On Hutchinson's conjecture

Yoshinori Mizuno

I report on a proof of a conjectual extention of Gross-Zagier's formula proposed by Tim Hutchinson.

1 Elliptic modular function $j(\tau)$

Let $\mathfrak{H} := \{\tau = u + iv; v > 0\}$ be the upper-half plane. The elliptic modular function $j(\tau)$ $(\tau \in \mathfrak{H})$ is defined by

$$j(\tau) := 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = \frac{1}{q} + 744 + 196884q + O(q^2) \quad (q = e^{2\pi i \tau}),$$

where $g_2(\tau)$, $g_3(\tau)$ are the Eisenstein series

$$g_2(\tau) := 60 \sum_{m,n\in\mathbb{Z},\ (m,n)\neq(0,0)} \frac{1}{(m\tau+n)^4}, \quad g_3(\tau) := 140 \sum_{m,n\in\mathbb{Z},\ (m,n)\neq(0,0)} \frac{1}{(m\tau+n)^6}.$$

The definition is rather simple, but this j function has many remarkable properties. It classifies isomorphism classes of elliptic curves defined over $\mathbb{C}$. Its Fourier coefficients are integers, and can be described analytically by Rademacher-Petersson's formula, arithmetically by Kaneko's formula. Congruences satisfied by Fourier coefficients are studies by several researchers. Moreover, Fourier coefficients of $j(\tau)$ relate dimensions of the irreducible representations of the monster group. We refer to [7] for more information.

Special values of $j(\tau)$ at imaginary quadratic irrationals are impressive. Let $\mathbb{X}_d^{(+)}$ be the set of all imaginary quadratic irrationals in $\mathfrak{H}$ of discriminant $d < 0$ defined by

$$\mathbb{X}_d^{(+)} := \left\{ \frac{b + i\sqrt{|d|}}{2a} \in \mathfrak{H}; a, b, c \in \mathbb{Z}, a > 0, (a, b, c) = 1, b^2 - 4ac = d \right\}.$$

It is well known that $j(\tau)$ for $\tau \in \mathbb{X}_d^{(+)}$ (called "singular modulus") is an algebraic interger of degree $h(d)$, where $h(d)$ is the class number of the order $\mathcal{O}_d = \mathbb{Z} + \frac{\sigma_d + i\sqrt{|d|}}{2}\mathbb{Z}$ ($\sigma_d \in \{0, 1\}$, $\sigma_d \equiv d \pmod 4$). We extract such special values from Cox [1]; $j(\sqrt{-1}) = 12^3$ $(h(-4) = 1)$, $j\left(\frac{1+\sqrt{-7}}{2}\right) = -(15)^3$ $(h(-7) = 1)$,

$$j(\sqrt{-14}) = 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3 \quad (h(-56) = 4).$$

As shown in [1], we can construct an abelian extension (called "ring class field") over the imaginary quadratic field $K := \mathbb{Q}(\sqrt{d})$ by adjoining an special value at $\tau \in \mathbb{X}_d^{(+)}$; $\mathrm{Gal}(K(j(\tau))/K) \cong \mathcal{C}_d$ (the ideal class group of the order $\mathcal{O}_d$) and $\mathrm{Gal}(K(j(\tau))/\mathbb{Q}) \cong \mathcal{C}_d \rtimes (\mathbb{Z}/2\mathbb{Z})$.

2 Work of Gross-Zagier

Let d_j $(j = 1, 2)$ be negative fundamental discriminants. Let $\mathfrak{X}^{(+)}_{d_j} = \mathbb{X}^{(+)}_{d_i} / \sim_+$ be the set of equivalence classes of quadratic irrationals in the upper-half plane $\mathfrak{H}$ of discriminant d_j with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$. The class of $\tau \in \mathbb{X}^{(+)}_{d_j}$ is denoted by $[\tau]_{\sim_+}$. The number $h(d_j) = \sharp(\mathfrak{X}^{(+)}_{d_j})$ is nothing but the class number of $\mathbb{Q}(\sqrt{d_j})$. Let $w_j = \sharp(\mathcal{O}^{\times}_{d_j})$ be the number of the units of discriminant d_j.[1]

2.1 Definition of $J(d_1, d_2)$

The value $J(d_1, d_2)$ is defined as the (modified) resultant of the class equations of discriminant d_j $(j = 1, 2)$ by

$$J(d_1, d_2) := \left(\prod_{[\tau_1]_{\sim_+} \in \mathfrak{X}^{(+)}_{d_1}} \prod_{[\tau_2]_{\sim_+} \in \mathfrak{X}^{(+)}_{d_2}} (j(\tau_1) - j(\tau_2)) \right)^{\frac{4}{w_1 w_2}} .$$

When d_1 and d_2 are coprime, it is known that the value $J(d_1, d_2)^2$ is an integer. In addition the resultant $J(d_1, d_2)$ itself is an integer, if both d_j $(j = 1, 2)$ are smaller than -4. Gross and Zagier established a closed formula of $J(d_1, d_2)^2 \in \mathbb{Z}$ when $(d_1, d_2) = 1$.

2.2 Definition of ϵ

To state the Gross-Zagier formula, let d_1 and d_2 be coprime negative fundamental discriminants as above.[2] Let q be a prime such that $\left(\frac{d_1 d_2}{q}\right) \neq -1$. According to [2], we put $\epsilon(q) := \begin{cases} \left(\frac{d_1}{q}\right) & \text{if } q \nmid d_1, \\ \left(\frac{d_2}{q}\right) & \text{if } q \nmid d_2. \end{cases}$ In general, the function ϵ is defined as a completely multiplicative function on the set of integers $m \geqq 1$ such that any prime $q \mid m$ satisfies $\left(\frac{d_1 d_2}{q}\right) \neq -1$.

2.3 Definition of $F(m)$

For m such that $\epsilon(m)$ is meaningful, we put $F(m) := \prod_{nn'=m,\ n,n'>0} n^{\epsilon(n')}$. Here the product is over all positive divisors n of m, and n' is defined as m/n.

2.4 Gross-Zagier formula (1985)

Gross and Zagier [2] establised the following closed formula of the value $J(d_1, d_2)^2$. If $d_j < 0$ $(j = 1, 2)$ are fundamental discriminants such that $(d_1, d_2) = 1$, then

$$J(d_1, d_2)^2 = \pm \prod_{x^2 < d_1 d_2,\ x^2 \equiv d_1 d_2 \pmod 4} F\left(\frac{d_1 d_2 - x^2}{4}\right).$$

We extract some examples from [2]. As mentioned there, the numbers are rather highly factorizable.

Example 1 ([2, p. 191])

$$J(-163, -4) = j\left(\frac{1 + i\sqrt{163}}{2}\right) - j(i) = -2^6 \cdot 3^6 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127^2 \cdot 163.$$

[1] w_j is 4 for $d_j = -4$, whereas 6 for $d_j = -3$, and $w_j = 2$ if $d_j < -4$

[2] So the product $d_1 d_2 > 0$ is a fundamental discriminant.

Example 2 ([2, p. 193])

$$J(-67,-163) = j\left(\frac{1+i\sqrt{67}}{2}\right) - j\left(\frac{1+i\sqrt{163}}{2}\right) = 2^{15}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 13\cdot 139\cdot 331.$$

2.5 Simple formula of $F(m)$

As noted in [2, p. 192], the arithmetical function $F(m)$ has the following simple description. Let $m\in\mathbb{N}$ be of the form $\frac{d_1d_2-x^2}{4}$ $(x\in\mathbb{Z})$. We consider the prime factorization of m and classify prime factors according to the values of ϵ. Depending on the number of prime factors p of m satisfying $2\nmid \mathrm{ord}_p(m)$ and $\epsilon(p)=-1$,[3] the simple formula for $F(m)$ is given as follows.

(i) When m has the form $m = p^{2a+1}p_1^{2a_1}\cdots p_r^{2a_r}q_1^{b_1}\cdots q_s^{b_s}$ $(\epsilon(p)=\epsilon(p_i)=-1,\ \epsilon(q_i)=1)$, then $F(m)=p^{(a+1)(b_1+1)\cdots(b_s+1)}$.

(ii) Otherwise, $F(m)=1$.

As a corollary, we can deduce that the primes dividing $J(d_1,d_2)^2$ are rather restricted ([2, p. 192]).

Corollary 1. *Let q be a prime with $q\mid J(d_1,d_2)^2$. Then*

(a) $\left(\frac{d_1}{q}\right)\neq -1$, $\left(\frac{d_2}{q}\right)\neq -1$,

(b) *q divides a nutural number of the form* $\frac{d_1d_2-x^2}{4}$,

(c) $q\leqq\frac{d_1d_2}{4}$. *More precisely,*

(i) *if $d_1d_2\equiv 1$ (mod 8) then* $q<\frac{d_1d_2}{8}$, (ii) *if $d_1\equiv d_2\equiv 5$ (mod 8) then* $q<\frac{d_1d_2}{16}$.

3 Sketch of the analytic proof due to Gross-Zagier

We shall review the analytic proof of the Gross-Zagier formula given in [2].[4] This proof consists of analytic and arithmetic techniques. First of all, it is noted that $F(m)=\prod_{nn'=m,\ n,n'>0} n^{\epsilon(n')} = \prod_{0<n|m} n^{-\epsilon(n)}$. Taking the logarithm, the Gross-Zagier formula is restated as

$$-\frac{4}{w_1w_1}\sum_{\substack{[\tau_1]_{\sim_+}\in\mathfrak{X}_{d_1}^{(+)}\\ [\tau_2]_{\sim_+}\in\mathfrak{X}_{d_2}^{(+)}}}\log|j(\tau_1)-j(\tau_2)|^2 = \sum_{x^2<d_1d_2,\ x^2\equiv d_1d_2\ (\mathrm{mod} 4)}\ \sum_{n|\frac{d_1d_2-x^2}{4}}\epsilon(n)\log n.$$

The left hand side is the CM average of $\log|j(z)-j(w)|^2$.

