An elliptic fibration arising from the Lagrange top

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Abstract

This paper is to investigate the complex algebro-geometric aspect of an elliptic fibration over $\mathbb{CP}^2$ arising from the Lagrange top. The discriminant locus of the elliptic fibration is described in detail. After suitable modifications of the base and the total spaces, the singular locus is concretely described and the singular fibres of the elliptic fibration are completely classified in view of R. Miranda's elliptic threefolds. Moreover a description of the monodromy of the elliptic fibration is obtained.

1 Introduction

This article is a summary of the paper [Ish] in preparation.

The heavy rigid body, i.e. the rotation of a rigid body around a fixed point under gravity, is one of the most typical problems in analytical mechanics. Among such heavy rigid bodies, it is known as a result by S. L. Ziglin [Zig82, Zig83] that the heavy rigid body dynamics is completely integrable only in the four particular cases: the Euler top, the Lagrange top, the Kowalevski top, and the Goryachev-Chaplygin top. See also [Aud99].

As for the Euler top, there are researches concerning the complex algebraic geometry of the fibrations by integral curves and by spectral curves as in the series of studies [NT12, TF14, FT15]. They have studied the complex algebraic geometry of the associated elliptic fibrations and applied their monodromy to the Birkhoff normal forms or the action coordinate.

The present article deals with the complex algebraic geometry of an elliptic fibration induced by the complexification of the energy-momentum map for the Lagrange top. The complexified energy-momentum map induces an elliptic threefold over $\mathbb{CP}^2$ in Weierstraß normal form. The main results of the present paper cover the detailed description of the discriminant locus of and the singular fibres the elliptic fibration, after suitable modifications of the base and the total spaces based on R. Miranda's method [Mir83]. Moreover the monodromy of the original elliptic fibration is described.

2 The Lagrange top

The heavy top is formulated as a Hamiltonian system on the cotangent bundle $T^*SO(3)$ to the three dimensional rotation group $SO(3)$, which can be identified with $SO(3) \times \mathfrak{so}(3)^*$ through the left-trivialization

$$T^*SO(3) \supset T_g^*SO(3) \ni \alpha_g \mapsto \left(g, L_g^*\alpha_g\right) \in SO(3) \times \mathfrak{so}(3)^*,$$

where $L_g \colon SO(3) \ni a \mapsto ga \in SO(3)$ is the left-translation. By Marsden-Weinstein Reduction Theorem [MW74] with respect to rotations about the axis collinear to the gravity vector, this Hamiltonian system can be reduced to the coadjoint orbits in $\left(\mathfrak{so}(3) \ltimes \mathbb{R}^3\right)^*$.

Through the natural identification $\left(\mathfrak{so}(3) \ltimes \mathbb{R}^3\right)^* \cong \mathbb{R}^3 \times \mathbb{R}^3$, the Hamilton equations for the heavy top are written on $\mathbb{R}^3 \times \mathbb{R}^3$ concretely as follows:

The Poisson bracket on $\mathbb{R}^3 \times \mathbb{R}^3$ is defined through

$$\{F, G\}(\Gamma, M) = -\langle \Gamma, (\nabla_M F) \times (\nabla_\Gamma G)\rangle - \langle \Gamma, (\nabla_\Gamma F) \times (\nabla_M G)\rangle - \langle M, (\nabla_M F) \times (\nabla_M G)\rangle,$$

where $(\Gamma, M) \in \mathbb{R}^3 \times \mathbb{R}^3$, $F, G \in \mathcal{C}^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^3$ and $(\nabla_\Gamma F, \nabla_M F)$ the gradient of the function F at (Γ, M) defined through

$$(\mathsf{d}F)_{(\Gamma,M)}(\xi,\eta) = \langle \nabla_\Gamma F(\Gamma, M), \xi\rangle + \langle \nabla_M F(\Gamma, M), \eta\rangle, \quad (\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong T_{(\Gamma,M)}\left(\mathbb{R}^3 \times \mathbb{R}^3\right).$$

