$\mathcal{R}$-boundedness for an integral operator in the half space and its application to the Stokes problems

岐阜大学 応用物理コース 梶原直人
Naoto Kajiwara
Applied Physics Course, Department of Electrical, Electronic and Computer Engineering,
Gifu University

Abstract

In this paper, we give a sufficient condition for $\mathcal{R}$-boundedness of an integral operator defined on the half-space. The assumption is simply bounded and holomorphic, which is easy to check. As applications, we derive resolvent estimate and maximal regularity for the Stokes equations with various boundary conditions. We can treat Dirichlet, Neumann and Robin boundary conditions in a unified manner.

Keywords : $\mathcal{R}$-boundedness, resolvent estimate, maximal regularity.

1 Introduction and main theorem

We consider the following integral operators on the half-space $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty)$;

$$
\begin{aligned}
T[m]f(x) &= \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m(\xi', x_N + y_N) \mathcal{F}_{x'} f](x, y_N) dy_N, \\
\tilde{T}_\gamma[m_\lambda]g(x,t) &= \mathcal{L}^{-1}_\lambda \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m_\lambda(\xi', x_N + y_N) \mathcal{F}_{x'} \mathcal{L} g](x, y_N, \lambda) dy_N, \\
&= [e^{\gamma t} \mathcal{F}^{-1}_{\tau \to t} T[m_\lambda] \mathcal{F}_{t \to \tau}(e^{-\gamma t} g)](x,t),
\end{aligned}
$$

where $\lambda = \gamma + i\tau$, symbols m, m_λ are $\mathbb{C}$-valued functions, and $f : \mathbb{R}^N_+ \to \mathbb{C}$ and $g : \mathbb{R}^N_+ \times \mathbb{R} \to \mathbb{C}$. Let $\mathcal{F}_{x'}$ and $\mathcal{F}^{-1}_{\xi'}$ denote the partial Fourier transform and its inverse;

$$
\mathcal{F}_{x'}[f](\xi', x_N) := \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N) dx', \quad \mathcal{F}^{-1}_{\xi'}[g](x) := \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N) d\xi'.
$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}_\lambda$ denote two-sided Laplace transform and its inverse, defined as

$$
\mathcal{L}[f](\lambda) = \int_{-\infty}^\infty e^{-\lambda t} f(t) dt = \mathcal{F}_{t\to\tau}[e^{-\gamma t} f](\lambda), \quad \mathcal{L}^{-1}_\lambda[g](t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{\lambda t} g(\lambda) d\tau = e^{\gamma t} \mathcal{F}^{-1}_{\tau \to t}[g](t),
$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$.

The main theorem of this paper is $L_q(\mathbb{R}^N_+)$-boundedness of the operator $T[m]$ and $\mathcal{R}$-boundedness of the operator of $T[m_\lambda]$. The latter is related to $L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))$ boundedness of the operator $\tilde{T}_\gamma[m_\lambda]$ with weights $e^{\pm\gamma t}$. Here $1 < p, q < \infty$.

Let us review some definitions. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from X to Y. We use $\mathcal{L}(X) := \mathcal{L}(X,X)$.

Definition 1.1. *A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded, if there exist constant $C > 0$ and $1 \le p < \infty$ such that for each $m \in \mathbb{N}, \{T_j\}_{j=1}^m \subset \mathcal{T}, \{x_j\}_{j=1}^m \subset X$ and for all sequences $\{\varepsilon_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \mathcal{A}, \mu)$ the inequality*

$$|\sum_{j=1}^m \varepsilon_j T_j x_j|_{L_p(\Omega,Y)} \le C|\sum_{j=1}^m \varepsilon_j x_j|_{L_p(\Omega,X)}$$

is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}(\mathcal{T})$.

Note that when X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is uniformly bounded.

To state our main theorem, we introduce two types of sector domain. Let $\gamma_0 \ge 0, \varepsilon, \eta \in (0, \pi/2)$ and

$$\begin{aligned} \Sigma_{\varepsilon,\gamma_0} &:= \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| \ge \gamma_0, |\arg\lambda| < \pi - \varepsilon\}, \\ \Sigma_\varepsilon &:= \Sigma_{\varepsilon,0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \varepsilon\}, \\ \tilde{\Sigma}_\eta &:= \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \eta\} \cup \{z \in \mathbb{C} \setminus \{0\} \mid \pi - \eta < |\arg z|\}. \end{aligned}$$

Let $H^\infty(D)$ be the space of bounded and holomorphic functions on an open set $D \subset \mathbb{C}^{N-1}$. The main theorem is as follows.

Theorem 1.2. (i) *Let m satisfy the following two conditions:*
(a) *There exists $\eta \in (0, \pi/2)$ such that $\{m(\cdot, x_N), x_N > 0\} \subset H^\infty(\tilde{\Sigma}_\eta^{N-1})$.*
(b) *There exist $\eta \in (0, \pi/2)$ and $C > 0$ such that $\sup_{\xi' \in \tilde{\Sigma}_\eta^{N-1}} |m(\xi', x_N)| \le C x_N^{-1}$ for all $x_N > 0$.*
Then $T[m]$ is a bounded linear operator on $L_q(\mathbb{R}^N_+)$ for every $1 < q < \infty$.
(ii) *Let $\gamma_0 \ge 0$ and let m_λ satisfy the following two conditions:*
(c) *There exists $\eta \in (0, \pi/2 - \varepsilon)$ such that for each $x_N > 0$ and $\gamma \ge \gamma_0$,*

$$\tilde{\Sigma}_\eta^N \ni (\tau, \xi') \mapsto m_\lambda(\xi', x_N) \in \mathbb{C}$$

is bounded and holomorphic.
(d) *There exist $\eta \in (0, \pi/2 - \varepsilon)$ and $C > 0$ such that $\sup\{|m_\lambda(\xi', x_N)| \mid (\tau, \xi') \in \tilde{\Sigma}_\eta^N\} \le C x_N^{-1}$ for all $\gamma \ge \gamma_0$ and $x_N > 0$.*
Then $T[m_\lambda](\in \mathcal{L}_q(\mathbb{R}^N_+))$ satisfies that there exists a $C > 0$ such that

$$\mathcal{R}\{(\tau\partial_\tau)^\beta T[m_\lambda] \mid \tau \in \mathbb{R}, \beta \in \{0,1\}\} \le C$$

for every $1 < q < \infty$. Also, we have that $\tilde{T}_\gamma[m_\lambda]$ satisfies

$$\|e^{-\gamma t}\tilde{T}_\gamma[m_\lambda]g\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} \le C\|e^{-\gamma t} g\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))}$$

for every $\gamma \ge \gamma_0$ and $1 < p, q < \infty$.

As applications of this theorem, we can consider the Stokes equations with various boundary conditions. For the sake of simplicity we set some external forces are zero although we can generalize them.

The resolvent Stokes problem on the half space is as follows;

$$\begin{cases} \lambda u - \Delta u + \nabla \pi = 0 & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^N_+, \end{cases} \tag{1}$$

with one of the following boundary conditions on $\mathbb{R}^N_0 = \mathbb{R}^{N-1} \times \{0\}$;

$$\text{(Dirichlet)} \begin{cases} u_j = h_j \quad (j = 1, \dots, N-1), \\ u_N = 0, \end{cases}$$

$$\text{(Neumann)} \begin{cases} -(\partial_N u_j + \partial_j u_N) = h_j \quad (j = 1, \dots, N-1), \\ -(2\partial_N u_N - \pi) = h_N, \end{cases}$$

$$\text{(Robin)} \begin{cases} \alpha u_j - \beta \partial_N u_j = h_j \quad (j = 1, \dots, N-1), \\ u_N = 0. \end{cases}$$

Here $\alpha \geq 0, \beta > 0$. Note that the end-point case $(\alpha, \beta) = (1, 0)$ in Robin boundary condition implies Dirichlet boundary condition. It is called that Dirichlet boundary is non-slip boundary condition, and Robin boundary is (partial-)slip boundary condition or Navier-boundary condition.