3.1 Automorphic Green function $G_s(\tau_1,\tau_2)$

Let us introduce an automorphic Green function $G_s(\tau_1,\tau_2)$ (cf. [5]). For $s\in\mathbb{C}$, $\Re(s)>0$, let $Q_{s-1}(t)$ $(t>1)$ be the Legendre function of the 2nd kind

$$Q_{s-1}(t) := \int_0^\infty\left(t+\sqrt{t^2-1}\cosh v\right)^{-s}dv,$$

or, in terms of the Gauss hypergeometric function, it is given as $(0<t<1)$

$$Q_{s-1}\left(\frac{1+t}{1-t}\right) = \frac{\Gamma(s)^2}{2\Gamma(2s)}(1-t)^sF(s,s;2s;1-t).$$

[3] $\mathrm{ord}_p(m)$ is a non-negative integer such that $p^{\mathrm{ord}_p(m)}$ is the highest power of p dividing m.

[4] There is a geometric proof. We refer to the first half of [2].

Using the Legendre function $Q_{s-1}(t)$ together with the hyperbolic distance function $d(\tau_1, \tau_2)$, we define $g_s(\tau_1, \tau_2)$ for $\tau_j = u_j + iv_j \in \mathfrak{H}$ $(j = 1, 2)$, $\tau_1 \neq \tau_2$ by

$$g_s(\tau_1, \tau_2) := -2Q_{s-1}(\cosh d(\tau_1, \tau_2)) = -2Q_{s-1}\left(\frac{(u_1 - u_2)^2 + v_1^2 + v_2^2}{2v_1v_2}\right).$$

The automorphic Green function $G_s(\tau_1, \tau_2)$ is then defined by

$$G_s(\tau_1, \tau_2) := \sum_{\gamma \in \Gamma} g_s(\tau_1, \gamma\tau_2), \qquad (\Gamma := \mathrm{PSL}_2(\mathbb{Z}),\ \Re(s) > 1).$$

Let $E(\tau, s)$ be the Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ and $\varphi(s)$ the function appears in its constant term;

$$E(\tau, s) := \frac{1}{2} \sum_{c,d \in \mathbb{Z}, (c,d)=1} \frac{v^s}{|c\tau + d|^{2s}} \ \ (\Re(s) > 1), \qquad \varphi(s) := \frac{\Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})\zeta(2s - 1)}{\Gamma(s)\zeta(2s)}.$$

Using these functions, Gross and Zagier show the following proposition.

Proposition 1. (*Gross-Zagier, Proposition 5.1*) *The series $G_s(\tau_1, \tau_2)$ can be continued meromorphically to the whole s-plane. For τ_1, $\tau_2 \in \mathfrak{H}$ such that $\tau_1 \not\sim_+ \tau_2$ ($\mathrm{SL}_2(\mathbb{Z})$-inequivalent), one has*

$$\log|j(\tau_1) - j(\tau_2)|^2 = \lim_{s \to 1, s \in \mathbb{R}} (G_s(\tau_1, \tau_2) + 4\pi E(\tau_1, s) + 4\pi E(\tau_2, s) - 4\pi\varphi(s)) - 24.$$

As functions of the variable $\tau_1 \in \mathfrak{H} \setminus \{\gamma\tau_2; \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ ($\tau_2 \in \mathfrak{H}$ fixed), both sides are harmonic and invariant with respect to $\mathrm{SL}_2(\mathbb{Z})$. The difference has an extension to a harmonic function on $\mathfrak{H}$ and it tends to zero as $\Im(\tau_1) \to \infty$. Hence the proposition can be deduced from the maximum principle of harmonic functions.[5]

3.2 CM average

We consider CM average of the automorphic Green function $G_s(z, w)$. Starting from the definition and using a kind of unfolding argument, we can rewrite the CM average as the sum of $g_s(\tau_1, \tau_2)$ over imaginary quadratic irrationals of discriminants d_1 and d_2 modulo the diagonal action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. That is

$$\frac{2}{w_1}\frac{2}{w_1} \sum_{[\tau_1]_{\sim_+} \in \mathfrak{X}_{d_1}^{(+)},\ [\tau_2]_{\sim_+} \in \mathfrak{X}_{d_2}^{(+)}} G_s(\tau_1, \tau_2) = \sum_{(\tau_1, \tau_2) \in \Gamma \backslash (\mathbb{X}_{d_1}^{(+)} \times \mathbb{X}_{d_2}^{(+)})} g_s(\tau_1, \tau_2).$$

If we write $\tau_j = \frac{b_j + \sqrt{d_j}}{2a_j}$, the corresponding binary form is $[a_j, b_j, c_j] := a_jx^2 + b_jxy + c_jy^2$ ($c_j = \frac{b_j^2 - d_j}{4a_j}$) and we see that

$$g_s(\tau_1, \tau_2) = -2Q_{s-1}\left(\frac{n}{\sqrt{\Delta}}\right),\ \ \Delta = d_1d_2,\ \ n = 2a_1c_2 + 2a_2c_1 - b_1b_2.$$

Collecting the terms according to the value n, one has

$$\frac{2}{w_1}\frac{2}{w_1} \sum_{[\tau_1]_{\sim_+} \in \mathfrak{X}_{d_1}^{(+)},\ [\tau_2]_{\sim_+} \in \mathfrak{X}_{d_2}^{(+)}} G_s(\tau_1, \tau_2) = -2 \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d_1, d_2, -n) Q_{s-1}\left(\frac{n}{\sqrt{\Delta}}\right),$$

[5] $\lim_{s \to 1, s \in \mathbb{R}}$ may be replaced by $\lim_{s \to 1}$.

where $h(d_1, d_2, -n)$ is the *class number of pairs*[6] defined by

$$h(d_1, d_2, -n) := \sharp(\{(f, F) \in \mathcal{Q}_{d_1}^{(+)} \times \mathcal{Q}_{d_2}^{(+)}; \mathcal{B}(f, F) = -n\}/\sim).$$

Here $\mathcal{Q}_{d_j}^{(+)}$ is the set of all positive-definite primitive binary quadratic forms of discriminant $d_j < 0$, we put $\mathcal{B}(f, F) = bB - 2aC - 2cA$ (*codiscriminant*) for $(f, F) \in \mathcal{Q}_{d_1}^{(+)} \times \mathcal{Q}_{d_2}^{(+)}$ with $f = [a, b, c]$, $F = [A, B, C]$, and for two pairs (f, F), $(g, G) \in \mathcal{Q}_{d_1}^{(+)} \times \mathcal{Q}_{d_2}^{(+)}$, we define $(f, F) \sim (g, G)$ by the diagonal action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_{d_1}^{(+)} \times \mathcal{Q}_{d_2}^{(+)}$.[7]

Recall that for fundamental discriminants $d_j < 0$ the CM average of the Eisenstein series can be described by the Dedekind zeta function as

$$\sum_{[\tau]_{\sim_+} \in \mathfrak{X}_{d_j}^{(+)}} \frac{E(\tau, s)}{w(d_j)} = \frac{1}{2} \left(\frac{|d_j|}{4}\right)^{s/2} \frac{\zeta_{\mathbb{Q}(\sqrt{d_j})}(s)}{\zeta(2s)}.$$

Together with Proposition 1, one has the following infinite series expression of the left hand side of the Gross-Zagier formula.

Proposition 2. (*Gross-Zagier, Proposition 5.3*) *Let $d_1 < 0$, $d_2 < 0$ be fundamental discriminants with $d_1 \neq d_2$.*[8] *Put $\Delta = d_1 d_2$, $h'_j = \frac{2}{w_j} h(d_j)$. Then*

$$\log |J(d_1, d_2)|^2 = \lim_{s \to 1, s \in \mathbb{R}} \left[-2 \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d_1, d_2, -n) Q_{s-1}\left(\frac{n}{\sqrt{\Delta}}\right) \right.$$
$$\left. + \frac{4\pi}{\zeta(2s)} \left(h'_2 \left|\frac{d_1}{4}\right|^{\frac{s}{2}} \zeta_{K_1}(s) + h'_1 \left|\frac{d_2}{4}\right|^{\frac{s}{2}} \zeta_{K_2}(s) - h'_1 h'_2 \frac{\Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s-1) \right) - 24 h'_1 h'_2 \right].$$

3.3 Some arithmetical results

To push forward, Gross and Zagier prepare some arithmetical results. The first is an explicit formula of the class number of pairs.