We consider the total energy H as the Hamiltonian function for the heavy top defined through

$$H(\Gamma, M) = \frac{1}{2}\langle M, \Omega\rangle + \langle \Gamma, \chi\rangle,\ \Omega = \mathcal{J}^{-1}M,$$

where $J = \mathrm{diag}(J_1, J_2, J_3)$ is the inertia matrix and $\chi \in \mathbb{R}^3$ stands for the center of mass. The Hamilton equations for the Hamiltonian H with respect to the Poisson bracket $\{\cdot,\cdot\}$ are written as

$$\begin{cases} \dot{\Gamma} = \Gamma \times \Omega, \\ \dot{M} = M \times \Omega + \Gamma \times \chi, \end{cases}$$

which are usually called the Euler-Poisson equations. As is well known, this system has the two Casimir functions C_1, C_2 given as

$$C_1(\Gamma, M) = \langle \Gamma, \Gamma\rangle,\ C_2(\Gamma, M) = \langle \Gamma, M\rangle,$$

and the common level sets defined through

$$\mathcal{O}_a := \left\{(\Gamma, M) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \langle\Gamma,\Gamma\rangle = 1, \langle \Gamma, M\rangle = a\right\},\ a \in \mathbb{R},$$

are the highest dimensional symplectic leaves of the Poisson space $\left(\mathbb{R}^3 \times \mathbb{R}^3, \{\cdot,\cdot\}\right)$. Moreover these level sets coincide with the generic coadjoint orbits in $\left(\mathfrak{so}(3) \ltimes \mathbb{R}^3\right)^*$. Restricting the Euler-Poisson equations to $\mathcal{O}_a$, we obtain a Hamiltonian system of two degrees of freedom.

The *Lagrange top* is a spacial case of the heavy top with inertia $\mathcal{J} = \mathrm{diag}(J_1, J_2, J_3)$ and the center of mass $\chi = (\chi_1, \chi_2, \chi_3)$ satisfying the following conditions:

$$J_1 = J_2,\ \chi_1 = \chi_2 = 0.$$

Furthermore it is easily checked that the two functions

$$H(\Gamma, M) = \frac{1}{2}\langle M, \Omega\rangle + \langle \Gamma, \chi\rangle,\ L(\Gamma, M) = -M_3,$$

are functionally independent constants of motion for the Lagrange top and thus, the Lagrange top is completely integrable in the sense of Liouville.

Now we replace the four constants of motion C_1, C_2, H, L by

$$\begin{aligned} H_1 &:= C_1 = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2, \\ H_2 &:= C_2/J_1 = \Gamma_1\Omega_1 + \Gamma_2\Omega_2 + (m+1)\Gamma_3\Omega_3, \\ H_3 &:= H/J_1 = \frac{1}{2}\left(\Omega_1^2 + \Omega_2^2 + (m+1)\Omega_3^2\right) - \Gamma_3, \\ H_4 &:= -L/J_3 = \Omega_3, \end{aligned}$$

for the later use, following [GZ98]. Here we assume $\chi_3/J_1 = -1$ by suitable reparametrizations and we put $m := (J_3 - J_2)/J_1$.

3 Elliptic fibrations

3.1 Weiertraß normal form

In this subsection, we consider elliptic fibrations defined in Weierstraß normal form. See [Mir83], [Kas77], [Nak88], [Nak02] for the details.

Let S be a (compact) complex manifold and $\mathcal{L}$ a holomorphic line bundle over it. Take holomorphic sections a and b of $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 6}$, respectively, such that $\Delta := a^3 - 27b^2$ is not identically zero on S. Moreover let $\mathcal{O}_S$ be the structure sheaf of S, which can be identified with the trivial line bundle over S. We consider the direct sum of line bundles $\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S$ and denote its projectification by $P\left(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S\right)$.

Definition 3.1. The Weierstraß normal form $\mathcal{W}$ is the divisor on the $\mathbb{CP}^2$-bundle $P\left(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S\right)$ over S defined through

$$Y^2 Z - 4X^3 + aXZ^2 + bZ^3 = 0,$$

where $(X : Y : Z)$ are the homogeneous fibre coordinates of $P\left(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S\right)$. The section $\Delta = a^3 - 27b^2 \in H^0\left(S, \mathcal{L}^{\otimes 12}\right)$ is called the discriminant, whereas the meromorphic function

$$J = \frac{a^3}{\Delta} = \frac{a^3}{a^3 - 27b^2}$$

on S is called the functional invariant of the Weierstraß normal form.

Restricting the canonical projection $\pi\colon P\left(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S\right) \to S$ to $\mathcal{W}$, we obtain an elliptic fibration $\pi_{\mathcal{W}}\colon \mathcal{W} \to S$. In this paper, this elliptic fibration is also called the Weierstraß normal form over S.