The non-stationary problem is as follows;

$$\begin{cases} \partial_t U - \Delta U + \nabla \Pi = 0 & \text{in } \mathbb{R}^N_+ \times (0, \infty), \\ \operatorname{div} U = 0 & \text{in } \mathbb{R}^N_+ \times (0, \infty), \end{cases} \tag{2}$$

with initial data $U|_{t=0} = U_0$ and with one of the following boundary conditions;

$$\text{(Dirichlet)} \begin{cases} U_j = H_j \quad (j = 1, \dots, N-1), \\ U_N = 0, \end{cases}$$

$$\text{(Neumann)} \begin{cases} -(\partial_N U_j + \partial_j U_N) = H_j \quad (j = 1, \dots, N-1), \\ -(2\partial_N U_N - \Pi) = H_N, \end{cases}$$

$$\text{(Robin)} \begin{cases} \alpha U_j - \beta \partial_N U_j = H_j \quad (j = 1, \dots, N-1), \\ U_N = 0. \end{cases}$$

It can be derived from theorem 1.2.

Theorem 1.3. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then for any $\lambda \in \Sigma_\varepsilon$, $h \in \begin{cases} W^2_q(\mathbb{R}^N_+) \text{ Dirichlet}, \\ W^1_q(\mathbb{R}^N_+) \text{ Neumann, Robin}, \end{cases}$ problem (1) admits a unique solution $(u, \pi) \in W^2_q(\mathbb{R}^N_+) \times \hat{W}^1_q(\mathbb{R}^N_+)$ with the resolvent estimate;*

$$\|(\lambda u, \lambda^{1/2} \nabla u, \nabla^2 u, \nabla \pi)\|_{L_q(\mathbb{R}^N_+)} \leq \begin{cases} C\|(\lambda h, \lambda^{1/2} \nabla h, \nabla^2 h)\|_{L_q(\mathbb{R}^N_+)} \text{ Dirichlet}, \\ C\|(\lambda^{1/2} h, \nabla h)\|_{L_q(\mathbb{R}^N_+)} \text{ Neumann}, \\ C'\|(\lambda^{1/2} h, \nabla h)\|_{L_q(\mathbb{R}^N_+)} \text{ Robin}, \end{cases}$$

for some constants $C = C_{N,q,\varepsilon}$ and $C' = C'_{N,q,\varepsilon,\alpha,\beta}$.

For $1 < p, q < \infty$, $m \in \mathbb{N}$, $\gamma_0 \geq 0$ and $s \geq 0$, let

$$
\begin{aligned}
\hat{W}^1_q(\mathbb{R}^N_+) &:= \{\pi \in L_{q,\mathrm{loc}}(\mathbb{R}^N_+) \mid \nabla\pi \in L_q(\mathbb{R}^N_+)\},\\
L_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbb{R}, X),\ f(t) = 0 \text{ for } t < 0\},\\
W^m_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f \in L_{p,0,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t}\partial_t^j f(t) \in L_p(\mathbb{R}, X),\ j = 1, \dots, m\},\\
H^s_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid \Lambda^s_\gamma f := \mathcal{L}^{-1}_\lambda[|\lambda|^s \mathcal{L}[f](\lambda)](t) \in L_{p,0,\gamma}(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_0\}.
\end{aligned}
$$

Theorem 1.4. *Let $1 < p, q < \infty$ and $\gamma_0 \geq 0$. Then for any*

$$
H \in \begin{cases} W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^N_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\mathbb{R}^N_+)) & \text{Dirichlet},\\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^N_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\mathbb{R}^N_+)) & \text{Neumann, Robin}, \end{cases}
$$

problem (2) *with $U_0 = 0$ admits a unique solution (U, Π) such that*

$$
\begin{aligned}
U &\in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^N_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\mathbb{R}^N_+)),\\
\Pi &\in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^1_q(\mathbb{R}^N_+))
\end{aligned}
$$

with the maximal L_p-L_q regularity;

$$
\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda^{1/2}_\gamma \nabla U, \nabla^2 U, \nabla\Pi)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} \leq \begin{cases} C\|e^{-\gamma t}(\partial_t H, \Lambda_\gamma \nabla H, \nabla^2 H)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} & \text{Dirichlet},\\ C\|e^{-\gamma t}(\Lambda^{1/2}_\gamma H, \nabla H)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} & \text{Neumann},\\ C'\|e^{-\gamma t}(\Lambda^{1/2}_\gamma H, \nabla H)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N_+))} & \text{Robin}, \end{cases}
$$

for any $\gamma \geq \gamma_0$ with some constants $C = C_{N,p,q,\gamma_0}$ and $C' = C'_{N,p,q,\gamma_0,\alpha,\beta}$.

At the end of this section, we see some references. In 2001, a sufficient condition for $L_p(\mathbb{R}, X)$-boundedness of Fourier multiplier operators was constructed by Weis [30] in terms of $\mathcal{R}$-bounded of the symbols under X is $\mathcal{HT}$ space. This breakthrough led a lot of results for the maximal regularity. For example, see the monographs by Denk–Hieber–Prüss [1] and Kunstmann–Weis [17]. These were applied to the elliptic operators. Weis's theorem was applied not only elliptic operators but also Stokes operator. It has shown by Geissert, Hech, Hieber and Sawada [6] that the existence of the Helmholtz decomposition implies the analyticity and maximal L_p-L_q regularity for the Stokes operators. Moreover we note that Farwig, Kozono and Sohr [3, 4] proved maximal L_p-$\tilde{L}_q$ regularity for general domains. A general explanation for the Stokes equations was given by [10]. We heavily depend on the results by Shibata et al. [16, 27]. It was also important for them to use the theorem due to Weis, where the methods seemed systematic ways in the sense that they got the resolvent estimate and the maximal regularity at the same time. Since then, there are a lot of results, e.g. for model problems with Neumann or free boundary conditions [24, 25, 27], Robin conditions [22, 28], two-phase problems [26]. For the case of general domains, see [19, 20, 21]. On the other hand, our method will show easier than them since the basis is bounded and holomorphic although essential ideas are similar. At last, see [18] for the comprehensive results about analyticity of semigroups, vector-valued harmonic analysis, maximal regularity, parabolic and Stokes equations and its applications to the free boundary problems. Almost all of our main theorems have already proved before, but we give a new simple approach to get resolvent estimates and maximal regularity estimates. Our method has already used for the Stokes equations with various boundary conditions

[11, 12] in the half space. Recently we proved the same results for the layer domain, which is applied for the Stokes equations with Dirichlet-Neumann boundary condition in [13], Neumann-Neumann boundary condition in [14], and for the heat equation with various boundary conditions in [15].

The structure of the paper is as follows. In section 2, we prepare some known definitions and theorems. In section 3, we prove the main theorem and how to apply $\mathcal{R}$-boundedness. In section 4, 5, we solve the equations in the half space by partial Fourier transforms. Three types of boundary conditions are treated similarly. The solution formula is Fourier multiplier type with the symbols of sum of heat part $e^{-\sqrt{\lambda+|\xi'|^2}x_N}$ and Stokes part $\mathcal{M}_\lambda(\xi', x_N)$ which is defined later. From so called Volevich's trick, the solutions are given by an integral form whose integrands are Fourier multiplier operators which act h and $\partial_N h$. Here, we decompose the symbols while paying attention to the desired estimates. Resolvent estimate is straightforward from the theorem prepared in section 2 and the estimates of $e^{-\sqrt{\lambda+|\xi'|^2}x_N}$ and $\mathcal{M}_\lambda(\xi', x_N)$. Maximal regularity estimates are also same as resolvent estimates since the symbols are $\mathcal{R}$-bounded in λ-variables. In section 6, we remark that the differences of previous works.

2 Preliminaries

Since key lemma in this paper is operator-valued Fourier multiplier theorem, we need some preparations for the base spaces and the symbols. Almost all of results in this section can be found in the book [18].