Proposition 3. (*Gross-Zagier, Proposition 6.1*) *Let $d_j < 0$ ($j = 1, 2$), $(d_1, d_2) = 1$ be as above and put $\Delta = d_1 d_2$. Take $\delta \in \mathbb{Z}$ such that $\delta < 0$, $\delta^2 > \Delta$, $\delta^2 \equiv \Delta$ (mod 4). Then*

$$h(d_1, d_2, \delta) = \sum_{\mu | \frac{\delta^2 - \Delta}{4}} \epsilon(\mu).$$

Another key observation is that several quantities in the Gross-Zagier formula have relation with Fourier coefficients of a Hilbert-Eisenstein series $E_s((z, z'))$ for the real quadratic field of discriminant $\Delta = d_1 d_2$. The precise definition of $E_s((z, z'))$ will be given in the next subsection. Let $\chi_{d_1, d_2}^{(\Delta)} : \mathcal{C}_\Delta^+ \to \{\pm 1\}$ be the genus character. For $\nu \in \frac{1}{\sqrt{\Delta}} \mathcal{O}_\Delta$, the ν-th Fourier coefficient of $E_s((z, z'))$ is given by

$$\sigma_{s, \chi_{d_1, d_2}^{(\Delta)}}((\sqrt{\Delta}\nu)) := \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_\Delta \\ (\nu\sqrt{\Delta}) \subset \mathfrak{b}}} \chi_{d_1, d_2}^{(\Delta)}(\mathfrak{b}) \mathfrak{N}_\Delta(\mathfrak{b})^s,$$

where the sum is over all ideals $\mathfrak{b}$ of $\mathcal{O}_\Delta$ such that $(\nu\sqrt{\Delta}) := \nu\sqrt{\Delta}\mathcal{O}_\Delta \subset \mathfrak{b}$.

[6] This terminology is not used in [2] but in [4].

[7] See subsection 7.1 for details.

[8] The proof works not only for $(d_1, d_2) = 1$ but also for $d_1 \neq d_2$.

Proposition 4. (*Gross-Zagier, p. 217, the 2nd displayed formula*) *Let* $\Delta = d_1 d_2$ *be as above. For* $\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta$ *with* $\mathrm{Tr}(\nu) := \nu + \nu' = 1$, $\nu > 0 > \nu'$ *(*ν' *: the conjugate of* ν*), there is* $x \in \mathbb{Z}$ *such that* $\sqrt{\Delta}\nu = \frac{x+\sqrt{\Delta}}{2}$, $x > \sqrt{\Delta}$, $x \equiv \Delta$ *(mod 2). With this setting, one has*

$$h(d_1, d_2, -x) = \sigma_{0,\chi^{(\Delta)}_{d_1,d_2}}((\sqrt{\Delta}\nu)).$$

The identity we want to prove is (see the beginning of section 3)

$$-\log|J(d_1,d_2)|^2 = \sum_{x^2<\Delta,\ x^2\equiv\Delta \pmod 4} \sum_{n|\frac{\Delta-x^2}{4}} \epsilon(n)\log n.$$

Using

- the bijection from $A_x = \{\mathfrak{b};\ \text{ideal of } \mathcal{O}_\Delta, \frac{x+\sqrt{\Delta}}{2} \in \mathfrak{b}\}$ to $B_x = \{n \in \mathbb{N}\ ;\ n \mid \frac{x^2-\Delta}{4}\}$ defined by the norm map $\mathfrak{b} \mapsto \mathfrak{N}_\Delta(\mathfrak{b}) := (\mathcal{O}_\Delta : \mathfrak{b})$,
- the relation $\epsilon(n) = \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})$ $(n = \mathfrak{N}_\Delta(\mathfrak{b}) := (\mathcal{O}_\Delta : \mathfrak{b}))$,

we can rewrite the above identity as

$$-\log|J(d_1,d_2)|^2 = \sum_{\substack{\nu\in\frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta \\ \mathrm{Tr}(\nu)=1,\nu\not\succ 0}} \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_\Delta \\ \nu\sqrt{\Delta}\in\mathfrak{b}}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})\log(\mathfrak{N}_\Delta(\mathfrak{b})).$$

Here $\nu \not\succ 0$ means $\nu > 0, \nu' > 0$. Gross and Zagier deduce this identity (hence the Gross-Zagier formula) from a suitable relation among the Fourier coefficients of $E_s((z,z'))$.

3.4 Hilbert-Eisenstein series for the real quadratic field of discriminant $d_1 d_2$

Let $d_j < 0$ $(j = 1,2)$ be fundamental discriminants such that $(d_1, d_2) = 1$. Then $\Delta = d_1 d_2 > 0$ is again a fundamental discriminant. The genus character $\chi^{(\Delta)}_{d_1,d_2} : \mathcal{C}^+_\Delta \to \{\pm 1\}$ corresponding to the decomposition $\Delta = d_1 d_2$ is defined from the value $\chi^{(\Delta)}_{d_1,d_2}(\mathfrak{p}) = \begin{cases} \left(\frac{d_1}{\mathfrak{N}_\Delta(\mathfrak{p})}\right) & \text{if } (\mathfrak{N}_\Delta(\mathfrak{p}), d_1) = 1, \\ \left(\frac{d_2}{\mathfrak{N}_\Delta(\mathfrak{p})}\right) & \text{if } (\mathfrak{N}_\Delta(\mathfrak{p}), d_2) = 1, \end{cases}$ through the unique factorization into prime ideals $\mathfrak{p}$. For an $\mathcal{O}_\Delta$-fractional ideal $\mathfrak{a}$ and for $s \in \mathbb{C}$ $(\Re(s) > 1)$, $(z,z') \in \mathfrak{H}\times\mathfrak{H}$, $z = x + iy$, $z' = x' + iy'$, the partial Eisenstein series for $\mathfrak{a}$ is defined by

$$E((z,z'),s,\mathfrak{a}) := \sum_{\{m,n\}\in\mathrm{Equ}(\mathfrak{a})} \frac{y^s y'^s}{(mz+n)(m'z'+n')|mz+n|^{2s}|m'z'+n'|^{2s}},$$

where the sum is over $\mathrm{Equ}(\mathfrak{a}) := (\mathfrak{a}\times\mathfrak{a}\setminus\{(0,0)\})/\sim$ with the equivalent relation "$\sim$" defined by the diagonal action of $\mathcal{O}^\times_\Delta$. Taking an average over wide ideal classes with twisting by the genus character, Gross and Zagier define the Eisenstein series to be required;

$$E_s((z,z')) := \sum_{[\mathfrak{a}]\in\mathcal{C}_\Delta} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{a})\mathfrak{N}_\Delta(\mathfrak{a})^{1+2s}E((z,z'),s,\mathfrak{a}).$$

Note that each summand is well defined by a nature of the genus character, and we extend $\mathfrak{N}_\Delta(\mathfrak{a}) := n^{-2}\mathfrak{N}_\Delta(n\mathfrak{a})$, where n is any natural number such that $n\mathfrak{a}$ is an ideal of $\mathcal{O}_\Delta$ and $\mathfrak{N}_\Delta(n\mathfrak{a}) = (\mathcal{O}_\Delta : n\mathfrak{a})$.

We have the Fourier expansion by a standard method ([2, p. 215]);

$$\begin{aligned} E_s((z,z')) &= L(2s+1,\chi^{(\Delta)}_{d_1,d_2})y^s y'^s + \frac{1}{\sqrt{\Delta}}L(2s,\chi^{(\Delta)}_{d_1,d_2})\Phi_s(0)^2 y^{-s}y'^{-s} \\ &+ \frac{1}{\sqrt{\Delta}}y^{-s}y'^{-s} \sum_{\nu\in\frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta\setminus\{0\}} \sigma_{-2s,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt{\Delta}))\Phi_s(\nu y)\Phi_s(\nu' y')e(\nu x + \nu' x'), \end{aligned}$$

where $L(s,\chi^{(\Delta)}_{d_1,d_2})$ is the L-function of $\chi^{(\Delta)}_{d_1,d_2}$, $\sigma_{s,\chi^{(\Delta)}_{d_1,d_2}}(\mathfrak{c}) := \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_\Delta\\ \mathfrak{c}\subset\mathfrak{b}}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b})\mathfrak{N}_\Delta(\mathfrak{b})^s$ for a non-zero ideal $\mathfrak{c}$ of $\mathcal{O}_\Delta$ is the ideal divisor function, $\Phi_s(t)$ is defined by $\Phi_s(t) := \int_{\mathbb{R}} \frac{e(-tu)}{(u+i)(u^2+1)^s}du$, which can be described in terms of the confluent hypergeometric function, and $e(z) := e^{2\pi i z}$ as usual.

Gross and Zagier compute the holomorphic projection of the diagonal restriction of the derivative at $s=0$ of the Hilbert-Eisenstein series. The Fourier expansion of $\partial_s E((z,z),s)|_{s=0}$ is given by ([2, p. 215])

$$\begin{aligned}\frac{\partial}{\partial s}E_s((z,z'))\Big|_{s=0} &= 2L(1,\chi^{(\Delta)}_{d_1,d_2})\log(yy') + 4C_{\chi^{(\Delta)}_{d_1,d_2}}\\ &+ 8\pi^2\Delta^{-\frac12}\sum_{\nu\in\frac{1}{\sqrt\Delta}\mathcal{O}_\Delta,\ \nu\gg 0}\sigma'_{\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))e(\nu z+\nu' z')\\ &- 4\pi^2\Delta^{-\frac12}\sum_{\nu\in\frac{1}{\sqrt\Delta}\mathcal{O}_\Delta,\ \nu>0>\nu'}\sigma_{0,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))\Phi(|\nu'|y')e(\nu z+\nu' z')\\ &- 4\pi^2\Delta^{-\frac12}\sum_{\nu\in\frac{1}{\sqrt\Delta}\mathcal{O}_\Delta,\ \nu<0<\nu'}\sigma_{0,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))\Phi(|\nu|y)e(\nu z+\nu' z'),\end{aligned}$$

where $C_{\chi^{(\Delta)}_{e_1,e_2}} = L'(1,\chi^{(\Delta)}_{d_1,d_2}) + \left(\frac12\log\Delta - \log\pi - \gamma\right)L(1,\chi^{(\Delta)}_{d_1,d_2})$ (γ is the Euler constant) and

$$\sigma'_{\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta)) = \frac{\partial}{\partial s}\sigma_{-2s,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))\Big|_{s=0},\quad \Phi(t) := \frac{i}{2\pi}e^{-2\pi t}\frac{\partial}{\partial s}\Phi_s(-t)\Big|_{s=0}.$$

The Fourier expansion of its diagonal restriction (with a suitable normalization) is given by ([2, p. 216])

$$\begin{aligned}F(z) := \frac{\sqrt\Delta}{8\pi^2}\frac{\partial}{\partial s}E_s((z,z))\Big|_{s=0} &= \frac{\sqrt\Delta}{2\pi^2}\left(L(1,\chi^{(\Delta)}_{d_1,d_2})\log y + C_{\chi^{(\Delta)}_{d_1,d_2}}\right)\\ + \sum_{\nu\in\frac{1}{\sqrt\Delta}\mathcal{O}_\Delta,\ \nu\gg0}\sigma'_{\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))e(\mathrm{Tr}(\nu)z) &- \sum_{\nu\in\frac{1}{\sqrt\Delta}\mathcal{O}_\Delta,\ \nu>0>\nu'}\sigma_{0,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt\Delta))\Phi(|\nu'|y)e(\mathrm{Tr}(\nu)z).\end{aligned}$$

Gross and Zagier apply the holomorphic projection lemma to this $F(z)$.