Proposition 3.2. ([Mir83, Proposition 2.1]) *Let A, B, and D be the divisors on S defined by $a = 0$, $b = 0$, and $\Delta = 0$, respectively. Moreover let $((X : Y : Z)\,; p)$ denote a point of $P\left(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6} \oplus \mathcal{O}_S\right)$ where $p \in S$. Then the following statements hold.*

1. *$\mathcal{W}$ is smooth at $Z = 0$ and the set given by $(X : Y : Z) = (0 : 1 : 0)$ defines a holomorphic section of the elliptic fibration $\pi_{\mathcal{W}}$.*
2. *If $\mathcal{W}$ is singular at $((X : Y : Z)\,; p)$, then we obtain $Y = 0$ and $Z \neq 0$.*
3. *$\mathcal{W}$ is singular at $((0 : 0 : Z)\,; p)$ if and only if both A and B contain p, and B is singular at p.*
4. *$\mathcal{W}$ is singular at $((X : 0 : Z)\,; p)$ with $X \neq 0$ if and only if neither A nor B contains p, but D contains p and is singular at p. In this case, we obtain $(X : 0 : Z) = (-3b : 0 : 2a)$.*

3.2 Miranda's elliptic threefolds

In this subsection, we explain the classification of singular fibres on the basis of R. Miranda's method for elliptic threefolds discussed in [Mir83].

Let A_0, B_0, and D_0 be the reduced divisors of A, B, and D, respectively. Blowing up the base space S if necessary, we may suppose that the following two conditions hold:

(A) The reduced discriminant locus D_0 permits only nodes as singularities.

(B) A_0 and B_0 also have only nodes as singularities.

If there exists a locally defined holomorphic function u on S satisfying $u^4 | a$ and $u^6 | b$, we may replace X and Y by $u^2 X$ and $u^3 Y$, respectively. Repeating this procedure if necessary, we also assume the following additional condition:

(C) If $u^4 | a$ and $u^6 | b$, then u is a unit.

On these assumptions, we may choose a local coordinate system (s_1, s_2) on S centered at $p \in \operatorname{supp}(D)$ such that a, b, and Δ are written as

$$a = s_1^{L_1} s_2^{L_2} \overline{a},\ b = s_1^{K_1} s_2^{K_2} \overline{b},\ \Delta = s_1^{N_1} s_2^{N_2} \overline{\Delta}, \tag{3.1}$$

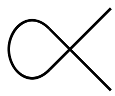
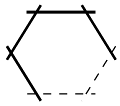
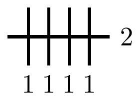
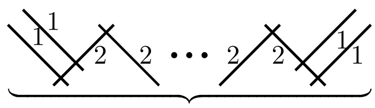
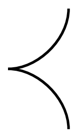
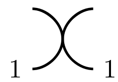
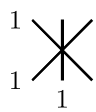
where $\overline{a}$, $\overline{b}$, and $\overline{\Delta}$ are local units at p.

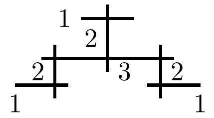
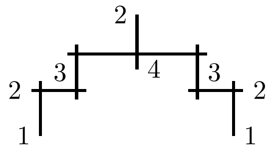
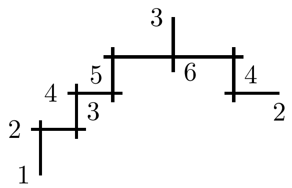
Singular fibres over smooth points of D_0 Let $p \in \operatorname{supp}(D)$ be a smooth point of D_0 and assume that $N_2 = 0$ at p. In this case, the singular fibres appear over the line $s_1 = 0$. Restricting a, b, and Δ onto the line $s_1 = 0$, we have

$$a = s_1^{L_1} \overline{a'},\ b = s_1^{K_1} \overline{b'},\ \Delta = s_1^{N_1} \overline{\Delta}.$$

Thus the types of singular fibres appearing over the point of $\operatorname{supp}(D)$ in a neighbourhood of p are of Kodaira type determined by the triple (L_1, K_1, N_1) in Table 1. See e.g. [Kod60, Kod63a, Kod63b], [Mir89, § 7.] for more details.