Definition 2.1. *A Banach space X is said to belong to the class $\mathcal{HT}$ if the Hilbert transform $\mathcal{H}$, defined by*

$$\mathcal{H}f := \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{|t-s|>\varepsilon}\frac{f(s)}{t-s}ds,$$

is a bounded linear operator on $L_p(\mathbb{R}, X)$ for some $1 < p < \infty$. In this case we write $X \in \mathcal{HT}$.

Let $\dot{\mathbb{R}} := \mathbb{R}\setminus\{0\}$ and $\dot{\mathbb{R}}^n = [\dot{\mathbb{R}}]^n$. Given $M \in C(\dot{\mathbb{R}}^n, \mathcal{L}(X, Y))$, we define an operator T_M : $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}^n, X) \to \mathcal{S}(\mathbb{R}^n, Y)$ by means of

$$T_M\phi := \mathcal{F}^{-1}M\mathcal{F}\phi, \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R}^n, X).$$

Theorem 2.2 (Weis [30])**.** *Let X, $Y \in \mathcal{HT}$ and $1 < p < \infty$. Let $M \in C^1(\dot{\mathbb{R}}, \mathcal{L}(X, Y))$ satisfy*

$$\mathcal{R}(\{(\xi\frac{d}{d\xi})^j M(\xi) \mid \xi \in \dot{\mathbb{R}}, j = 0, 1\}) = \kappa < \infty.$$

Then the operator T_M is a bounded linear operator from $L_p(\mathbb{R}, X)$ to $L_p(\mathbb{R}, Y)$. Moreover

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on X, Y and p.

Definition 2.3. *A Banach space X is said to have property (α) if there exists a constant $\alpha > 0$ such that*

$$|\sum_{i,j=1}^m \alpha_{ij}\varepsilon_i\varepsilon'_j x_{ij}|_{L_2(\Omega\times\Omega',X)} \le \alpha|\sum_{i,j=1}^m \varepsilon_i\varepsilon'_j x_{ij}|_{L_2(\Omega\times\Omega',X)},$$

for all $\alpha_{ij} \in \{-1, 1\}, x_{ij} \in X, m \in \mathbb{N}$, and all symmetric independent $\{-1, 1\}$-valued random variables $\{\varepsilon_i\}_{i=1}^m$ resp. $\{\varepsilon'_j\}_{j=1}^m$ on a probability space $(\Omega, \mathcal{A}, \mu)$ resp. $(\Omega', \mathcal{A}', \mu')$. The class $\mathcal{HT}(\alpha)$ denotes the set of all Banach spaces which belong to $\mathcal{HT}$ and have property (α).

Remark 2.4. *For any Hilbert space E, we have $E \in \mathcal{HT}(\alpha)$. If (S, Σ, σ) is a sigma-finite measure space and $1 < p < \infty$, then $L_p(S, E) \in \mathcal{HT}(\alpha)$ as well.*

Theorem 2.5. *Let $1 < p < \infty$, X, $Y \in \mathcal{HT}(\alpha)$, and suppose that the family of multipliers $\mathcal{M} \subset C^n(\dot{\mathbb{R}}^n, \mathcal{L}(X, Y))$ satisfies*

$$\mathcal{R}(\{\xi^\alpha \partial_\xi^\alpha M(\xi) \mid \xi \in \dot{\mathbb{R}}^n, \alpha \in \{0,1\}^n, M \in \mathcal{M}\}) =: \kappa < \infty.$$

Then the family of operators $\mathcal{T} := \{T_M \mid M \in \mathcal{M}\} \subset \mathcal{L}(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$ is $\mathcal{R}$-bounded with $\mathcal{R}(\mathcal{T}) \leq C\kappa$, where $C > 0$ only depends on X, Y and p.

To verify the Lizorkin condition in above theorem, a useful sufficient condition is known in terms of holomorphic and boundedness, which is denoted by the class H^∞.

Theorem 2.6 ([18, Proposition 4.3.10]). *Let X, Y be Banach spaces and suppose that, for some $0 < \eta < \pi/2$, the family of multipliers $\mathcal{M} \subset H^\infty(\tilde{\Sigma}_\eta^n, \mathcal{L}(X, Y))$ satisfies*

$$\mathcal{R}(\{M(z) \mid z \in \tilde{\Sigma}_\eta^n, M \in \mathcal{M}\}) =: \kappa < \infty.$$

Then

$$\mathcal{R}(\{\xi^\alpha \partial_\xi^\alpha M(\xi) \mid \xi \in \dot{\mathbb{R}}^n, |\alpha| = k, M \in \mathcal{M}\}) \leq \kappa/(\sin\eta)^k,$$

for each $k \in \mathbb{N}_0$.

From above theorem, we do not need to show $\mathcal{R}$-boundedness of the derivatives when multipliers are bounded and holomorphic. To take over $\mathcal{R}$-boundedness, we need a dominated theorem below.

Theorem 2.7 ([18, Proposition 4.1.5]). *Let X, Y be Banach spaces, $D \subset \mathbb{R}^n$, and $1 < p < \infty$. Suppose $\mathcal{K} \subset \mathcal{L}(L_p(D, X), L_p(D, Y))$ is a family of kernel operators in the sense that*

$$Kf(x) = \int_D k(x, x') f(x') dx', \quad x \in D, f \in L_p(D, X),$$

for each $K \in \mathcal{K}$, where the kernels $k : D \times D \to \mathcal{L}(X, Y)$ are measurable, with

$$\mathcal{R}(\{k(x, x') : K \in \mathcal{K}\}) \leq k_0(x, x'), \quad x, x' \in D,$$

and the operator K_0 with scalar kernel k_0 is bounded in $L_p(D)$. Then $\mathcal{K} \subset \mathcal{L}(L_p(D, X), L_p(D, Y))$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{K}) \leq \|K_0\|_{L_p(D)}$.

Moreover we use the following theorem of the boundedness of a kernel operator.

Lemma 2.8 ([27, Lemma 5.5], [16, Proposition 1.4.16]). *Let X be a Banach space, $k(t, s)$ be a function defined on $(0, \infty) \times (0, \infty)$ which satisfies the condition: $k(\lambda t, \lambda s) = \lambda^{-1} k(t, s)$ for any $\lambda > 0$ and $(t, s) \in (0, \infty) \times (0, \infty)$. In addition, we assume that for some $1 \leq q < \infty$*

$$\int_0^\infty |k(1, s)| s^{-1/q} ds =: A_q < \infty.$$

If we define the integral operator T by the formula:

$$[Tf](t) = \int_0^\infty k(t, s) f(s) ds,$$

then T is a bounded linear operator on $L_q(\mathbb{R}_+, X)$ and

$$\|Tf\|_{L_q(\mathbb{R}_+, X)} \leq A_q \|f\|_{L_q(\mathbb{R}_+, X)}.$$

We use the theorem for $k(t, s) = (t + s)^{-1}$ which satisfies the assumption.

3 Proof of theorem 1.2 (Sufficient condition for $\mathcal{R}$-boundedness)

Proof. (i) From the assumptions, equivalence of uniformly boundedness and $\mathcal{R}$-boundedness on Hilbert space $\mathbb{C}$ and Theorem 2.6, we have

$$\mathcal{R}(\{\xi'^{\alpha}\partial_{\xi'}^{\alpha} m(\xi', x_N) \mid \xi' \in \dot{\mathbb{R}}^{N-1}, \alpha \in \{0,1\}^{N-1}\}) \leq C_\eta x_N^{-1}.$$