Proposition 5. (*Gross-Zagier, Proposition 7.3*) *Suppose that a continuaous function $F:\mathfrak{H}\to\mathbb{C}$ satisfies*

(i) $F(\gamma z) = (cz+d)^2F(z)$ for all $z\in\mathfrak{H}, \gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$,

(ii) there are constants A, B and a positive number ϵ such that $F(z) = A\log y + B + O(y^{-\epsilon})$ as $y := \Im(z)\to\infty$ (uniformly in $x := \Re(z)$),

(iii) F has an absolutely convergent Fourier series $F(z) = \sum_{m=-\infty}^{\infty} a_m(y)e(mx)$.

Then it holds that

$$\lim_{s\to0,\ \Re(s)>0}\left(4\pi\int_0^\infty a_1(y)e^{-4\pi y}y^s dy + \frac{24A}{s}\right) = 24A\left(2\frac{\zeta'}{\zeta}(2)+1+\log 4\right) - 24B.$$

This proposition is applicable to

$$F(z) = \frac{\sqrt\Delta}{8\pi^2}\frac{\partial}{\partial s}E_s((z,z))\Big|_{s=0}.$$

In this case the required datum in Proposition 5 are $A = \dfrac{\sqrt{\Delta}}{2\pi^2} L(1, \chi^{(\Delta)}_{d_1,d_2})$, $B = \dfrac{\sqrt{\Delta}}{2\pi^2} C_{\chi^{(\Delta)}_{d_1,d_2}}$. The 1st Fourier coefficient $a_1(y)$ of $F(z)$ is given by

$$a_1(y) = \sum_{\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta, \nu \succ 0, \mathrm{Tr}(\nu)=1} \sigma'_{\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt{\Delta})) - \sum_{\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta, \nu > 0 > \nu', \mathrm{Tr}(\nu)=1} \sigma_{0,\chi^{(\Delta)}_{d_1,d_2}}((\nu\sqrt{\Delta}))\Phi(|\nu'|y)$$

$$= \sum_{\substack{\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta \\ \mathrm{Tr}(\nu)=1,\ \nu \succ 0}} \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_\Delta \\ \nu\sqrt{\Delta} \in \mathfrak{b}}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b}) \log(\mathfrak{N}_\Delta(\mathfrak{b})) - \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d_1, d_2, -n)\Phi(|\nu'|y).$$

Here Proposition 4 is used for the second term. We put

$$R := \sum_{\substack{\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta \\ \mathrm{Tr}(\nu)=1,\ \nu \succ 0}} \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_\Delta \\ \nu\sqrt{\Delta} \in \mathfrak{b}}} \chi^{(\Delta)}_{d_1,d_2}(\mathfrak{b}) \log(\mathfrak{N}_\Delta(\mathfrak{b})).$$

Notice that

(i) the identity we want to establish is

$$-\log |J(d_1, d_2)|^2 = R \quad \text{(the Gross-Zagier formula, cf. subsection 3.3).}$$

(ii) The integral transform of $a_1(y)$ in Proposition 5 is

$$4\pi \int_0^\infty a_1(y) e^{-4\pi y} y^s dy = \frac{\Gamma(s+1)}{(4\pi)^s} R - \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d, D, -n) \Psi_s\left(\frac{n - \sqrt{\Delta}}{2\sqrt{\Delta}}\right),$$

where $\Psi_s(\lambda) := 4\pi \int_0^\infty \Phi(\lambda y) e^{-4\pi y} y^s dy$ $(\lambda > 0)$.

Proposition 5 gives a closed form of the finite part of the integral transform of $a_1(y)$ in (ii) at $s = 0$. Together with some manipulation on Ψ_s, it follows that

$$R = \lim_{s \to 0, \Re(s) > 0} \left(2 \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d_1, d_2, -n) Q_{s-1}\left(\frac{n}{\sqrt{\Delta}}\right) - \frac{12\sqrt{\Delta}}{\pi^2} \frac{L(1, \chi^{(\Delta)}_{d_1,d_2})}{s-1} \right)$$
$$+ \frac{12\sqrt{\Delta}}{\pi^2} L(1, \chi^{(\Delta)}_{d_1,d_2}) \left(2 + 2\frac{\zeta'}{\zeta}(1) - \frac{1}{2}\log\Delta + \gamma + \log 4 + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} \right) - \frac{12\sqrt{\Delta}}{\pi^2} L'(1, \chi^{(\Delta)}_{d_1,d_2}).$$

Recall that we have established (Proposition 2)

$$\log |J(d_1, d_2)|^2 = \lim_{s \to 1, s \in \mathbb{R}} \Bigg[-2 \sum_{n > \sqrt{\Delta},\ n \equiv \Delta \pmod 2} h(d_1, d_2, -n) Q_{s-1}\left(\frac{n}{\sqrt{\Delta}}\right)$$
$$+ \frac{4\pi}{\zeta(2s)} \left(h'(d_2) \left|\frac{d_1}{4}\right|^{\frac{s}{2}} \zeta_{\mathbb{Q}(\sqrt{d_1})}(s) + h'(d_1) \left|\frac{d_2}{4}\right|^{\frac{s}{2}} \zeta_{\mathbb{Q}(\sqrt{d_2})}(s) \right.$$
$$\left. - h'(d_1)h'(d_2) \frac{\Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s-1) \right) - 24 h'(d_1) h'(d_2) \Bigg],$$

where $h'(d_j) = \frac{2}{w_j} h(d_j)$. In these expressions for $\log |J(d_1, d_2)|^2$ and R, we apply

- $\zeta_{\mathbb{Q}(\sqrt{d_j})}(s) = \zeta(s) L(s, \chi_{d_j})$, $L(s, \chi^{(\Delta)}_{d_1,d_2}) = L(s, \chi_{d_1}) L(s, \chi_{d_2})$,
- $L(1, \chi_d) = \frac{\pi}{\sqrt{|d|}} h'(d) \quad (h'(d) = \frac{2}{w(d)} h(d))$.

Combined with the Laurent expansions of $\zeta(s)$, $L(s, \chi_d)$ and $\Gamma(s)$, we can confirm the coincidence of the above expressions for $-\log |J(d_1, d_2)|^2$ and R. This finishes Gross-Zagier's analytic proof.

4 Where is the assumption $(d_1, d_2) = 1$ used?

Tim Hutchinson [6] tried to generalize Gross-Zagier's formula to the case when d_1, d_2 are not necessarily fundamental and (d_1, d_2) is a power of a prime not dividing the product of the conductors of d_1 and d_2. Before we discuss his conjecture, we summarize typical places that d_1 and d_2 are coprime is used; (1) definition of ϵ, (2) to prove $h(d_1, d_2, -x) = \sum_{\mu | \frac{x^2 - d_1 d_2}{4}} \epsilon(\mu)$, (3) ideal theory and genus theory of the fundamental discriminant $d_1 d_2$. In particular $L(s, \chi^{(d_1 d_2)}_{d_1, d_2}) = L(s, \chi_{d_1}) L(s, \chi_{d_2})$, $\epsilon(\mathfrak{N}_{d_1 d_2}(\mathfrak{b})) = \chi^{(d_1 d_2)}_{d_1, d_2}(\mathfrak{b})$, $h(d_1, d_2, -x) = \sigma_{0, \chi^{(d_1 d_2)}_{d_1, d_2}}((\sqrt{d_1 d_2} \nu))$ (Proposition 4).

5 Work of Hutchinson (1998)

5.1 Hutchinson's conjecture

We discuss the easiest case of the conjectual extension presented by Hutchinson [6, Conjecture 3.2, p. 258]. Suppose that d_j *are again two different negative fundamental discriminants, and that the greatest common divisor of* d_1 *and* d_2 *is a power of a prime* l.[9] Under these assumptions, possible values for (d_1, d_2) is an odd prime or 4 or 8. Note that $d_1 d_2$ is a non-fundamental discriminant, in particular, a prime q divides the conductor of $d_1 d_2$ iff $q = l$. By numerical computations, Hutchinson made a conjecture saying that, by a *suitable modification of* $F(m)$ presented in the next subsection 5.2, a Gross-Zagier type formula should hold true;

$$|J(d_1, d_2)|^2 = \prod_{x^2 < d_1 d_2,\ x^2 \equiv d_1 d_2 \pmod 4} F\left(\frac{d_1 d_2 - x^2}{4}\right).$$

5.2 Modification of $F(m)$ due to Hutchinson

Under the above setting on d_j $(j = 1, 2)$ and l, suppose that $m \in \mathbb{N}$ is of the form $\frac{d_1 d_2 - x^2}{4}$ $(x \in \mathbb{Z})$. We consider the prime factorization of m. Except for the prime l, we classify prime factors of m according to the values of ϵ.[10] Depending on the number of prime factors p of m satisfying $2 \nmid \mathrm{ord}_p(m)$ and $\epsilon(p) = -1$, the modification of $F(m)$ due to Hutchinson is as follows.