Table 1: Kodaira's list

Kodaira's notation	Dynkin diagram	(L, K, N) types	types of singular fibres
I_0	–	$(L \geqslant 0, 0, 0)$ or $(0, K \geqslant 0, 0)$	Smooth elliptic curve
I_1	A_0	$(0, 0, 1)$	 Nodal rational curve
I_N	A_{N-1}	$(0, 0, N \geqslant 2)$	 cycle of N smooth rational curve
I_0^*	D_4	$(L \geqslant 2, K \geqslant 3, 6)$	
I_{N-6}^*	D_{N-2}	$(2, 3, N \geqslant 7)$	 $N-5$ multiplicity 2 components
II	–	$(L \geqslant 1, 1, 2)$	 cuspidal rational curve
III	A_1	$(1, K \geqslant 2, 3)$	
IV	A_2	$(L \geqslant 2, 2, 4)$	

IV^*	E_6	$(L \geqslant 3, 4, 8)$	
III^*	E_7	$(3, K \geqslant 5, 9)$	
II^*	E_8	$(L \geqslant 4, 5, 10)$	

Singular fibres over singular points of D_0 Let $p \in \operatorname{supp}(D)$ be a singular point of D_0 and (s_1, s_2) a local coordinate system centred at p. As D_0 is singular at $(s_1, s_2) = (0, 0)$, we have $N_1 > 0, N_2 > 0$. In this case, the reduced discriminant locus D_0 is defined through $s_1 s_2 = 0$ around p.

The total space $\mathcal{W}$ of the elliptic fibration is now described as

$$Y^2 Z = 4X^3 - s_1^{L_1} s_2^{L_2} \overline{a} X Z^2 - s_1^{K_1} s_2^{K_2} \overline{b} Z^3, \tag{3.2}$$

whose discriminant is written as

$$\Delta = s_1^{N_1} s_2^{N_2} \overline{\Delta}. \tag{3.3}$$

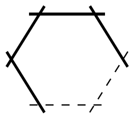
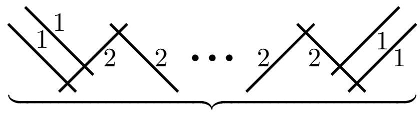
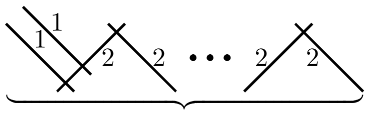
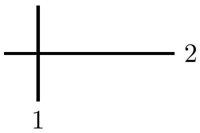
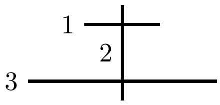
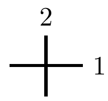
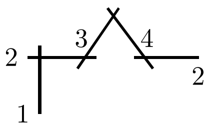
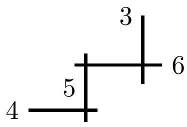
From the above argument, the types of singular fibres over the lines $s_1 = 0$ and $s_2 = 0$ except for the node $s_1 = s_2 = 0$ are determined by the triples (L_1, K_1, N_1) and (L_2, K_2, N_2), respectively.

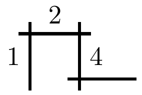
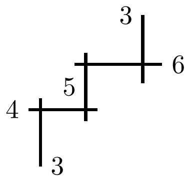
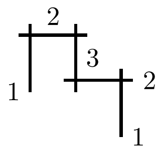
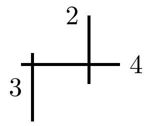
We take a blowing-up $S_1 \to S$ of S at p and the pull-back $\pi_1 \colon \mathcal{W}_1 \to S_1$ of the elliptic fibration $\pi \colon \mathcal{W} \to S$ via the projection of the blowing-up. The following proposition describe the types of singular fibres of π_1 over the exceptional divisor E on S_1.

Proposition 3.3. [Mir83, Proposition 9.1] *Assume that the defining equation of $\mathcal{W}$ and the discriminant around p are written as in the forms* (3.2) *and* (3.3). *Then, after a change of coordinates for π_1 if necessary, we can assume that the condition* (C) *is satisfied and then the types of the singular fibres of π_1 over the exceptional divisor E in S_1 are determined by the triple $(L_1 + L_2, K_1 + K_2, N_1 + N_2)$ modulo* $(4, 6, 12)$.

After suitable blowing-ups of S and a resolution of singularities $\widehat{\mathcal{W}} \to \mathcal{W}$, R. Miranda [Mir83] classifies the possible collisions between the irreducible components $s_1 = 0$ and $s_2 = 0$ of D_0 in a neighbourhood of p, and the explicit description of the singular fibres of the elliptic fibration $\pi_{\widehat{\mathcal{W}}} \colon \widehat{\mathcal{W}} \to S$ over the collision point for each colliding type as in Table 2.