This means that for each $x_N > 0$,

$$\begin{aligned}\|T[m]f(\cdot, x_N)\|_{L_q(\mathbb{R}^{N-1})} &\leq \int_0^\infty \|\mathcal{F}_{\xi'}^{-1} m(\xi', x_N + y_N)\mathcal{F}_{x'} f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})} dy_N \\ &\leq C \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_q(\mathbb{R}^{N-1})}}{x_N + y_N} dy_N\end{aligned}$$

from Fourier multiplier theorem. And then, by Lemma 2.8,

$$\|T[m]f\|_{L_q(\mathbb{R}_+^N)} \leq C\|f\|_{L_q(\mathbb{R}_+^N)}$$

for some $C > 0$.
(ii) From the assumptions, equivalence of uniformly boundedness and $\mathcal{R}$-boundedness on Hilbert space $\mathbb{C} \in \mathcal{HT}(\alpha)$, and Theorem 2.6, we have

$$\mathcal{R}(\{\xi'^{\alpha}\partial_{\xi'}^{\alpha}(\tau\partial_\tau)^\beta m_\lambda(\xi', x_N) \mid \xi' \in \dot{\mathbb{R}}^{N-1}, \tau \in \dot{\mathbb{R}}, \alpha \in \{0,1\}^{N-1}, \beta \in \{0,1\}\}) \leq C_\eta x_N^{-1},$$

where $\gamma \geq \gamma_0$. This means that for each $x_N > 0, \gamma \geq \gamma_0$,

$$\{(\tau\partial_\tau)^\beta \mathcal{F}_{\xi'}^{-1} m_\lambda(\xi', x_N)\mathcal{F}_{x'} \mid \tau \in \dot{\mathbb{R}}, \beta \in \{0,1\}\} \subset \mathcal{L}(L_q(\mathbb{R}^{N-1}))$$

is $\mathcal{R}$-bounded and its $\mathcal{R}$-norm is less than Cx_N^{-1} by theorem 2.5. Combining theorem 2.7 with $D = (0,\infty), X = Y = L_q(\mathbb{R}^{N-1}), k_0(x_N, y_N) = (x_N + y_N)^{-1}$ and Lemma 2.8, we have

$$\{(\tau\partial_\tau)^\beta T[m_\lambda] \mid \tau \in \dot{\mathbb{R}}, \beta \in \{0,1\}\} \subset \mathcal{L}(L_q(\mathbb{R}_+^N))$$

is $\mathcal{R}$-bounded. We use operator-valued Fourier multiplier theorem to get $\mathcal{F}_{\tau\to t}^{-1} T[m_\lambda]\mathcal{F}_{t\to\tau}$ is a bounded linear operator on $L_p(\mathbb{R}, L_q(\mathbb{R}_+^N))$, which conclude

$$\|e^{-\gamma t}\tilde{T}_\gamma[m_\lambda]g\|_{L_p(\mathbb{R},L_q(\mathbb{R}_+^N))} \leq C\|e^{-\gamma t} g\|_{L_p(\mathbb{R},L_q(\mathbb{R}_+^N))}$$

for every $\gamma \geq \gamma_0$ and $1 < p, q < \infty$. □

4 Solution formulas for the Stokes equations on the half space

We give the solution of the resolvent problem (1) with $\lambda \in \Sigma_\varepsilon$ by Fourier multipliers for each boundary condition. We apply partial Fourier transform with respect to tangential direction $x' \in \mathbb{R}^{N-1}$. In this section and section 5 the index j runs from 1 to $N-1$ if we do not indicate. We use $A := \sqrt{\sum_{j=1}^{N-1} \xi_j^2}$ and $B := \sqrt{\lambda + A^2}$ with positive real parts.

4.1 Dirichlet boundary

In this subsection we focus on Dirichlet boundary condition. By partial Fourier transform, we have the following second order ordinary differential equations;

$$\begin{cases} (B^2 - \partial_N^2)\hat{u}_j + i\xi_j \hat{\pi} = 0 & \text{in } x_N > 0, \\ (B^2 - \partial_N^2)\hat{u}_N + \partial_N \hat{\pi} = 0 & \text{in } x_N > 0, \\ \displaystyle\sum_{j=1}^{N-1} i\xi_j \hat{u}_j + \partial_N \hat{u}_N = 0 & \text{in } x_N > 0, \\ \hat{u} = \hat{h} & \text{on } x_N = 0. \end{cases}$$

We find the solution of the form

$$\hat{u}_j(\xi', x_N) = \alpha_j e^{-Ax_N} + \beta_j e^{-Bx_N} \ (j = 1, \dots, N), \qquad \hat{\pi}(\xi', x_N) = \gamma e^{-Ax_N}.$$

Then, the equations are

$$\begin{cases} \{\alpha_j (B^2 - A^2) + i\xi_j \gamma\} e^{-Ax_N} = 0, \\ \{\alpha_N (B^2 - A^2) - A\gamma\} e^{-Ax_N} = 0, \\ (\displaystyle\sum_{j=1}^{N-1} i\alpha_j \xi_j - A\alpha_N) e^{-Ax_N} + (\sum_{j=1}^{N-1} i\beta_j \xi_j - B\beta_N) e^{-Bx_N} = 0, \\ \alpha_j + \beta_j = \hat{h}_j, \quad \alpha_N + \beta_N = \hat{h}_N. \end{cases}$$

By the linear independence of e^{-Ax_N} and e^{-Bx_N}, we are able to find the coefficients α_j, β_j and γ;

$$\alpha_j = \frac{i\xi_j}{A(B-A)}(i\xi' \cdot \hat{h}' - B\hat{h}_N), \quad \alpha_N = -\frac{i\xi' \cdot \hat{h}' - B\hat{h}_N}{B - A},$$
$$\beta_j = \hat{h}_j - \alpha_j, \quad \beta_N = \hat{h}_N - \alpha_N, \quad \gamma = -\frac{A+B}{A}(i\xi' \cdot \hat{h}' - B\hat{h}_N),$$

where $\xi' \cdot \hat{h}' = \sum_{k=1}^{N-1} \xi_k \hat{h}_k$. We introduce the new notation

$$\mathcal{M}_\lambda(\xi', x_N) = \frac{e^{-Bx_N} - e^{-Ax_N}}{B - A}$$

to treat $B - A$ in the denominator. Then, we have

$$\hat{u}_j(\xi', x_N) = \sum_{k=1}^{N-1} \left\{ \delta_{jk} e^{-Bx_N} + \frac{\xi_j \xi_k}{A} \mathcal{M}_\lambda(\xi', x_N) \right\} \hat{h}_k(\xi', 0) + \frac{i\xi_j B}{A} \mathcal{M}_\lambda(\xi', x_N) \hat{h}_N(\xi', 0),$$
$$\hat{u}_N(\xi', x_N) = \sum_{k=1}^{N-1} \{ i\xi_k \mathcal{M}_\lambda(\xi, x_N) \} \hat{h}_k(\xi', 0) + (e^{-Bx_N} - B\mathcal{M}_\lambda(\xi', x_N)) \hat{h}_N(\xi', 0),$$
$$\hat{\pi}(\xi', x_N) = \sum_{k=1}^{N-1} \left\{ -\frac{i\xi_k (A+B)}{A} e^{-Ax_N} \right\} \hat{h}_k(\xi', 0) + \frac{(A+B)B}{A} e^{-Ax_N} \hat{h}_N(\xi', 0),$$

To simplify, we define the symbols;

$$\phi^D_{j,k}(\lambda,\xi',x_N)=\delta_{jk}e^{-Bx_N}+\frac{\xi_j\xi_k}{A}\mathcal{M}_\lambda(\xi',x_N),\quad \phi^D_{j,N}(\lambda,\xi',x_N)=\frac{i\xi_jB}{A}\mathcal{M}_\lambda(\xi',x_N),$$
$$\phi^D_{N,k}(\lambda,\xi',x_N)=i\xi_k\mathcal{M}_\lambda(\xi,x_N),\qquad \phi^D_{N,N}(\lambda,\xi',x_N)=e^{-Bx_N}-B\mathcal{M}_\lambda(\xi',x_N),$$
$$\chi^D_j(\lambda,\xi',x_N)=-\frac{i\xi_j(A+B)}{A}e^{-Ax_N},\qquad \chi^D_N(\lambda,\xi',x_N)=\frac{(A+B)B}{A}e^{-Ax_N},$$

which derives the solution formula;

$$\hat{u}_j(\xi',x_N)=\sum_{k=1}^N\phi^D_{j,k}(\lambda,\xi',x_N)\hat{h}_k(\xi',0)\quad (j=1,\dots,N),$$
$$\hat{\pi}(\xi',x_N)=\sum_{k=1}^N\chi^D_k(\lambda,\xi',x_N)\hat{h}_k(\xi',0).$$

