Averaging operators on L^p-spaces

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1 Introduction

In this article, some properties on averaging operators between L^p-spaces are investigated, and an announcement of some results based on the research with P. Górka is contained. Let $X = (X, d, \mu)$ be a metric measure space such that for every point $x \in X$ and every positive number $r > 0$, $0 < \mu(B(x, r)) < \infty$, where d is a metric, μ is a measure and $B(x, r)$ is the closed ball centered at x with radius r. Furthermore, we assume that X is *Borel-regular*, which means that all Borel sets in X are measurable and any subset is contained in some Borel set with the same measure. Let $L^0(X)$ be the space of real-valued measurable functions on X, and for $1 \leq p < \infty$, let

$$L^p(X) = \left\{ f \in L^0(X) \;\middle|\; \int_X |f(x)|^p d\mu(x) < \infty \right\}$$

be a normed linear space endowed with a norm $\|\cdot\|_p$ defined by

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p},$$

where two functions that are coincident almost everywhere (a.e.) are identified. For each $r > 0$, we define $A_r : L^1(X) \to L^0(X)$ by

$$A_r f(x) = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f(y) d\mu(y)$$

for any $f \in L^1(X)$, which is called the *averaging operator*, and we call $A_r f$ the *average function* of f. The operator A_r is linear. Averaging operators play important roles in analysis.

2 Some properties on average functions

In this section, we shall show some basic properties on average functions, refer to [8] for details. On the measurability of average functions, we have the following:

Proposition 2.1. *For each $f \in L^1(X)$, $A_r f \in L^0(X)$.*

Denote the set of real numbers by $\mathbb{R}$, and let $\mathbb{R}^n = (\mathbb{R}^n, \rho, \nu)$ be the Euclidean space with the Euclidean metric ρ and the Lebesgue measure ν. The continuity on $A_s f(x)$ at $x \in X$ and $s \in (0, \infty)$ follows form the one of $\mu(B(x,s))$.

Proposition 2.2. *Let $f \in L^1(X)$.*

(1) *For any fixed $s \in (0,\infty)$, if the function $X \ni x \mapsto \mu(B(x,s)) \in (0,\infty)$ is continuous, then the function $A_s f : X \ni x \mapsto A_s f(x) \in \mathbb{R}$ is also continuous.*

(2) *For any fixed $x \in X$, if the function $(0,\infty) \ni s \mapsto \mu(B(x,s)) \in (0,\infty)$ is continuous, then the function $(0,\infty) \ni s \mapsto A_s f(x) \in \mathbb{R}$ is also continuous.*

Remark 1. We can consider the following conditions between metrics and measures on X:

($\star$) For each $x \in X$, $(0,\infty) \ni s \mapsto \mu(B(x,s)) \in (0,\infty)$ is continuous;

($*$) For any $x \in X$ and any $s \in (0,\infty)$, $\mu(B(x,s) \triangle B(y,s)) \to 0$ as $y \to x$;[1]

($\dagger$) For every $s \in (0,\infty)$, $X \ni x \mapsto \mu(B(x,s)) \in (0,\infty)$ is continuous.

As is easily observed, the implications ($\star$) $\Rightarrow$ ($*$) $\Rightarrow$ ($\dagger$) hold. Note that the Euclidean space $\mathbb{R}^n$ satisfies the property ($\star$).

A metric measure space X is *doubling* if there is $\gamma \geq 1$ such that

$$\mu(B(x,2s)) \leq \gamma \mu(B(x,s))$$

for every point $x \in X$ and every positive number $s > 0$, where γ is said to be *the doubling constant*. Remark that $\mathbb{R}^n$ is doubling. We introduce the Lebesgue differentiation theorem, which is a Lebesgue version of the fundamental theorem of calculus.

Theorem 2.3. *Let X be doubling and $f \in L^1(X)$. Then $A_s f$ converges to f a.e. as s tends to 0, that is, there exists $E \subset X$ of measure 0 such that for every point $x \in X \setminus E$,*

$$\lim_{s \to 0} A_s f(x) = f(x).$$

For $p \in [1,\infty)$ and $s \in (0,\infty)$, put

$$\|A_s\|_p = \sup\{\|A_s f\|_p \mid f \in L^p(X) \text{ with } \|f\|_p \leq 1\}.$$

When X is a doubling metric measure space with the doubling constant γ,

$$\|A_s\|_p \leq \gamma^{1/p}$$

for any $s > 0$, see [5, Lemma 2.1]. In the paper [1], the L^p-version of the Lebesgue differentiation theorem is shown as follows:

Theorem 2.4. *Let $p \in [1,\infty)$ and $f \in L^p(X)$. If there is a constant $c > 0$ such that $\sup_{s>0} \|A_s\|_p \leq c$, then*

$$\lim_{s \to 0} \|A_s f - f\|_p = 0.$$

Hence in the case where X is doubling, the above limit equation holds.