Table 2: List of Miranda's singular fibres

colliding types	fibre of $\pi_{\widehat{\mathcal{W}}}$ over p	corresponding Kodaira's types	contracted components
$I_{M_1} + I_{M_2}$	 cycle of $(M_1 + M_2)$ smooth rational curves	$I_{M_1+M_2}$	none
$I_{M_1} + I^*_{M_2}$ (M_1: even)	 $\left(M_2 + \frac{M_1}{2} + 1\right)$ components with multiplicity 2	$I^*_{M_1+M_2}$	$\frac{M_1}{2}$ components with multiplicity 2
$I_{M_1} + I^*_{M_2}$ (M_1: odd)	 $\left(M_2 + \frac{M_1-1}{2} + 1\right)$ components with multiplicity 2	$I^*_{M_1+M_2}$	$\frac{M_1-1}{2}$ components with multiplicity 2 and 2 components with multiplicity 1
$II + IV$		I^*_0	3 components with multiplicity 1
$II + I^*_0$		IV^*	two of the three  components
$II + IV^*$		II^*	 components

$IV + I_0^*$		II^*	 components
$III + I_0^*$		III^*	 components

4 An elliptic threefold arising from the Lagrange top

4.1 The complexified energy-momentum map and a family of affine cubic curves

Let $\mathcal{O}_a$ be the common level set of H_1 and H_2:

$$\mathcal{O}_a \coloneqq \left\{ (\Gamma, M) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; H_1(\Gamma, M) = 1, H_2(\Gamma, M) = a \right\}.$$

The other two constants of motion H_3, H_4 can naturally be restricted to $\mathcal{O}_a$. To study the energy-momentum map $\mathcal{EM} \coloneqq (H_3, H_4) : \mathcal{O}_a \to \mathbb{R}^2$ as a fibration from the viewpoint of complex algebraic geometry, we complexify all the above settings. The complexified energy-momentum map, which is also denoted by $\mathcal{EM}$, is defined as the $\mathbb{C}^2$-valued function on

$$\mathcal{O}_a^{\mathbb{C}} \coloneqq \left\{ (\Gamma, M) \in \mathbb{C}^3 \times \mathbb{C}^3 \;\middle|\; H_1(\Gamma, M) = 1, H_2(\Gamma, M) = a \right\}. \tag{4.1}$$

Let $\mathbb{C}^* \cong \mathbb{C}/2\pi i\mathbb{Z}$ denote the complexified group of transformations defined by the Hamiltonian flow associated to H_4. It is known that $\mathbb{C}^*$ acts freely on the generic fibre

$$\mathcal{EM}^{-1}(h_3, h_4) = \left\{ (\Gamma, M) \in \mathcal{O}_a^{\mathbb{C}} \;\middle|\; H_3(\Gamma, M) = h_3, H_4(\Gamma, M) = h_4 \right\}$$

of $\mathcal{EM} \colon \mathcal{O}_a^{\mathbb{C}} \to \mathbb{C}^2$. Moreover the quotient manifold $\mathcal{EM}^{-1}(h_3, h_4)/\mathbb{C}^*$ is isomorphic to the affine part of the elliptic curve defined through the cubic equation in Weierstraß normal form

$$y^2 = 4x^3 - g_2 x - g_3, \tag{4.2}$$

where

$$g_2 = 1 + \frac{a_2^2}{12} - \frac{\alpha}{4} a_1, \; g_3 = \frac{a_2^3}{216} + \frac{a_1^2}{16} - \frac{\alpha}{48} a_1 a_2 - \frac{1}{6} a_2 + \frac{\alpha^2}{16}. \tag{4.3}$$

Here a_1, a_2, and α are given as

$$a_1 = 2(1+m) h_4, \; a_2 = 2h_3 + (1+m) m h_4^2, \; \alpha = -2a.$$

As will be seen in the next subsection, we construct the (singular) elliptic fibration W induced by the family of the cubic curves $C_{(a_1, a_2)}$ defined as in (4.2) parametrized by $(a_1, a_2) \in \mathbb{C}^2$.

4.2 Formulation of W as the elliptic fibration over $\mathbb{CP}^2$ and its singular locus

Let $C_{(a_1,a_2)}$ be the affine cubic curve defined through (4.2) and (4.3). We can construct the elliptic fibration W over $\mathbb{CP}^2$ induced by the family of cubic curves $\{C_{(a_1,a_2)}\}_{(a_1,a_2)\in\mathbb{C}^2}$ as follows:

Consider the affine coordinates $(a_1, a_2) \in \mathbb{C}^2$ as inhomogeneous coordinates of $\mathbb{CP}^2$. Namely setting $a_1 = A_1/A_0$, $a_2 = A_2/A_0$, $(A_0 : A_1 : A_2)$ denotes the homogeneous coordinates of $\mathbb{CP}^2$. Then g_2 and g_3 in (4.3) induce the holomorphic sections

$$g_2^* \in H^0\left(\mathbb{CP}^2, \mathcal{O}\left(\mathcal{L}^{\otimes 4}\right)\right),\; g_3^* \in H^0\left(\mathbb{CP}^2, \mathcal{O}\left(\mathcal{L}^{\otimes 6}\right)\right),$$

where $\mathcal{L}$ denotes the hyperplane bundle $\mathcal{O}_{\mathbb{CP}^2}(1)$ over $\mathbb{CP}^2$. Let $(X : Y : Z)$ be the homogeneous fibre coordinates of the projective bundle $P\left(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_{\mathbb{CP}^2}\right)$ over $\mathbb{CP}^2$. We consider the hypersurface W of $P\left(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_{\mathbb{CP}^2}\right)$ defined through

$$YZ^2 = 4X^3 - g_2^* XZ^2 - g_3^* Z^3.$$

Then, restricting the canonical projection $\pi\colon P\left(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_{\mathbb{CP}^2}\right) \to \mathbb{CP}^2$ to W, we obtain an elliptic fibration $\pi_W : W \to \mathbb{CP}^2$.

From Proposition 3.2, the singular locus of the elliptic fibration π_W is given by the divisor D on $\mathbb{CP}^2$ defined through

$$\left(g_2^*\right)^3 - 27\left(g_3^*\right)^2 = 0,$$

which is called the *discriminat locus*. The details on the divisor D is described as in the following theorem.

Theorem 4.1. ([Ish]) *The discriminant locus D consists of a line defined by $A_0 = 0$ with multiplicity 7 and a singular quintic curve which has four cusps and two nodes as its singularities. Moreover this quintic curve is tangent to the line $A_0 = 0$ at $(0 : 1 : 0)$ and $(0 : 0 : 1)$.*

Notation 4.2. We denote the two components of D which are the line $A_0 = 0$ and the singular quintic curve respectively by L and Q.

4.3 Singular fibres of a smooth model of W as a Miranda elliptic threefold

In the previous subsection, we see that the elliptic fibration $\pi_W : W \to \mathbb{CP}^2$ does not satisfy the conditions (A) and (B) discussed in Subsection 3.2.

We take a suitable modification $\sigma\colon \widehat{\mathbb{CP}^2} \to \mathbb{CP}^2$ of the base space and modify the total space W along the corresponding locus. Moreover, we change the homogeneous fibre coordinates as discussed in Subsection 3.2 if necessary. Then we obtain the new elliptic fibration $\pi_{\mathcal{W}}\colon \mathcal{W} \to \widehat{\mathbb{CP}^2}$ in Weierstraß normal form satisfying the following conditions:

- The Weierstraß normal form $\pi_{\mathcal{W}}\colon \mathcal{W} \to \widehat{\mathbb{CP}^2}$ is birationally isomorphic to $\pi_W : W \to \mathbb{CP}^2$.
- The elliptic fibration $\pi_{\mathcal{W}}\colon \mathcal{W} \to \widehat{\mathbb{CP}^2}$ satisfies the condition (A)–(C) in Subsection 3.2.
- All the colliding types appearing in $\mathcal{W}$ are on Miranda's list.

In what follows, we describe the types of singular fibres of the elliptic fibration $\pi_{\mathcal{W}}\colon \mathcal{W} \to \widehat{\mathbb{CP}^2}$ according to the three different types of singular points of D_0.

(a) By blowing-up three times at each cusp p of D, we have a picture of the reduced total transform of D as in Figure 1.

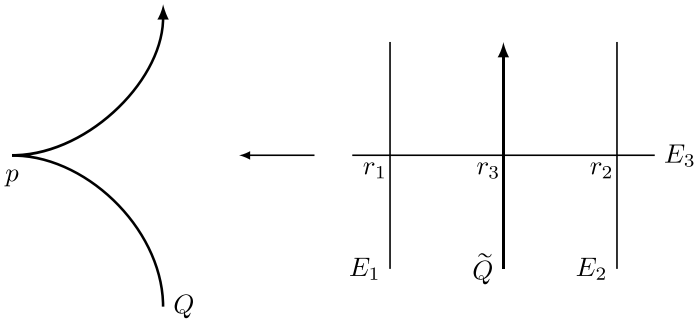


Figure 1: Blowing up at the cusp

The singular fibres over generic points of E_1, E_2, E_3, and $\widetilde{Q}$ are of type II, III, I_0^*, and I_1, respectively, in Kodaira's notation. However, the singular fibres over the collision points r_1, r_2, and r_3 do not belong to Kodaira's list. On Miranda's list, the dual graphs of each singular fibre are written as in Figures 2, 3, 4.