In the next step, we use the Volevich trick $a(\xi',0)=-\int_0^\infty\partial_N a(\xi',y_N)dy_N$ for a suitable decaying function a. We obtain the solution formula;

$$\begin{aligned}u_j(x)=-\sum_{k=1}^N\Big\{&\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[(\partial_N\phi^D_{j,k}(\lambda,\xi',x_N+y_N))\mathcal{F}_{x'}h_k\right](x,y_N)dy_N\\&+\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[\phi^D_{j,k}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_Nh_k)\right](x,y_N)dy_N\Big\}\quad (j=1,\dots,N),\\
\pi(x)=-\sum_{k=1}^N\Big\{&\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[(\partial_N\chi^D_k(\lambda,\xi',x_N+y_N))\mathcal{F}_{x'}h_k\right](x,y_N)dy_N\\&+\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[\chi^D_k(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_Nh_k)\right](x,y_N)dy_N\Big\}.\end{aligned}$$

Since Laplace transformed non-stationary Stokes equations (2) with $F=G=0$ on $\mathbb{R}$ are the resolvent problem (1), we have the following formula for Dirichlet boundary condition;

$$\begin{aligned}U_j(x,t)=-\mathcal{L}^{-1}_\lambda\sum_{k=1}^N\Big\{&\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[(\partial_N\phi^D_{j,k}(\lambda,\xi',x_N+y_N))\mathcal{F}_{x'}\mathcal{L}H_k\right](x,y_N)dy_N\\&+\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[\phi^D_{j,k}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}\mathcal{L}(\partial_NH_k)\right](x,y_N)dy_N\Big\}\quad (j=1,\dots,N),\\
\Pi(x,t)=-\mathcal{L}^{-1}_\lambda\sum_{k=1}^N\Big\{&\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[(\partial_N\chi^D_k(\lambda,\xi',x_N+y_N))\mathcal{F}_{x'}\mathcal{L}H_k\right](x,y_N)dy_N\\&+\int_0^\infty\mathcal{F}^{-1}_{\xi'}\left[\chi^D_k(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}\mathcal{L}(\partial_NH_k)\right](x,y_N)dy_N\Big\}.\end{aligned}$$

4.2 Neumann boundary

The corresponding ordinary differential equations are as follows;

$$\begin{cases} (B^2-\partial_N^2)\hat{u}_j + i\xi_j\hat{\pi} = 0 & \text{in } x_N>0, \\ (B^2-\partial_N^2)\hat{u}_N + \partial_N\hat{\pi} = 0 & \text{in } x_N>0, \\ \displaystyle\sum_{j=1}^{N-1} i\xi_j\hat{u}_j + \partial_N\hat{u}_N = 0 & \text{in } x_N>0, \\ -(\partial_N\hat{u}_j + i\xi_j\hat{u}_N) = \hat{h}_j, \quad -(2\partial_N\hat{u}_N - \hat{\pi}) = \hat{h}_N & \text{on } x_N=0. \end{cases}$$

The solutions are given by

$$\begin{aligned} \hat{u}_j(\xi', x_N) &= \sum_{k=1}^{N-1}\left\{(\frac{\delta_{jk}}{B} - \frac{\xi_j\xi_k(3B-A)}{D(A,B)B})e^{-Bx_N} + \frac{2\xi_j\xi_k B}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N)\right\}\hat{h}_k(\xi', 0) \\ &\quad + \left(\frac{-i(B-A)}{D(A,B)}\xi_j e^{-Bx_N} + \frac{i\xi_j(A^2+B^2)}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N)\right)\hat{h}_N(\xi', 0), \\ \hat{u}_N(\xi', x_N) &= \sum_{k=1}^{N-1}\left\{\frac{i\xi_k(B-A)}{D(A,B)}e^{-Bx_N} + \frac{2i\xi_k AB}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N)\right\}\hat{h}_k(\xi', 0) \\ &\quad + \left(\frac{A(A+B)}{D(A,B)}e^{-Bx_N} - \frac{A(A^2+B^2)}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N)\right)\hat{h}_N(\xi', 0), \\ \hat{\pi}(\xi', x_N) &= \sum_{k=1}^{N-1}\left\{\frac{-2i\xi_k B(A+B)}{D(A,B)}e^{-Ax_N}\right\}\hat{h}_k(\xi', 0) + \frac{(A+B)(A^2+B^2)}{D(A,B)}e^{-Ax_N}\hat{h}_N(\xi', 0), \end{aligned}$$

where $D(A,B) = B^3 + AB^2 + 3A^2B - A^3$. It is known that $D(A,B)\neq 0$ for $\lambda\in\Sigma_\varepsilon$, $\xi'\in\mathbb{R}^{N-1}\setminus\{0\}$ in [23, Lemma 4.4]. For the details of Neumann boundary condition, see also [27]. Let

$$\begin{aligned} \phi^N_{j,k}(\lambda, \xi', x_N) &= (\frac{\delta_{jk}}{B} - \frac{\xi_j\xi_k(3B-A)}{D(A,B)B})e^{-Bx_N} + \frac{2\xi_j\xi_k B}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N), \\ \phi^N_{j,N}(\lambda, \xi', x_N) &= \frac{-i\xi_j(B-A)}{D(A,B)}e^{-Bx_N} + \frac{i\xi_j(A^2+B^2)}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N), \\ \phi^N_{N,k}(\lambda, \xi', x_N) &= \frac{i\xi_k(B-A)}{D(A,B)}e^{-Bx_N} + \frac{2i\xi_k AB}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N), \\ \phi^N_{N,N}(\lambda, \xi', x_N) &= \frac{A(A+B)}{D(A,B)}e^{-Bx_N} - \frac{A(A^2+B^2)}{D(A,B)}\mathcal{M}_\lambda(\xi', x_N), \\ \chi^N_j(\lambda, \xi', x_N) &= \frac{-2i\xi_j B(A+B)}{D(A,B)}e^{-Ax_N}, \quad \chi^N_N(\lambda, \xi', x_N) = \frac{(A+B)(A^2+B^2)}{D(A,B)}e^{-Ax_N}, \end{aligned}$$

then, the solution formula is written as Dirichlet boundary.

4.3 Robin boundary

The symbols of the solutions are given by

$$\phi^R_{j,k}(\lambda, \xi', x_N) = (\frac{\delta_{jk}}{\alpha+\beta B} - \frac{\beta\xi_j\xi_k}{(\alpha+\beta B)(\alpha+\beta(A+B))A})e^{-Bx_N} + \frac{\xi_j\xi_k}{(\alpha+\beta(A+B))A}\mathcal{M}_\lambda(\xi', x_N),$$

$$\phi^R_{j,N}(\lambda,\xi',x_N) = -\frac{i\beta\xi_j B}{(\alpha+\beta(A+B))A}e^{-Bx_N} + \frac{i\xi_j B(\alpha+\beta B)}{(\alpha+\beta(A+B))A}\mathcal{M}_\lambda(\xi',x_N)$$

$$\phi^R_{N,k}(\lambda,\xi',x_N) = \frac{i\xi_k}{\alpha+\beta(A+B)}\mathcal{M}_\lambda(\xi',x_N),$$

$$\phi^R_{N,N}(\lambda,\xi',x_N) = e^{-Bx_N} - \frac{B(\alpha+\beta B)}{\alpha+\beta(A+B)}\mathcal{M}_\lambda(\xi',x_N),$$

$$\chi^R_j(\lambda,\xi',x_N) = \frac{-i\xi_j(A+B)}{(\alpha+\beta(A+B))A}e^{-Ax_N}, \quad \chi^R_N(\lambda,\xi',x_N) = \frac{B(A+B)(\alpha+\beta B)}{(\alpha+\beta(A+B))A}e^{-Ax_N}.$$

For the details of Robin boundary condition, see [22, 28] although they only treated the case $h_N = 0$.