(i) When m has the form $m = l^e p_1^{2a_1} \cdots p_r^{2a_r} q_1^{b_1} \cdots q_s^{b_s}$ $(\epsilon(p_i) = -1,\ \epsilon(q_i) = 1,\ a_i, b_j \geqq 0,\ e \geqq 1)$, then $F(m) = l^{e(b_1+1)\cdots(b_s+1)}$.

(ii) When m has the form $m = l^e p^{2a+1} p_1^{2a_1} \cdots p_r^{2a_r} q_1^{b_1} \cdots q_s^{b_s}$ $(\epsilon(p) = \epsilon(p_i) = -1,\ \epsilon(q_i) = 1,\ a, a_i, b_j \geqq 0,\ e \geqq 0)$, then $F(m) = p^{(a+1)(b_1+1)\cdots(b_s+1)2^\chi}$.

Here $2^\chi = \begin{cases} 2 & \text{if } e \geqq 1\ (l \mid m), \\ 1 & \text{if } e = 0\ (l \nmid m), \end{cases}$ and the exponent is $(a+1)(b_1+1)\cdots(b_s+1)2^\chi$.

(iii) Otherwise, $F(m) = 1$.

6 Main result and reformulation of the conjecture using ϵ_m

The main result in this talk is to prove the Hutchinson conjecture.

Theorem 1. *Hutchinson's conjecture stated above holds true.*

Our proof proceeds along the lines of the analytic proof of Gross-Zagier.[11] A non-trivial differ-

[9] A general case is investigated in [6, Conjecture 3.8, p. 262].

[10] ϵ is the same as in Gross-Zagier's case.

[11] The geometric proof is generalized to general discriminants by Lauter and Viray [9]. As they say that "his formulations are very different from ours", it is not clear whether their formula implies Hutchinson's conjectural formula immediately.

ence is that, in Gross-Zagier's case, the proof uses *"the defining form of $F(m)$"*, that is, $F(m) := \prod_{nn'=m,\ n,n'>0} n^{\epsilon(n')}$. In Hutchinson's case, the conjecture is formulated through a generalization of *"the simple formula of $F(m)$"* (cf. subsection 2.5). This means that an analog of "the defining form of $F(m)$" is not clear. In particular, any suitable analog of ϵ is not clear to rewrite the conjectual simple formula to the defining form in order to adapt the analytic proof.[12] Key ingredients to prove Hutchinson's conjecture is generalizing the following contents to *non-fundamental discriminant* d_1d_2;

- defining an analog of ϵ and relating it to the genus character $\chi_{d_1,d_2}^{(d_1d_2)}$,
- an explicit formula of the class number of pairs $h(d_1, d_2, -x)$,
- a decomposition formula of L-function $L(s, \chi_{d_1,d_2}^{(d_1d_2)})$.

In the following we rewrite the above d_1 and d_2 ($d_1 \neq d_2$) as $d_1 = d$, $d_2 = D$, and suppose that (d, D) is a power of a prime l, $\mathrm{ord}_2(d) \leqq \mathrm{ord}_2(D)$.

6.1 Definition of ϵ_m

First, we present a function ϵ_m, which makes us possible to rewrite the conjectual simple formula $F(m)$ to the defining form similar to Gross-Zagier's case. We denote it by ϵ_m since its definition depends on m unlike Gross-Zagier's ϵ. For any integer $m \in \{\pm\frac{dD-x^2}{4}; x \in \mathbb{Z}, x^2 \equiv dD (\mathrm{mod}\ 4)\}$, we define $\epsilon_m : \{n \in \mathbb{N}; n \mid m\} \to \{0, 1, -1\}$ as a multiplicative function such that

(i) $\epsilon_m(1) := 1$

(ii) For a prime $q \mid m$, $q \neq l$, put $\epsilon_m(q) := \begin{cases} \left(\frac{d}{q}\right) & \text{if } q \nmid d, \\ \left(\frac{D}{q}\right) & \text{if } q \nmid D, \end{cases}$ and $\epsilon_m(q^r) := \epsilon_m(q)^r$ $(r \geqq 2)$.

(iii) For $l \mid m$ with $r_l := \mathrm{ord}_l(m)$, we put $\epsilon_m(l^r) := \begin{cases} 0 & \text{if } 1 \leqq r < r_l, \\ \left(\frac{d/l^*}{l^{r_l}}\right)\left(\frac{l^*}{-m/l^{r_l}}\right) & \text{if } r = r_l, \end{cases}$

where $l^* \in \{(-1)^{\frac{l-1}{2}} l \text{ (if } l \text{ is odd)}, -4, \pm 8 \text{ (if } l \text{ is even)}\}$ such that d/l^* is a discriminant.

We state the 1st main proposition in our proof of the conjecture.

Proposition 6. (*1st main point in this work*) *For $m \in \{\pm\frac{dD-x^2}{4}; x \in \mathbb{Z}, x^2 \equiv dD (\mathrm{mod}\ 4)\}$ and $n \in \mathbb{N}$ with $n \mid m$, we put*

$$G_m(n) := \prod_{a \mid n} a^{\epsilon_m(a)}.$$

Then, Hutchinson's $F(m)$ coincides with $G_m(m)^{-1}$.

6.2 Reformulation of Hutchinson's conjecture

It follows from Proposition 6 that Hutchinson's conjecture can be stated as

$$|J(d, D)|^2 = \prod_{x^2 < dD,\ x^2 \equiv dD \pmod 4} G_{\frac{dD-x^2}{4}}\left(\frac{dD-x^2}{4}\right)^{-1}.$$

By taking the logarithm, the conjecture is

$$-\log |J(d, D)|^2 = \sum_{x^2 < dD,\ x^2 \equiv dD \pmod 4} \ \sum_{n \mid \frac{dD-x^2}{4}} \epsilon_{\frac{dD-x^2}{4}}(n) \log n.$$

[12]At the first glance, this author expected to define $\epsilon(l) := 0$ since $\left(\frac{d_1}{l}\right) = \left(\frac{d_2}{l}\right) = 0$. But this does not give us a correct answer.

As in Gross-Zagier's case, we intend to see that both sides are related to Fourier coefficients of a certain Hilbert-Eisenstein series for the non-maximal quadratic order $\mathcal{O}_{dD}$ of discriminant dD. We shall modify the results used in the analytic proof suitably to the case $l \mid (d, D) \neq 1$. Especially, we need to study (1) an explicit formula of the class number of pairs $h(d, D, \delta)$, (2) the L-function $L(s, \chi_{d,D}^{(dD)})$, (3) the Hilbert-Eisenstein series $E_s((z, z'))$ for the quadratic order $\mathcal{O}_{dD}$.

7 Sketch of the proof of the conjecture

7.1 Equivalence of pairs of forms

Let us introduce the class numbers of pairs of forms. Let $d < 0$, $D < 0$ be fundamental discriminants,[13] $\mathcal{Q}_d^{(+)}$ (resp. $\mathcal{Q}_D^{(+)}$) the set of all positive-definite primitive integral binary quadratic forms of discriminant d (resp. D). The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d^{(+)}$ and $\mathcal{Q}_D^{(+)}$ respectively by the linear transformation of the variables as $(\gamma f)(x, y) := f(x', y')$, where $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $(f, F) \in \mathcal{Q}_d^{(+)} \times \mathcal{Q}_D^{(+)}$ diagonally by $\gamma(f, F) := (\gamma f, \gamma F)$. The equivalence relation of pairs of forms is defined by this diagonal action; (f, F), $(g, G) \in \mathcal{Q}_d^{(+)} \times \mathcal{Q}_D^{(+)}$ are equivalent ($(f, F) \sim (g, G)$) iff there is $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(f, F) = (g, G)$.

Next, we introduce a notion of *codiscriminant* of a pair. For $f = [a, b, c]$, $F = [A, B, C]$, we call $\mathcal{B}(f, F) := bB - 2aC - 2cA$ the codiscriminant. If $(f, F) \in \mathcal{Q}_d^{(+)} \times \mathcal{Q}_D^{(+)}$, then the value $\delta = \mathcal{B}(f, F)$ must satisfy

$$(1)\ \delta < 0, \quad (2)\ \delta^2 \geqq dD, \quad (3)\ \delta^2 \equiv dD \pmod 4.$$

For such a number δ, we define the *class number of pairs* as the number of equivalence classes of pairs having the codiscriminant δ;

$$h(d, D, \delta) := \sharp(\{(f, F) \in \mathcal{Q}_d^{(+)} \times \mathcal{Q}_D^{(+)}; \mathcal{B}(f, F) = \delta\}/\sim).$$

It is known that $h(d, D, \delta)$ is finite.

Next theorem is the 2nd main point in the proof of the conjecture.