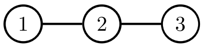


Figure 2: The singular fibre over r_1

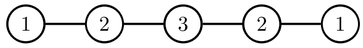


Figure 3: The singular fibre over r_2

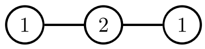


Figure 4: The singular fibre over r_3

(b) By suitable blowing-ups at $(0 : 1 : 0)$, we have a picture of the reduced total transform of D as in Figure 5.

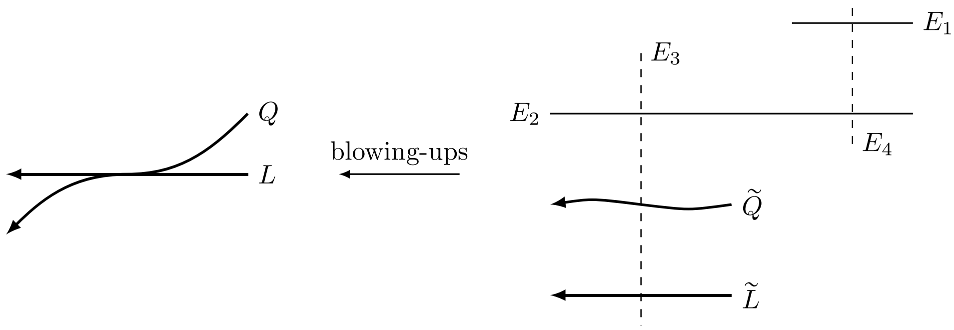


Figure 5: Blowing up at $(0 : 1 : 0)$

The singular fibres over E_1, E_2, $\widetilde{L}$, and $\widetilde{Q}$ are of type IV^*, IV, I_1^*, and I_1, respectively. Note that the fibres over E_3 and E_4 are smooth elliptic curves.

(c) By blowing-up two times at $(0 : 0 : 1)$, we have a picture of the reduced total transform of D as in Figure 6.

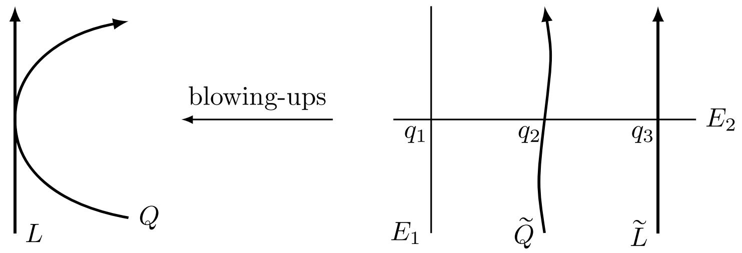


Figure 6: Blowing up at $(0:0:1)$

The singular fibres over generic points of E_1, E_2, $\widetilde{L}$, and $\widetilde{Q}$ are of type I_2^*, I_4, I_1^*, and I_1, respectively. Moreover, on Miranda's list, the singular fibres over q_1, q_2, and q_3 are of type I_4^*, I_5, I_3^*, respectively.

(d) By Theorem 4.1, the singular quintic curve Q has two nodes as its singularities. The singular fibres over generic points of Q are of type I_1. Hence the singular fibres over these two points are of type I_2 on Miranda's list.

To sum up, we have the following theorem.

Theorem 4.3. ([Ish]) *The singular elliptic fibration $\pi_W\colon W \to \mathbb{CP}^2$ is birationally equivalent to a Miranda elliptic threefold*

$$\pi_{\widehat{\mathcal{W}}}\colon \widehat{\mathcal{W}} \to \widehat{\mathbb{CP}^2},$$

with the discriminant locus $\widehat{D}$ whose support is given by

$$\operatorname{supp}\left(\widehat{D}\right) = \bigcup_{i=1}^{4} (E_{1,p_i} \cup E_{2,p_i} \cup E_{3,p_i}) \cup \left(\widetilde{E}_{1,(0:1:0)} \cup \widetilde{E}_{2,(0:1:0)}\right) \cup \left(E_{1,(0:0:1)} \cup E_{2,(0:0:1)}\right) \cup \widetilde{L} \cup \widetilde{Q},$$

where E_{1,p_i}, E_{2,p_i}, and E_{3,p_i} are exceptional divisors over the four cusps of the original discriminant locus p_i, $(i = 1, 2, 3, 4)$ as in Figure 1. *The singular fibres of $\pi_{\widehat{\mathcal{W}}}$ are described as follows:*