5 Proof of the resolvent estimates and maximal regularity estimates

5.1 Dirichlet boundary

In section 4, we obtained the solution formulas for Dirichlet boundary condition. We use the following identity;

$$B^2 = \lambda + \sum_{m=1}^{N-1}\xi_m^2, \qquad 1 = \frac{B^2}{B^2} = \frac{\lambda^{1/2}}{B^2}\lambda^{1/2} - \sum_{m=1}^{N-1}\frac{i\xi_m}{B^2}(i\xi_m).$$

From now, we restrict the case $h_N = 0$ for simplicity. We decompose the solution operator so that the independent variables become the right-hand side of the estimates;

$$\begin{aligned}
u_j(x) = -\sum_{k=1}^{N-1}\Big\{&\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[B^{-2}\partial_N\phi^D_{j,k}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}((\lambda-\Delta')h_k)\right](x,y_N)dy_N\\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\lambda^{1/2}B^{-2}\phi^D_{j,k}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}\partial_N h_k)\right](x,y_N)dy_N\\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[i\xi_m B^{-2}\phi^D_{j,k}(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m\partial_N h_k)\right](x,y_N)dy_N\Big\} \quad (j=1,\dots,N),\\
\pi(x) = -\sum_{k=1}^{N-1}\Big\{&\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[B^{-2}\partial_N\chi^D_k(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}((\lambda-\Delta')h_k)\right](x,y_N)dy_N\\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\lambda^{1/2}B^{-2}\chi^D_k(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}\partial_N h_k)\right](x,y_N)dy_N\\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[i\xi_m B^{-2}\chi^D_k(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m\partial_N h_k)\right](x,y_N)dy_N\Big\}.
\end{aligned}$$

Let $S^D_{u_j}(\lambda,\xi',x_N)$ and $S^D_\pi(\lambda,\xi',x_N)$ be any of symbols;

$$S^D_{u_j}(\lambda,\xi',x_N) := \begin{cases} B^{-2}\partial_N\phi^D_{j,k}(\lambda,\xi',x_N),\\ \lambda^{1/2}B^{-2}\phi^D_{j,k}(\lambda,\xi',x_N),\\ i\xi_m B^{-2}\phi^D_{j,k}(\lambda,\xi',x_N) \qquad m\in\{1,\dots,N-1\},\end{cases}$$

$$S^D_\pi(\lambda,\xi',x_N) := \begin{cases} B^{-2}\partial_N\chi^D_k(\lambda,\xi',x_N), \\ \lambda^{1/2}B^{-2}\chi^D_k(\lambda,\xi',x_N), \\ i\xi_m B^{-2}\chi^D_k(\lambda,\xi',x_N) \qquad m\in\{1,\dots,N-1\}. \end{cases}$$

We are able to confirm that all of the symbols are bounded in the sense that

$$\begin{aligned}&\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde\Sigma^{N-1}_\eta\\ \ell,\ell'=1,\dots,N-1}} \Big\{(|\lambda|+|\lambda|^{1/2}|\xi_\ell|+|\xi_\ell||\xi_{\ell'}|)|S^D_{u_j}|+(|\lambda|^{1/2}+|\xi_\ell|)|\partial_N S^D_{u_j}|+|\partial_N^2 S^D_{u_j}|+|\xi_\ell||S^D_\pi|+|\partial_N S^D_\pi|\Big\}\\ &< Cx_N^{-1} \end{aligned} \tag{3}$$

for any $j\in\{1,\dots,N\}$, $0<\varepsilon<\pi/2$ and $0<\eta<\min\{\pi/4,\varepsilon\}$, because of the identity

$$\begin{aligned} \partial_N\mathcal{M}_\lambda(\xi',x_N) &= -e^{-Bx_N}-A\mathcal{M}_\lambda(\xi',x_N),\\ \partial_N^2\mathcal{M}_\lambda(\xi',x_N) &= (A+B)e^{-Bx_N}+A^2\mathcal{M}_\lambda(\xi',x_N),\\ \partial_N^3\mathcal{M}_\lambda(\xi',x_N) &= -(A^2+AB+B^2)e^{-Bx_N}-A^3\mathcal{M}_\lambda(\xi',x_N) \end{aligned}$$

and the estimate, essentially given by Shibata–Shimizu [27, Lemma 5.3]; Let $\tilde A := \sqrt{\sum_{j=1}^{N-1}|\xi_j|^2}$.

Lemma 5.1 ([11]). *Let $0<\varepsilon<\pi/2$, $0<\eta<\varepsilon/2$ and $m=0,1,2,3$. Then for any $(\lambda,\xi',x_N)\in\Sigma_\varepsilon\times\tilde\Sigma^{N-1}_\eta\times(0,\infty)$, we have*

$$c\tilde A\le \operatorname{Re}A\le|A|\le\tilde A, \tag{a}$$
$$c(|\lambda|^{1/2}+\tilde A)\le\operatorname{Re}B\le|B|\le|\lambda|^{1/2}+\tilde A, \tag{b}$$
$$|\partial_N^m e^{-Bx_N}|\le(|\lambda|^{1/2}+\tilde A)^m e^{-c(|\lambda|^{1/2}+\tilde A)x_N}\le C(|\lambda|^{1/2}+\tilde A)^{-1+m}x_N^{-1}, \tag{c}$$
$$|\partial_N^m e^{-Ax_N}|\le\tilde A^m e^{-c\tilde A x_N}\le C\tilde A^{-1+m}x_N^{-1}, \tag{d}$$
$$|\mathcal{M}_\lambda(\xi',x_N)|\le C(|\lambda|^{1/2}+\tilde A)^{-1}\tilde A^{-1}x_N^{-1}, \tag{e}$$
$$|\partial_N^m\mathcal{M}_\lambda(\xi',x_N)|\le C(|\lambda|^{1/2}+\tilde A)^{-2+m}x_N^{-1}(m\ne0), \tag{f}$$

with positive constants c and C, which are independent of λ,ξ',x_N.

We remark that the paper [27] treated for $\xi'\in\mathbb{R}^{N-1}\setminus\{0\}$ although above theorem is $\xi'\in\tilde\Sigma^{N-1}_\eta$. Since we have prepared theorem 1.2, we do not need the estimate of derivatives of the symbols.

From this lemma, we have that

$$\begin{aligned} |\partial_N^m\phi^D_{j,k}|,|\partial_N^m\phi^D_{N,k}|&\le C(|\lambda|^{1/2}+\tilde A)^{-1+m}x_N^{-1},\\ |\partial_N^m\chi^D_j|&\le C(|\lambda|^{1/2}+\tilde A)\tilde A^{-1+m}x_N^{-1}, \end{aligned}$$

for $m=0,1,2,3$, therefore the inequality (3) holds. The inequality (3) corresponds to the estimates λu, $\lambda^{1/2}\partial_\ell u$, $\partial_\ell\partial_{\ell'}u$, $\lambda^{1/2}\partial_N u$, $\partial_\ell\partial_N u$, $\partial_N^2 u$, and $\partial_\ell\pi$ and $\partial_N\pi$ respectively.

We also see that the new symbols $S^D_{u_j}$ and S^D_π, multiplied λ, ξ_ℓ and ∂_N, are holomorphic in $(\tau,\xi')\in\tilde\Sigma^N_\eta$. Therefore we are able to use theorem 1.2. This derives the existence part of theorem 1.3 with Dirichlet boundary condition. The uniqueness was proved in [16, p.121] where they considered the homogeneous equation and the dual problem.

For the non-stationary Stokes equations we have, by theorem 1.2 again.