Theorem 2. (*2nd main point in this work*) *Let $d < 0$, $D < 0$ be any fundamental discriminants with $\mathrm{ord}_2(d) \leqq \mathrm{ord}_2(D)$, and let $\delta \in \mathbb{Z}$ such that $\delta < 0$, $\delta^2 > dD$, $\delta^2 \equiv dD \pmod 4$. Then we have*

$$h(d, D, \delta) = \sum_{0 < \mu \mid \frac{\delta^2 - dD}{4}} E_{\frac{\delta^2 - dD}{4}}(\mu),$$

where for $m = \frac{\delta^2 - dD}{4} \in \mathbb{N}$ and $0 < \mu \mid m = \mu\nu$, we put

$$E_m(\mu) := \begin{cases} \prod_{p^* \in P(d)} \text{“a non-zero value between } \left(\frac{p^*}{\mu}\right) \text{ or } \left(\frac{p^*}{\nu}\right)\text{”}, & \text{if } p \nmid \mu \text{ or } p \nmid \nu \text{ for all } p^* \in P(d)), \\ 0, & \text{if there is } p^* \in P(d) \text{ such that } p \mid \mu \text{ and } p \mid \nu. \end{cases}$$

Here $P(d) := \{p^*; \text{prime discriminant}, \ p^* \mid d \text{ and } d/p^* \equiv 0, 1 \pmod 4\}$.[14]

We mention some prior researches on explicit formulas of class number of pairs. Gross and Zagier [2] imposed the assumption that d and D are coprime (cf. Proposition 3). Hardy and Williams [4] studied class number of pairs independent of the paper [2]. They imposed the assumption that (dD, δ) is a power of 2. We used their ideas to prove Theorem 2. There is another approach due to Morales [10] and Nakagawa [11]. It might be possible to obtain Theorem 2 by

[13] Except for Proposition 7, we allow $d = D$ in this subsection 7.1.

[14] A prime discriminant is an element of $\{(-1)^{\frac{p-1}{2}} p \ (p \text{ odd primes}), -4, \pm 8\}$.

their methods.[15] However, to obtain explicit results, some additional assumptions are imposed, and moreover, quadratic forms $ax^2+2bxy+cy^2$ are main object of their studies.

[Example] ($d=-7$, $D=-7$, $\delta=-21$)
Representatives are the next four pairs (hence $h(-7,-7,-21)=4$) as given in [4, p. 105];

$$\begin{aligned}&\{(f,F)\in\mathcal{Q}^{(+)}_{-7}\times\mathcal{Q}^{(+)}_{-7};\mathcal{B}(f,F)=-21\}/\sim\\&=\{([1,1,2],[2,-5,4]),([1,1,2],\ [2,9,11]),([1,1,2],\ [4,-3,1]),([1,1,2],\ [4,11,8])\}.\end{aligned}$$

In Theorem 2, we see that $P(-7)=\{-7\}$, $\frac{\delta^2-dD}{4}=98$,

$$E_{98}(\mu):=\begin{cases}\text{a non-zero value between } \left(\frac{-7}{\mu}\right) \text{ or } \left(\frac{-7}{\nu}\right), & \text{if } 7\nmid\mu \text{ or } 7\nmid\nu,\\ 0, & \text{if } 7\mid\mu \text{ and } 7\mid\nu.\end{cases}$$

(μ,ν)	$(1,98)$	$(2,49)$	$(7,14)$	$(14,7)$	$(49,2)$	$(98,1)$
$E_{98}(\mu)$	$\left(\frac{-7}{1}\right)=1$	$\left(\frac{-7}{2}\right)=1$	0	0	$\left(\frac{-7}{2}\right)=1$	$\left(\frac{-7}{1}\right)=1$

By Theorem 2, the class number of pairs is the sum of these $E_{98}(\mu)$ values, which is 4 as expected.

We state a relation between ϵ_m (used in the reformulation of the conjecture, subsection 6.1) and E_m (appeared in the class number formula of pairs). We can show that these are essentially same by some manipulation of Kronecker symbols. Hence, the class number of pairs can be written in terms of ϵ_m. This gives a generalization of the important identity (Proposition 3) due to Gross and Zagier.

Proposition 7. (*Relation between* ϵ_m *and* E_m) *Suppose* $d<0$, $D<0$, $l\mid(d,D)$ *are the same as in the conjecture with* $\mathrm{ord}_2(d)\leqq\mathrm{ord}_2(D)$. *Let* $\delta\in\mathbb{Z}$ *be an integer such that* $\delta<0$, $\delta^2>dD$, $\delta^2\equiv dD$ *(mod 4).*

(1) *If* $m\in\mathbb{N}$ *has the form* $m=\frac{\delta^2-dD}{4}>0$, *then* $\epsilon_{-m}(\mu)=E_m(\mu)$ *for* $0<\mu\mid m$.

(2) *One has* $h(d,D,\delta)=\sum_{\mu\mid\frac{\delta^2-dD}{4}}\epsilon_{\frac{dD-\delta^2}{4}}(\mu)$ *by (1) and Theorem 2.*

7.2 Correspondences

We recall the well-known correspondences involving binary quadratic forms. We follow the exposition in [3].[16] Let Δ be a quadratic discriminant, that is, Δ in a non-square integer such that $\Delta\equiv 0,1 \pmod 4$. We put

$$\mathbb{X}_\Delta:=\left\{\xi=\frac{b+\sqrt{\Delta}}{2a};a,b,c\in\mathbb{Z},a\neq0,(a,b,c)=1,b^2-4ac=\Delta\right\}.$$

We define $\mathfrak{X}_\Delta:=\mathbb{X}_\Delta/\sim$ (resp. $\mathfrak{X}^+_\Delta=\mathbb{X}_\Delta/\sim_+$) as the set of all $\mathrm{GL}_2(\mathbb{Z})$ (resp. $\mathrm{SL}_2(\mathbb{Z})$) equivalence classes of quadratic irrationals of discriminant Δ. We denote by $[\xi]_\sim$ (resp. $[\xi]_{\sim_+}$) the equivalence class of ξ. Here the equivalence relations are defined as follows; putting $M\langle z\rangle=\frac{\alpha z+\beta}{\gamma z+\delta}$ for $M=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)$, z_1, $z_2\in\mathbb{X}_\Delta$ are equivalent ($z_1\sim z_2$) iff there exists $A\in\mathrm{GL}_2(\mathbb{Z})$ such that $z_1=A\langle z_2\rangle$, and z_1, z_2 are proper equivalent ($z_1\sim_+z_2$) iff there exists $A\in\mathrm{SL}_2(\mathbb{Z})$ such that $z_1=A\langle z_2\rangle$.

[15] In addition, their methods are applicable to indefinite forms.

[16] This is just to introduce general genus characters $\chi^{(\Delta)}_{d_1,d_2}$ and its L-function $L(s,\chi^{(\Delta)}_{d_1,d_2})$.

We write $f = [a, b, c]$ for an integral binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$. As usual "primitive" means $(a, b, c) = 1$. Put $M_f = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$. Then we see that $(x\ y)M_f \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = f(x, y)$ and that $\gamma f = g$ iff ${}^t\gamma M_f \gamma = M_g$.[17] We denote by $\mathfrak{F}_\Delta$ the $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes of "positive definite or indefinite" primitive integral binary quadratic forms of discriminant Δ. We denote by $[\![f]\!] = [\![a, b, c]\!]$ the equivalence class of $f = [a, b, c]$. For $f = [a, b, c]$, we put $\xi_f := \frac{b+\sqrt{\Delta}}{2a}$ which is one of roots of $f(X, -1) = 0$.

Finally, we recall the definition of $\mathcal{C}_\Delta$ and $\mathcal{C}_\Delta^+$. Let $\mathcal{O}_\Delta = [1, \frac{\sigma_\Delta+\sqrt{\Delta}}{2}] := \mathbb{Z} + \mathbb{Z}\frac{\sigma_\Delta+\sqrt{\Delta}}{2}$ be the quadratic order of discriminant Δ, where $\sigma_\Delta \in \{0, 1\}$, $\sigma_\Delta \equiv \Delta \pmod 4$. Put $K = \mathbb{Q}(\sqrt{\Delta})$. For lattices $\mathfrak{a}$, $\mathfrak{b}$ in K,[18] $\mathfrak{a}$, $\mathfrak{b}$ are equivalent ($\mathfrak{a} \sim \mathfrak{b}$) iff there is $\lambda \in K^\times$ such that $\mathfrak{a} = \lambda\mathfrak{b}$, and $[\mathfrak{a}]$ denotes the class of $\mathfrak{a}$. Whereas $\mathfrak{a}$, $\mathfrak{b}$ are narrow equivalent ($\mathfrak{a} \sim_+ \mathfrak{b}$) iff there is $\lambda \in K^\times$ with $\mathcal{N}(\lambda) := \lambda\lambda' > 0$ such that $\mathfrak{a} = \lambda\mathfrak{b}$,[19] and $[\mathfrak{a}]^+$ denotes the narrow class of $\mathfrak{a}$. For a lattice $\mathfrak{a}$ in K, put $\mathcal{R}(\mathfrak{a}) := \{\lambda \in K; \lambda\mathfrak{a} \subset \mathfrak{a}\}$. We restrict our consideration to $\mathfrak{a}$ such that $\mathcal{R}(\mathfrak{a}) = \mathcal{O}_\Delta$. This condition is equivalent to say that $\mathfrak{a}$ is an $\mathcal{O}_\Delta$-invertible $\mathcal{O}_\Delta$-fractional ideal. We define the ideal class group $\mathcal{C}_\Delta$ as the set of all $\mathcal{O}_\Delta$-invertible classes $[\mathfrak{a}]$, and the narrow ideal class group $\mathcal{C}_\Delta^+$ as the set of all $\mathcal{O}_\Delta$-invertible narrow classes $[\mathfrak{a}]^+$. For $\xi = \frac{b+\sqrt{\Delta}}{2a} \in \mathbb{X}_\Delta$, the lattice $I(\xi) := [a, a\xi] = \mathbb{Z}a + \mathbb{Z}a\xi$ in K is an ideal of $\mathcal{O}_\Delta$. The ideal $I(\xi)$ is $\mathcal{O}_\Delta$-primitive, $\mathcal{O}_\Delta$-invertible and $(\mathcal{O}_\Delta : I(\xi)) = |a|$.