- *The singular fibres over generic points of $\widetilde{Q}$ are of type I_1.*
- *The singular fibres over $\widetilde{L}$ are of type I_1^*.*
- *The singular fibres over generic points of E_{1,p_i} are of type II.*
- *The singular fibres over generic points of E_{2,p_i} are of type III.*
- *The singular fibres over generic points of E_{3,p_i} are of type I_0^*.*
- *The singular fibres over the intersection point of E_{1,p_i} with E_{3,p_i}, the one of E_{2,p_i} with E_{3,p_i}, and the one of $\widetilde{Q}$ with E_{3,p_i} are displayed as in* Figure 2, 3, *and* 4, *respectively.*
- *The singular fibres over $\widetilde{E}_{1,(0:1:0)}$ are of type IV^*.*
- *The singular fibres over $\widetilde{E}_{2,(0:1:0)}$ are of type IV.*
- *The singular fibres over generic points of $E_{1,(0:0:1)}$ are of type I_2^*.*
- *The singular fibres over generic points of $E_{2,(0:0:1)}$ are of type I_4.*
- *The singular fibres over the intersection point of $E_{1,(0:0:1)}$ with $E_{2,(0:0:1)}$, the one of $\widetilde{Q}$ with $E_{2,(0:0:1)}$, and the one of $\widetilde{L}$ with $E_{2,(0:0:1)}$ are of type I_4^*, I_5, and I_3^*, respectively.*
- *The singular fibres over the two nodes of $\widetilde{Q}$ are of type I_2.*

4.4 Monodromy of π_W

In this subsection, we describe the monodromy of the original elliptic fibration $\pi_W : W \to \mathbb{CP}^2$ discussed in Subsection 4.2.

Since the regular locus $\mathbb{CP}^2 \setminus \mathrm{supp}\,(D)$ can be identified with $\mathbb{C}^2 \setminus Q_{\mathrm{aff}}$, where Q_{aff} denotes the affine part of the singular quintic curve Q, we have the following isomorphism:

$$\pi_1\left(\mathbb{CP}^2 \setminus \mathrm{supp}\,(D)\,, *\right) \cong \pi_1\left(\mathbb{C}^2 \setminus Q_{\mathrm{aff}}, *\right).$$

Let $\mathrm{Reg}\,(Q_{\mathrm{aff}})$ be the set of smooth points of Q_{aff}. To each connected component of $\mathrm{Reg}\,(Q_{\mathrm{aff}})$, we can associate a generator of $\pi_1\left(\mathbb{C}^2 \setminus Q_{\mathrm{aff}}\right)$. We denote the generators corresponding to the two components around a node of Q_{aff} by a_1, b_1 as in Figure 7 and around a cusp by a_2, b_2 as in Figure 8.

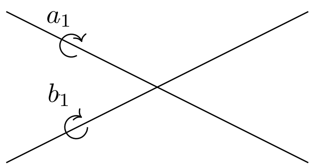


Figure 7: Node

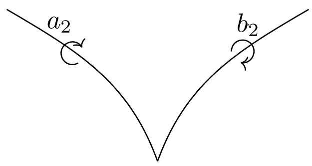


Figure 8: Cusp

By Zariski–van Kampen theorem [Zar29, Kam33], the generators a_1, b_1 around a node and the generators a_2, b_2 around a cusp satisfy

$$a_1 b_1 = b_1 a_1, \tag{4.4}$$
$$a_2 b_2 a_2 = b_2 a_2 b_2, \tag{4.5}$$

respectively.

The monodromy matrices of all the types of singular fibres in elliptic surfaces were found by K. Kodaira [Kod60, Kod63a, Kod63b]. Since singular fibres over the smooth points of Q_{aff} are of type I_1 in Kodaira's notation, the corresponding monodromy matrices are given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ up to conjugation by $SL\,(2, \mathbb{Z})$. Then we have the following theorem.

Theorem 4.4. ([Ish]) *With respect to a suitable choice of basis for $H_1\left(\pi_W^{-1}(p)\,, \mathbb{Z}\right)$ where p is the reference point of $\pi_1\left(\mathbb{C}^2 \setminus Q_{\mathrm{aff}}, p\right)$, the monodromy representation of the original elliptic fibration π_W is characterized as follows:*

- *For each connected component of* $\mathrm{Reg}\,(Q_{\mathrm{aff}})$, *the corresponding monodromy matrix is either*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

- *When two connected components share the same node in their closure, the above monodromy matrices are the same.*
- *When two connected components share the same cusp in their closure, the above matrices are distinct.*

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