5.2 Neumann boundary

Using the result in section 4, we have the following form

$$\begin{aligned}
u_j(x) = -\sum_{k=1}^{N}\Bigg\{&\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}B^{-2}\partial_N\phi_{j,k}^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}h_k)\right](x,y_N)dy_N\\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_m B^{-2}\partial_N\phi_{j,k}^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m h_k)\right](x,y_N)dy_N\\
&+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\phi_{j,k}^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_N h_k)\right](x,y_N)dy_N\Bigg\} \quad (j=1,\dots,N),\\
\pi(x) = -\sum_{k=1}^{N}\Bigg\{&\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}B^{-2}\partial_N\chi_k^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}h_k)\right](x,y_N)dy_N\\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_m\partial_N\chi_k^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m h_k)\right](x,y_N)dy_N\\
&+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\chi_k^N(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_N h_k)\right](x,y_N)dy_N\Bigg\}.
\end{aligned}$$

Let $S_{u_j}^N(\lambda,\xi',x_N)$ and $S_\pi^N(\lambda,\xi',x_N)$ be any of symbols;

$$S_{u_j}^N(\lambda,\xi',x_N) := \begin{cases}\lambda^{1/2}B^{-2}\partial_N\phi_{j,k}^N(\lambda,\xi',x_N) & \text{or,}\\ i\xi_m B^{-2}\partial_N\phi_{j,k}^N(\lambda,\xi',x_N) & \text{or,}\\ \phi_{j,k}^N(\lambda,\xi',x_N),\end{cases}$$

$$S_\pi^N(\lambda,\xi',x_N) := \begin{cases}\lambda^{1/2}B^{-2}\partial_N\chi_k^N(\lambda,\xi',x_N) & \text{or,}\\ i\xi_m B^{-2}\partial_N\chi_k^N(\lambda,\xi',x_N) & \text{or,}\\ \chi_k^N(\lambda,\xi',x_N).\end{cases}$$

We are able to confirm that all of the symbols are bounded in the sense that

$$\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde\Sigma_\eta^{N-1}\\ \ell,\ell'=1,\dots,N-1}}\left\{(|\lambda|+|\lambda|^{1/2}|\xi_\ell|+|\xi_\ell||\xi_{\ell'}|)|S_{u_j}^N|+(|\lambda|^{1/2}+|\xi_\ell|)|\partial_N S_{u_j}^N|+|\partial_N^2 S_{u_j}^N|+|\xi_\ell||S_\pi^N|+|\partial_N S_\pi^N|\right\}$$
$$< Cx_N^{-1}$$

by the estimates in lemma 5.1 and the following lemma.

Lemma 5.2 ([11]). *Let $0<\varepsilon<\pi/2$. Then there exist $\eta\in(0,\varepsilon/2)$ and a positive constant c such that*

$$c(|\lambda|^{1/2}+\tilde A)^3 \le |D(A,B)| \qquad (\lambda\in\Sigma_\varepsilon, \xi'\in\tilde\Sigma_\eta^{N-1}),$$

where $D(A,B) = B^3+AB^2+3AB^2-A^3$.

This is a generalization of [23, Lemma 4.4] in which they proved for $\xi'\in\mathbb{R}^{N-1}\setminus\{0\}$.

The symbols of Neumann boundary conditions satisfy

$$|\partial_N^m\phi_{j,k}^N|, |\partial_N^m\phi_{N,k}^N|, |\partial_N^m\phi_{j,N}^N|, |\partial_N^m\phi_{N,N}^N| \le C(|\lambda|^{1/2}+\tilde A)^{-2+m}x_N^{-1},$$

$$|\partial_N^m \chi_j^N| \le C(|\lambda|^{1/2}+\tilde{A})^{-1}\tilde{A}^m x_N^{-1},$$
$$|\partial_N^m \chi_N^N| \le C\tilde{A}^{-1+m} x_N^{-1}$$

for $m = 0,1,2,3$. Since the new symbols are holomorphic in $(\tau,\xi') \in \tilde{\Sigma}_\eta^N$, we apply theorem 1.2 for Neumann boundary condition.

We can prove existence of the solution. The uniqueness is proved in [27]. Non-stationary problem can be treated similarly.

5.3 Robin boundary

Using the result in section 4, we decompose as follows;

$$\begin{aligned}
u_j(x) = &-\sum_{k=1}^{N-1}\Bigg\{\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}B^{-2}\partial_N\phi_{j,k}^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{\xi'}(\lambda^{1/2}h_k)\right](x,y_N)dy_N \\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_m B^{-2}\partial_N\phi_{j,k}^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m h_k)\right](x,y_N)dy_N \\
&+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\phi_{j,k}^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_N h_k)\right](x,y_N)dy_N\Bigg\} \quad (j=1,\dots,N), \\
\pi(x) = &-\sum_{k=1}^{N-1}\Bigg\{\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}B^{-2}\partial_N\chi_k^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\lambda^{1/2}h_k)\right](x,y_N)dy_N \\
&-\sum_{m=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[i\xi_m B^{-2}\partial_N\chi_k^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_m h_k)\right](x,y_N)dy_N \\
&+\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\chi_k^R(\lambda,\xi',x_N+y_N)\mathcal{F}_{x'}(\partial_N h_k)\right](x,y_N)dy_N\Bigg\}.
\end{aligned}$$

Let $S_{u_j}^R(\lambda,\xi',x_N)$ and $S_\pi^R(\lambda,\xi',x_N)$ be any of symbols;

$$S_{u_j}^R(\lambda,\xi',x_N) := \begin{cases} \lambda^{1/2}B^{-2}\partial_N\phi_{j,k}^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\}, \\ i\xi_m B^{-2}\partial_N\phi_{j,k}^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\},\ m\in\{1,\dots,N-1\}, \\ \phi_{j,k}^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\}, \end{cases}$$
$$S_\pi^R(\lambda,\xi',x_N) := \begin{cases} \lambda^{1/2}B^{-2}\partial_N\chi_k^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\}, \\ i\xi_m B^{-2}\partial_N\chi_k^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\},\ m\in\{1,\dots,N-1\}, \\ \chi_k^R(\lambda,\xi',x_N) & k\in\{1,\dots,N-1\}. \end{cases}$$

Using the inequality $|\alpha+\beta(A+B)| \ge C_{\alpha,\beta}|B|$, we are able to confirm that all of the symbols are bounded in the sense that

$$\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde{\Sigma}_\eta^{N-1}\\ \ell,\ell'=1,\dots,N-1}}\Big\{(|\lambda|+|\lambda|^{1/2}|\xi_\ell|+|\xi_\ell||\xi_{\ell'}|)|S_{u_j}^R|+(|\lambda|^{1/2}+|\xi_\ell|)|\partial_N S_{u_j}^R|+|\partial_N^2 S_{u_j}^R|+|\xi_\ell||S_\pi^R|+|\partial_N S_\pi^R|\Big\}$$
$$< Cx_N^{-1}$$

The symbols of Robin boundary conditions satisfy

$$|\partial_N^m \phi_{j,k}^R|, |\partial_N^m \phi_{N,k}^R| \leq C(|\lambda|^{1/2} + \tilde{A})^{-2+m} x_N^{-1} \ (k = 1, \ldots, N-1),$$
$$|\partial_N^m \chi_j^R| \leq C(|\lambda|^{1/2} + \tilde{A})^{-1} \tilde{A}^m x_N^{-1},$$

for $m = 0, 1, 2, 3$. Since the new symbols are holomorphic in $(\tau, \xi') \in \tilde{\Sigma}_\eta^N$, we apply theorem 1.2 for Robin boundary condition.

This proved the existence of the theorem. The uniqueness is proved in [22]. Non-stationary problem can be treated similarly.

6 Remark

At last we remark that the differences of the previous works. Shibata et al. considered some theorems as follows.