Proposition 8. *Let Δ be a quadratic discriminant.*
(i) *For $\Delta > 0$, the map $\vartheta_\Delta : \mathfrak{F}_\Delta \to \mathfrak{X}_\Delta^+$ given by $\vartheta_\Delta([\![f]\!]) := [\xi_f]_{\sim_+}$ is a bijection.*
For $\Delta < 0$, the map $\vartheta_\Delta : \mathfrak{F}_\Delta \to \mathfrak{X}_\Delta$ given by $\vartheta_\Delta([\![f]\!]) := [\xi_f]_\sim$ is a bijection.
(ii) *For any Δ, the map $\iota_\Delta : \mathfrak{X}_\Delta \to \mathcal{C}_\Delta$ given by $\iota_\Delta([\xi]_\sim) := [I(\xi)]$ is a bijection.*
For $\Delta > 0$, the map $\iota_\Delta : \mathfrak{X}_\Delta^+ \to \mathcal{C}_\Delta^+$ given by $\iota_\Delta([\xi]_{\sim_+}) := [I(\xi)\sqrt{\Delta}^{(1-\mathrm{sign}(a))/2}]^+$ is a bijection.
(iii) *For any Δ, the map $\Phi_\Delta : \mathfrak{F}_\Delta \to \mathcal{C}_\Delta^+$ given by $\Phi_\Delta([\![f]\!]) := [I(\xi_f)\sqrt{\Delta}^{(1-\mathrm{sign}(a))/2}]^+$ is a bijection.*

Here in (ii) "a" comes from $\xi = \frac{b+\sqrt{\Delta}}{2a} \in \mathbb{X}_\Delta$ and in (iii) "a" comes from $f = [a, b, c]$.

The group structure of $\mathcal{C}_\Delta^+$ induces that of $\mathfrak{F}_\Delta$.

7.3 Genus character $\chi_{e_1,e_2}^{(\Delta)}$

Let e_1, e_2 ($e_1 \neq e_2$) be (positive or negative) fundamental discriminants, and put $\Delta = e_1 e_2 f_0^2$ with $f_0 \in \mathbb{N}$. This Δ is a quadratic discriminant. For a primitive binary quadratic form $f = [a, b, c]$ of discriminant Δ, we can take $m \in \mathbb{Z}$ represented by f with $(\Delta, m) = 1$. We then define

$$\chi_{e_1,e_2}^{(\Delta)}(f) := \left(\frac{e_1}{m}\right).$$

This induces group homomorphisms $\chi_{e_1,e_2}^{(\Delta)} : \mathfrak{F}_\Delta \to \{\pm 1\}$ and $\chi_{e_1,e_2}^{(\Delta)} : \mathfrak{F}_\Delta / \mathfrak{F}_\Delta^2 \to \{\pm 1\}$. It is known that $\chi_{e_1,e_2}^{(\Delta)} = \chi_{e_2,e_1}^{(\Delta)}$. We can compute the value $\chi_{e_1,e_2}^{(\Delta)}(f)$ directly from a primitive form $f = [a, b, c]$ as

$$\chi_{e_1,e_2}^{(\Delta)}([a, b, c]) = \prod_{q^* \in P(e_1)} \chi^{(q^*)}([a, b, c]),$$

$$\chi^{(q^*)}([a, b, c]) := \begin{cases} \chi_{q^*}(a) & \text{if } (a, q^*) = 1, \\ \chi_{q^*}(c) & \text{if } (c, q^*) = 1. \end{cases}$$

See Theorem 2 for $P(e_1)$ and $\chi_{q^*}(m) := \left(\frac{q^*}{m}\right)$ is the Kroecker symbol.

[17] We put $(\gamma f)(x, y) := f(x', y')$ for $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ as in subsection 7.1.
[18] $\mathfrak{a}$ is a lattice in K iff $\mathfrak{a}$ is of the form $\mathfrak{a} = [\omega_1, \omega_2] := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in K$ linear independent over $\mathbb{Q}$.
[19] λ' is the conjugate of λ.

7.4 Genus character L-function

By the group isomorphism $\Phi_\Delta : \mathfrak{F}_\Delta \to \mathcal{C}_\Delta^+$, we identify $\mathcal{C}_\Delta^+$ and $\mathfrak{F}_\Delta$. Thus we have a narrow class character $\chi_{e_1,e_2}^{(\Delta)} : \mathcal{C}_\Delta^+ \to \{\pm 1\}$. The genus character L-function is defined by

$$L(s, \chi_{e_1,e_2}^{(\Delta)}) := \sum_{\mathfrak{a}\subset\mathcal{O}_\Delta,\ \mathcal{O}_\Delta\text{-invertible}} \frac{\chi_{e_1,e_2}^{(\Delta)}(\mathfrak{a})}{\mathfrak{N}_\Delta(\mathfrak{a})^s} \quad (\Re(s) > 1),$$

where the sum is taken over all $\mathcal{O}_\Delta$-invertible ideal $\mathfrak{a}$ of $\mathcal{O}_\Delta$ and $\mathfrak{N}_\Delta(\mathfrak{a}) = (\mathcal{O}_\Delta : \mathfrak{a})$ is the absolute norm.

Theorem 3. (*3rd main point in this work (Chinta-Offen, Kaneko-M[8], Ibukiyama)*) *Let e_1, e_2 ($e_1 \neq e_2$) be any fundamental discriminants and put $\Delta = e_1 e_2 f_0^2$ with $f_0 \in \mathbb{N}$. Then*

$$L(s, \chi_{e_1,e_2}^{(\Delta)}) = L(s, \chi_{e_1})L(s, \chi_{e_2})$$
$$\times \prod_{\substack{p\mid f_0 \\ p:\text{prime}}} \frac{(1-\chi_{e_1}(p)p^{-s})(1-\chi_{e_2}(p)p^{-s}) - p^{m_p-1-2m_p s}(p^{1-s}-\chi_{e_1}(p))(p^{1-s}-\chi_{e_2}(p))}{1-p^{1-2s}}.$$

Here $L(s, \chi_d)$ is the Dirichlet L-function of the Kronecker character $\chi_d(*) = \left(\frac{d}{*}\right)$, the product on the right runs over the prime factors p of f_0 and $m_p := \mathrm{ord}_p(f_0)$ if $f_0 > 1$. Whereas the empty product is understood as being 1 if $f_0 = 1$.

In particular, one has $L(s, \chi_{e_1,e_2}^{(e_1 e_2)}) = L(s, \chi_{e_1})L(s, \chi_{e_2})$ as the special case of Theorem 3.

7.5 Relation with Hilbert-Eisenstein series for the non-maximal order $\mathcal{O}_{dD}$

Suppose that $d < 0$, $D < 0$, $l \mid (d, D)$ are as in the Hutchinson conjecture. Associated to the genus character $\chi_{d,D}^{(\Delta)} : \mathcal{C}_\Delta^+ \to \{\pm 1\}$ of non-fundamental discriminant $\Delta = dD$ (cf. subsection 7.3), a Hilbert-Eisenstein series $E_s((z, z'))$ of the non-maximal quadratic order $\mathcal{O}_\Delta$ can be defined and it is not difficult to see that its Fourier expansion has almost the same form as in subsection 3.4. For $\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta$, the ν-th Fourier coefficient of $E_s((z, z'))$ is given by

$$\sigma_{s,\chi_{d,D}^{(\Delta)}}((\sqrt{\Delta}\nu)) := \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_\Delta,\ \mathcal{O}_\Delta\text{-invertible} \\ (\nu\sqrt{\Delta})\subset\mathfrak{b}}} \chi_{d,D}^{(\Delta)}(\mathfrak{b})\mathfrak{N}_\Delta(\mathfrak{b})^s,$$

where the sum is over all $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{b}$ of $\mathcal{O}_\Delta$ such that $(\nu\sqrt{\Delta}) := \nu\sqrt{\Delta}\mathcal{O}_\Delta \subset \mathfrak{b}$.

Proposition 9. *Suppose that $d < 0$, $D < 0$, $l \mid (d, D)$ are as in the Hutchinson conjecture and that* $\mathrm{ord}_2(d) \leqq \mathrm{ord}_2(D)$. *Put $\Delta = dD$. For $\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta$ with $\mathrm{Tr}(\nu) = \nu + \nu' = 1$, $\nu > 0 > \nu'$, we can take $x \in \mathbb{Z}$ such that $\sqrt{\Delta}\nu = \frac{x+\sqrt{\Delta}}{2}$, $x > \sqrt{\Delta}$, $x \equiv \Delta$ (mod 2). Then we have*

$$h(d, D, -x) = \sigma_{0,\chi_{d,D}^{(\Delta)}}((\sqrt{\Delta}\nu)).$$

The constant term of $E_s((z, z'))$ is described by the L-function $L(s, \chi_{d,D}^{(\Delta)})$ of $\chi_{d,D}^{(\Delta)}$, which coincides with $L(s, \chi_d)L(s, \chi_D)$ with the help of Theorem 3.

