(\omega_{\ell,m}(g,h)\phi)(x) = \alpha(g)^{-\frac{m}{2}} \alpha(h)^{\frac{\ell}{2}} \phi(g^{-1}xh)$$

for $(g,h) \in \mathrm{GL}_\ell(\mathbb{A}) \times \mathrm{GL}_m(\mathbb{A})$ and $\phi \in \mathcal{S}_{\ell,m}(\mathbb{A})$, where $\alpha(g) = |\det g|$. Let ξ be a unitary character of $\mathbb{A}^\times / F^\times$. We regard ξ as a character of $Z_m(\mathbb{A})$, where Z_m denotes the center of GL_m. For $s \in \mathbb{C}$ and $\phi \in \mathcal{S}_{\ell,m}(\mathbb{A})$, we define theta functions $\theta(s, \xi, \phi)$ by

$$\theta(s, \xi, \phi) = \int_{Z_m(F)\backslash Z_m(\mathbb{A})} \xi(z)\, \alpha(z)^{s+\frac{\ell}{2}} \sum_{x \in \mathrm{Mat}_{\ell,m}(F),\, x \neq 0} \phi(zx)\, dz$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$. It has a holomorphic continuation to $\mathbb{C}$ by [3, Lemma 11.5, 11.6].

Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ with μ_σ its central character. Then for $\phi \in \mathcal{S}_{m,m}(\mathbb{A})$ and $s \in \mathbb{C}$, we define theta lift $\theta(\varphi, \phi, s)$ of $\varphi \in V_\pi$ to $\mathrm{GL}_m(\mathbb{A})$ by

$$\theta(\varphi, \phi, s)(g) := \int_{Z_m(\mathbb{A})\mathrm{GL}_m(F)\backslash \mathrm{GL}_m(\mathbb{A})} \varphi(h)\alpha(h)^s \theta(s, \mu_\pi, \omega_{\ell,m}(g,h)\phi)\, dh.$$

Then, as remarked in [14, p.707], $\theta(\varphi, \phi, s)$ is cusp form on $\mathrm{GL}_\ell(\mathbb{A})$ for $\mathrm{Re}(s) \gg 0$. Moreover, it has a holomorphic continuation to $\mathbb{C}$ as remarked above, and we simply write $\theta(\varphi, \phi) = \theta(\varphi, \phi, 0)$. For $\phi_i \in \mathcal{S}_{m,m}(\mathbb{A})$, we define

$$(\phi_1, \phi_2) := \int_{\mathrm{Mat}_{m\times m}(\mathbb{A})} \phi_1(x)\overline{\phi_2(x)}\, dx.$$

Then we set

$$\Phi^s_\phi(g) = (\omega_{m,m}(1,g)(\phi \cdot \alpha^s), \phi \cdot \alpha^s)$$

for $\phi \in \mathcal{S}_{m,m}(\mathbb{A})$. An inner product formula of our theta lift is given as follows.

Theorem 2.2. *Let $\phi = \otimes \phi_v \in \mathcal{S}_{m,m}(\mathbb{A})$ and $0 \neq \varphi = \otimes \varphi_v \in \sigma$. Then we have*

$$\frac{\langle \theta(\varphi, \phi), \theta(\varphi, \phi)\rangle}{\langle \varphi, \varphi \rangle} = C_1 \cdot L\left(\frac{1}{2}, \sigma\right) L\left(\frac{1}{2}, \sigma^\vee\right) \prod_v Z_v(\varphi_v, \phi_v, 0)$$

where $\sigma^\vee$ denotes the contragredient of σ. Here we define

$$Z_v(\varphi_v, \phi_v, s) := \frac{1}{L\left(s+\frac{1}{2}, \sigma_v\right) L\left(s+\frac{1}{2}, \sigma_v^\vee\right)} \int_{\mathrm{GL}_m(F_v)} \frac{\langle \pi_v(g_v)\varphi_v, \varphi_v\rangle_v}{\langle \varphi_v, \varphi_v\rangle_v} \cdot \Phi^s_{\phi_v}(g_v)\, dg_v$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$ and has a holomorphic continuation to $\mathbb{C}$.

Acknowledgements. This paper is based on the author's talk given at 2023 RIMS conference *Analytic and arithmetic aspects of automorphic representations*. The author thanks the organizers of the conference, Tadashi Miyazaki and Hiroki Aoki, for giving him the opportunity to talk.

The research of the author was supported in part by JSPS KAKENHI Grant Number 17K14166 and 21K03164.

References

[1] M. Furusawa and K. Morimoto, *Refined global Gross-Prasad conjecture on special Bessel periods and Böcherer's conjecture.* J. Eur. Math. Soc. (JEMS) **23** (2021), no. 4, 1295–1331.
[2] M. Furusawa and K. Morimoto, *On the Gan-Gross-Prasad conjecture and its refinement for* $(\mathrm{U}(2n), \mathrm{U}(1))$. preprint, arXiv:2205.09471
[3] R. Godement and H. Jacquet, *Zeta Functions of Simple Algebras,* Lecture Notes in Mathematics, Vol. **260**. Springer-Verlag, Berlin-New York, 1972. ix+188 pp.
[4] T. Kaletha, A. Minguez, S.W. Shin, and P.J. White, *Endoscopic classification of representations: Inner forms of unitary groups.* Preprint, arXiv:1409.3731
[5] S. Kudla, *Splitting metaplectic covers of dual reductive pairs.* Israel J. Math. **87** (1994), 361–401.
[6] E. Lapid and Z. Mao, *A conjecture on Whittaker-Fourier coefficients of cusp forms.* J. Number Theory **146** (2015), 448–505.
[7] E. Lapid and S. Rallis, *On the local factors of representations of classical groups.* In: Automorphic Representations, L-Functions and Applications: Progress and Prospects, pp. 309–359. de Gruyter, Berlin (2005).

[8] Y. Liu, *Refined Gan-Gross-Prasad conjecture for Bessel periods* J. Reine Angew. Math. **717** (2016) 133–194.
[9] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups.* Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248 pp.
[10] K. Morimoto, *On Gan-Gross-Prasad conjecture for* $(\mathrm{U}(2n), \mathrm{U}(1))$ *and* $(\mathrm{SO}(5), \mathrm{SO}(2))$. 数理解析研究所講究録, 2197, 1-8 (2020)
[11] K. Morimoto, *On a certain local identity for Lapid-Mao's conjecture and formal degree conjecture: even unitary group case.* J. Inst. Math. Jussieu **21** (2022), no. 4, 1107–1161.
[12] K. Morimoto, *On Ichino-Ikeda type formula of Whittaker periods for unitary groups.* preprint.
[13] F. Shahidi, *On certain L-functions.* Amer. J. Math. **103** (1981) 297–355.
[14] T. Watanabe, *Global Theta Liftings of General Linear Groups.* J. Math. Sci. Univ. Tokyo **3** (1996), 699–711.
[15] S. Yamana, *L-functions and theta correspondence for classical groups.* Invent. Math. **196** (2014), no. 3, 651–732.