Theorem 6.1 ([16, Lemma 5.3.5]). *Let $\varepsilon \in (0, \pi/2)$, $1 < q < \infty$, $\gamma_0 \geq 0$. Let m_i $(i = 1, 2)$ satisfies that*

$$|\partial_{\xi'}^{\alpha'}((\tau\partial_\tau)^\ell m_1(\lambda, \xi'))| \leq C_{\alpha'}|B|^{-|\alpha'|},$$
$$|\partial_{\xi'}^{\alpha'}((\tau\partial_\tau)^\ell m_2(\lambda, \xi'))| \leq C_{\alpha'}|A|^{-|\alpha'|}$$

for any $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$, $\alpha' \in \mathbb{N}_0^{N-1}$, $\ell = 0, 1$. Define

$$[K_1(\lambda, m_1)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_1(\lambda, \xi')|\lambda|^{1/2} e^{-B(x_N+y_N)} \mathcal{F}_{x'} g(\xi', y_N)\right](x, y_N) dy_N,$$
$$[K_2(\lambda, m_2)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi') A e^{-B(x_N+y_N)} \mathcal{F}_{x'} g(\xi', y_N)\right](x, y_N) dy_N,$$
$$[K_3(\lambda, m_2)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi') A e^{-A(x_N+y_N)} \mathcal{F}_{x'} g(\xi', y_N)\right](x, y_N) dy_N,$$
$$[K_4(\lambda, m_2)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi') A^2 \mathcal{M}_\lambda(\xi', x_N + y_N) \mathcal{F}_{x'} g(\xi', y_N)\right](x, y_N) dy_N,$$
$$[K_5(\lambda, m_2)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')|\lambda|^{1/2} A \mathcal{M}_\lambda(\xi', x_N + y_N) \mathcal{F}_{x'} g(\xi', y_N)\right](x, y_N) dy_N,$$

then, the following holds

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell K_1(\lambda, m_1) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C,$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell K_j(\lambda, m_2) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C$$

for any $\ell = 0, 1$, $j = 2, \ldots, 5$.

On the other hand our theorem 1.2 claims that we do not need distinguish as above under holomorphic condition.

Theories of Prüss et et al. is based on $\mathcal{H}^\infty$ property for *operators* not $\mathbb{C}$-valued *functions*. The following theorem is known as Kalton-Weis theorem. We change the notation;

$$\Sigma_\phi := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \phi\}.$$

Theorem 6.2 ([18, Theorem 4.5.6]). *Let X be a Banach space, $A \in \mathcal{H}^\infty(X)$, $\phi > \phi_A$, and let $\mathcal{F}$ be an operator-valued family $\mathcal{F} \subset H^\infty(\Sigma_\phi; \mathcal{L}(X))$ such that*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1} F(\lambda), \quad \mu \in \rho(A),\ \lambda \in \Sigma_\phi,\ F \in \mathcal{F}.$$

Then there is a constant $C_A > 0$ depending only on A and X such that
(i) If $\sup_{F \in \mathcal{F}} \mathcal{R}(F(\Sigma_\phi)) < \infty$, then $\mathcal{F}(A) := \{F(A) \mid F \in \mathcal{F}\} \subset \mathcal{L}(X)$ and

$$|F(A)|_{\mathcal{L}(X)} \leq C_A \mathcal{R}(F(\Sigma_\phi)),\ F \in \mathcal{F}.$$

(ii) If X has property (α) and $\mathcal{R}\{F(z) \mid z \in \Sigma_\phi, F \in \mathcal{F}\} < \infty$, then the operator family $\mathcal{F}(A)$ is $\mathcal{R}$-bounded, and

$$\mathcal{R}(\mathcal{F}(A)) \leq C_A \mathcal{R}\{F(z) \mid z \in \Sigma_\phi, F \in \mathcal{F}\}.$$

They often use this theorem as $A = d/dt$ whose H^∞-angle is $\pi/2$. For the detailed definitions and their methods, see [18]. On the other hand, theorem 1.2 does not require any knowledge of *operators* with $\mathcal{H}^\infty$-property.

References

[1] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Memoirs of AMS. Vol 166. No. 788. 2003.

[2] R. Denk, M. Hieber and J. Prüss, Optimal L^p-L^q-estimates for parabolic problems with inhomogeneous boundary data, Math. Z., 257(1), 2007, 193–224.

[3] R. Farwig, H. Kozono and H. Sohr, An L^q-approach to Stokes and Navier–Stokes equations in general domains, Acta Math. **195**, (2005), 21–53.

[4] R. Farwig, H. Kozono and H. Sohr, The Stokes operator in general unbounded domains, Hokkaido Math. J. **38**, (2009), 111–136.

[5] G. P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady State Problems, 2nd edn. (Splinger, New York, 2011)

[6] M. Geissert, H. Heck, M. Hieber and O. Sawada, Weak Neumann implies Stokes, J. Reine Angew. Math. **669**, (2012), 75–100.

[7] M. Geissert, M. Hess, M. Hieber, C. Schwartz and K. Stavrakidis, Maximal L^p-L^q-estimates for the Stokes equation: a short proof of Solonnikov's Theorem, J. Math. Fluid. Mech. **12**, (2010), 47–60.

[8] Y. Giga, Domains of fractional powers of the Stokes operator in L_r spaces, Arch. Ration. Mech. Anal. **89** (1985), 251–265.

[9] Y. Giga and H. Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains, J. Funct. Anal. **102** (1991), 72–94.

[10] M. Hieber and J. Saal, The Stokes equation in the L^p setting: well-poseedness and regularity properties, Handbook of mathematical analysis in mechanics of viscous fluid, Splinger, Cham, 2018, 117–206.

[11] N. Kajiwara, Maximal L_p-L_q regularity for the Stokes equations with various boundary conditions in the half space, preprint.

[12] N. Kajiwara, Solution formula for generalized two-phase Stokes equations and its applications to maximal regularity; model problems, preprint.

[13] N. Kajiwara, Maximal L_p-L_q regularity for the Stokes equations with Dirichlet-Neumann boundary condition in an infinite layer, preprint.

[14] N. Kajiwara and Y. Kiyomizu, Maximal regularity for the Stokes equations with Neumann boundary condition in an infinite layer, in preparation.

[15] N. Kajiwara and A. Matsui, Maximal regularity for the heat equation with various boundary condition in an infinite layer, in preparation.

[16] T. Kubo and Y. Shibata, Nonlinear differential equations, Asakura Shoten, Tokyo, 2012, (in Japanese).

[17] P. C. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, Functional analytic methods for evolution equations, Lecture Notes in Math., 1855, Splinger, Berlin, 2004, 65–311.

[18] J. Prüss and G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Birkhauser Monographs in Mathematics, 2016, ISBN: 978-3-319-27698-4

[19] Y. Shibata, On the $\mathcal{R}$-bounded solution operators in the study of free boundary problem for the Navier–Stokes equations, Springer Proceedings in Mathematics & Statistics Vol. 183 2016, Mathematical Fluid Dynamics, Present and Future, Tokyo, Japan, November 204, ed. Y. Shibata and Y. Suzuki, 203–285.

[20] Y. Shibata, On the $\mathcal{R}$-boundedness of solution operators for the Stokes equations with free boundary conditions, Differ. Integral Equ. **27**, (2014), 313–368.

[21] Y. Shibata, $\mathcal{R}$ boundedness, maximal regularity and free boundary problems for free boundary problems for the Navier Stokes equations, Mathematical analysis of the Navier–Stokes equations, Lecture Notes in Math., **2254**, Fond. CIME/CIME Found. Subser, (2020), 193–462.

[22] Y. Shibata and R. Shimada, On a generalized resolvent estimate for the Stokes system with Robin boundary conditions, J. Math. Soc. Jpn. **59** (2) (2007), 469–519.

[23] Y. Shibata and S. Shimizu, On a resolvent estimate for the Stokes system with Neumann boundary condition, Diff. Int. Eqns. **16** (4) (2003), 385–426.

[24] Y. Shibata and S. Shimizu, On the L_p-L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain, J. Reine Angew. Math. **615** (2008), 157–209.

[25] Y. Shibata and S. Shimizu, On a resolvent estimate of the Stokes system in a half space arising from a free boundary problem for the Navier-Stokes equations, Math. Nachr. **282** (2009), 482–499.

[26] Y. Shibata and S. Shimizu, Maximal L^p-L^q-regularity for the two phase Stokes equations; model problems, J. Difer. Equ. **251**, (2011), 373–419.

[27] Y. Shibata and S. Shimizu, On the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition; model problems, J. Math. Soc. Japan **64** (2) (2012), 561–626.

[28] R. Shimada, On the L^p-L^q maximal regularity for the Stkoes equations with Robin boundary conditions in a bounded domain, Math. Methods. Appl. Sci, **30**, (2007), 257–289.

[29] V. A. Solonnikov, Estimates for solutions of nonstationary Navier–Stokes equations, J. Sov. Math. **8**, (1977) 467–529.

[30] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. **319** (2001), 735–758.