7.6 Relation between ϵ_m and $\chi_{d,D}^{(\Delta)}$

We prepare a lemma about the index appeared in the sum $\sigma_{-2s,\chi_{d,D}^{(\Delta)}}((\nu\sqrt{\Delta}))$, that is, $\mathcal{O}_\Delta$-invertible ideals $\mathfrak{b}$ of $\mathcal{O}_\Delta$ such that $\nu\sqrt{\Delta} \in \mathfrak{b}$ ($\Delta = dD$).

Proposition 10. *Suppose that $d < 0$, $D < 0$, $l \mid (d, D)$ are as in the Hutchinson conjecture and that $\mathrm{ord}_2(d) \leqq \mathrm{ord}_2(D)$. Put $\Delta = dD$ and suppose $\nu \in \frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta$, $\mathrm{Tr}(\nu) = 1$.*

(1) *There is $x \in \mathbb{Z}$ such that $x \equiv \Delta \pmod 2$, $\nu = \frac{x+\sqrt{\Delta}}{2\sqrt{\Delta}}$.*

(2) *Write $\nu\sqrt{\Delta} = \frac{x+\sqrt{\Delta}}{2}$ as in (1). Put $r_l := \mathrm{ord}_l(\frac{x^2-\Delta}{4})$. The map $\mathfrak{b} \mapsto \mathfrak{N}_\Delta(\mathfrak{b})$ gives a bijection from $A_x = \{\mathfrak{b};\ \mathcal{O}_\Delta\text{-invertible ideal of } \mathcal{O}_\Delta,\ \frac{x+\sqrt{\Delta}}{2} \in \mathfrak{b}\}$ to $B_x = \{n \in \mathbb{N};\ n \mid \frac{x^2-\Delta}{4}, \mathrm{ord}_l(n) \in \{0, r_l\}\}$.*

(3) *For $\mathfrak{b} \in A_x$, put $n = \mathfrak{N}_\Delta(\mathfrak{b})$. Then $\chi_{d,D}^{(\Delta)}(\mathfrak{b}) = \epsilon_{\frac{\Delta-x^2}{4}}(n)$.*

To show Proposition 10, we need ideal theory of the order $\mathcal{O}_{dD}$.

Lemma 1. *As in Proposition 10, let $\Delta = dD$, $\nu\sqrt{\Delta} = \frac{x+\sqrt{\Delta}}{2}$ with $x \in \mathbb{Z}$.*

(1) *$|\frac{x^2-\Delta}{4}|$ has the prime factorization of the form $|\frac{x^2-\Delta}{4}| = l^{r_l}\prod_{(\frac{\Delta}{p})=1} p^{r_p} \prod_{(\frac{\Delta}{q})=0, q\neq l, q \mid \frac{x^2-\Delta}{4}} q$ with $r_l \geqq 0, r_p \geqq 0$. Here and in the following, the empty product is undestood as being 1.*

(2) *$(\nu\sqrt{\Delta})$ is an $\mathcal{O}_\Delta$-regular ideal*[20] *and has the unique factorization into $\mathcal{O}_\Delta$-regular ideals with mutually coprime prime-power norms of the form $(\frac{x+\sqrt{\Delta}}{2}) = \mathfrak{a}^{(l)}\prod_{(\frac{\Delta}{p})=1}\mathfrak{p}^{r_p}\prod_{(\frac{\Delta}{q})=0, q\neq l, q\mid\frac{x^2-\Delta}{4}}\mathfrak{q}$, where $\mathfrak{a}^{(l)} = [l^{r_l}, \frac{x+\sqrt{\Delta}}{2}]$, $\mathfrak{p} = [p, \frac{x+\sqrt{\Delta}}{2}]$, $\mathfrak{q} = [q, \frac{x+\sqrt{\Delta}}{2}]$.*

(3) *Put $A_x = \{\mathfrak{b}; \mathcal{O}_\Delta\text{-invertible ideal of } \mathcal{O}_\Delta, \frac{x+\sqrt{\Delta}}{2} \in \mathfrak{b}\}$ and $B_x = \{n \in \mathbb{N}; n \mid \frac{x^2-\Delta}{4}, \mathrm{ord}_l(n) \in \{0, r_l\}\}$. Then $\mathfrak{b} \in A_x$ has the unique factorization into $\mathcal{O}_\Delta$-regular ideals with mutually coprime prime-power norms of the form $\mathfrak{b} = (\mathfrak{a}^{(l)})^{\delta_l}\prod_{(\frac{\Delta}{p})=1}\mathfrak{p}^{a_p}\prod_{(\frac{\Delta}{q})=0, q\neq l, q\mid\frac{x^2-\Delta}{4}}\mathfrak{q}^{a_q}$, where $0 \leqq a_p \leqq r_p$, $a_q \in \{0,1\}$, $\delta_l \in \{0,1\}$. In particular, $\mathrm{ord}_l(\mathfrak{N}_\Delta(\mathfrak{b})) \in \{0, r_l\}$ and $\mathfrak{N}_\Delta(\mathfrak{b}) \in B_x$.*

First of all, we shall determine the value $\chi_{d,D}^{(\Delta)}(\mathfrak{p})$ for $\mathfrak{p} = [p, \frac{x+\sqrt{\Delta}}{2}]$, $p \neq l$, $(\frac{\Delta}{p}) \neq -1$ to deduce Proposition 10 (3). By definition, $\chi_{d,D}^{(\Delta)}(\mathfrak{p}) = \chi_{d,D}^{(\Delta)}([p, x, \frac{x^2-\Delta}{4p}]) = \prod_{t^*\in P(d)}\chi^{(t^*)}([p, x, \frac{x^2-\Delta}{4p}])$. Since $p \neq l$, we have $p \nmid d$ or $p \nmid D$. If $p \nmid d$, then $\chi^{(t^*)}([p, x, \frac{x^2-\Delta}{4p}]) = \chi_{t^*}(p)$ for any $t^* \in P(d)$ and thus $\chi_{d,D}^{(\Delta)}(\mathfrak{p}) = \prod_{t^*\in P(d)}\chi_{t^*}(p) = \chi_d(p) = \epsilon_{\frac{\Delta-x^2}{4}}(p)$ with $\chi_{t^*}(a) = (\frac{t^*}{a})$. If $p \nmid D$, we may use $\chi_{d,D}^{(\Delta)}(\mathfrak{p}) = \chi_{D,d}^{(\Delta)}(\mathfrak{p})$. The value $\chi_{d,D}^{(\Delta)}(\mathfrak{a}^{(l)})$ can be determined as $\chi_{d,D}^{(\Delta)}(\mathfrak{a}^{(l)}) = \chi_{d,D}^{(\Delta)}([l^{r_l}, x, \frac{x^2-\Delta}{4l^{r_l}}]) = \prod_{t^*\in P(d)}\chi^{(t^*)}([l^{r_l}, x, \frac{x^2-\Delta}{4l^{r_l}}]) = \chi_{d/l^*}(l^{r_l})\chi_{l^*}(\frac{x^2-\Delta}{4l^{r_l}}) = \left(\frac{d/l^*}{l^{r_l}}\right)\left(\frac{l^*}{-(\Delta-x^2)/4l^{r_l}}\right) = \epsilon_{\frac{\Delta-x^2}{4}}(l^{r_l})$ with $l^* \in P(d)$ such that $l \mid l^*$.

The identity $h(d, D, -x) = \sigma_{0,\chi_{d,D}^{(\Delta)}}((\sqrt{\Delta}\nu))$ in Proposition 9 follows from Propositions 7 and 10 as $h(d, D, -x) = \sum_{n\in B_x}\epsilon_{\frac{\Delta-x^2}{4}}(n) = \sum_{\mathfrak{b}\in A_x}\chi_{d,D}^{(\Delta)}(\mathfrak{b}) = \sigma_{0,\chi_{d,D}^{(\Delta)}}((\sqrt{\Delta}\nu))$.

7.7 Further reformulation of Hutchinson's conjecture

Let $\Delta = dD$ be as above. In subsection 6.2, Hutchinson's conjecture has been reformulated as

$$-\log|J(d,D)|^2 = \sum_{x^2<\Delta,\ x^2\equiv\Delta \pmod 4}\ \sum_{n\mid\frac{\Delta-x^2}{4}}\epsilon_{\frac{\Delta-x^2}{4}}(n)\log n.$$

[20] "$\mathcal{O}_\Delta$-regular ideal" means $\mathcal{O}_\Delta$-primitive and $\mathcal{O}_\Delta$-invertible ideal of $\mathcal{O}_\Delta$.

Using Proposition 10, the Hutchinson conjecture can be restated as

$$-\log|J(d,D)|^2 = \sum_{\substack{\nu\in\frac{1}{\sqrt{\Delta}}\mathcal{O}_\Delta \\ \mathrm{Tr}(\nu)=1,\ \nu\gg 0}} \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_\Delta,\ \mathcal{O}_\Delta\text{-invertible} \\ \nu\sqrt{\Delta}\in\mathfrak{b}}} \chi_{d,D}^{(\Delta)}(\mathfrak{b})\log(\mathfrak{N}_\Delta(\mathfrak{b})).$$

This identity can be proved along the lines of Gross-Zagier's analytic proof (by applying the holomorphic projection lemma to the Fourier expansion of $E_s((z,z'))$ together with preparations so far). Details will be published elsewhere.

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Yoshinori Mizuno
Graduate School of Technology, Industrial and Social Sciences
Tokushima University
2-1, Minami-josanjima-cho, Tokushima, 770-8506, Japan
e-mail: mizuno.yoshinori@tokushima-u.ac.jp