[1]Given subsets $A, B \subset X$, let $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

3 Applications of averaging operators to characterizing compact sets in L^p-spaces

Using averaging operators, we can give criteria for compact sets in $L^p(X)$. For a subset $E \subset X$, the characteristic function of E is denoted by χ_E, and for a function $f : X \to \mathbb{R}$, $f\chi_E$ is defined by $f\chi_E(x) = f(x) \cdot \chi_E(x)$. A.N. Kolmogorov [4] and M. Riesz [7] characterized compact subsets of $L^p(\mathbb{R}^n)$, which is an L^p-version of the Ascoli-Arzelà theorem, see [3, Theorem 5].

Theorem 3.1. *A bounded subset $F \subset L^p(\mathbb{R}^n)$ is relatively compact if and only if the following conditions are satisfied.*

(1) *For every $\epsilon > 0$, there exists $\delta > 0$ such that $\|\tau_a f - f\|_p < \epsilon$ for any $f \in F$ and $a \in \mathbb{R}^n$ with $|a| < \delta$.*[2]

(2) *For each $\epsilon > 0$, there is $s > 0$ such that $\|f\chi_{\mathbb{R}^n \setminus B(\mathbf{0},s)}\|_p < \epsilon$ for any $f \in F$.*

By virtue of averaging operators, P. Górka and A. Macios [2] extended the Kolmogorov-Riesz theorem to doubling metric measure spaces in the case that $p > 1$ as follows:

Theorem 3.2. *Let X be a doubling metric measure space and $F \subset L^p(X)$ be a bounded set, where $p > 1$. Suppose that X satisfies the following condition:*

(♯) $\inf\{\mu(B(x, s)) \mid x \in X\} > 0$ *for every $s > 0$.*

Then F is relatively compact if and only if the following are satisfied.

(1) *For each $\epsilon > 0$, there is $\delta > 0$ such that for any $f \in F$ and any $s \in (0, \delta)$, $\|A_s f - f\|_p < \epsilon$.*

(2) *For every $\epsilon > 0$, there exists a bounded subset E of X such that $\|f\chi_{X\setminus E}\|_p < \epsilon$ for any $f \in F$.*

In the case that $p \geq 1$, the author [5] established the following characterization.

Theorem 3.3. *Let any $p \in [1, \infty)$. Suppose that X is a doubling metric measure space satisfying the condition $(*)$ and F is a bounded subset of $L^p(X)$. Then F is relatively compact if and only if the following hold.*

(1) *For any $\epsilon > 0$, there exists $\delta > 0$ such that for each $f \in F$ and each $s \in (0, \delta)$, $\|A_s f - f\|_p < \epsilon$.*

(2) *For every $\epsilon > 0$, there is a bounded set E in X such that $\|f\chi_{X\setminus E}\|_p < \epsilon$ for any $f \in F$.*

Example 1. There are no implications between the assumptions (♯) and $(*)$ in Theorems 3.2 and 3.3, see the following instances which are Examples 1 and 2 of [5].

[2]For each $f : \mathbb{R}^n \to \mathbb{R}$ and each $a \in \mathbb{R}^n$, let $\tau_a f$ be a translation defined by $\tau_a f(x) = f(x - a)$.

(1) Let $X = \{0\} \cup [1, \infty)$ with the usual metric ρ and the measure μ defined by

$$\mu(A) = \begin{cases} \nu(A) & \text{if } 0 \notin A, \\ \nu(A) + 1 & \text{if } 0 \in A \end{cases}$$

for any Lebesgue measurable set $A \subset X$, where ν is the Lebesgue measure on $\mathbb{R}$. Then X is a doubling metric measure space satisfying ($\sharp$) but not ($*$).

(2) Let $X = [1, \infty)$ with the usual metric ρ and the measure μ induced by

$$\mu((a, b]) = \sqrt{b} - \sqrt{a}$$

for any $1 \leq a < b < \infty$. Then X is a doubling metric measure space that satisfies ($*$) but not ($\sharp$).

4 The boundedness and compactness of averaging operators

This section is devoted to investigating the boundedness and compactness of averaging operators. Given normed linear spaces $Y = (Y, \|\cdot\|_Y)$ and $Z = (Z, \|\cdot\|_Z)$, a linear operator $T : Y \to Z$ is *bounded* provided that

$$\|T\| = \sup\{\|T(y)\|_Z \mid y \in Y \text{ with } \|y\|_Y \leq 1\} < \infty.$$

It is known that the boundedness is equivalent to the continuity, refer to [6, Theorem 1.4.2]. We say that a linear operator T is *compact* if $T(\{y \in Y \mid \|y\|_Y \leq 1\})$ is relatively compact in Z. For a positive number $s > 0$, a metric measure space X is *s-doubling* if there is a constant $\gamma(s) \geq 1$ such that

$$\mu(B(x, 2s)) \leq \gamma(s)\mu(B(x, s))$$

for every point $x \in X$. The above $\gamma(s)$ is called the *s-doubling constant.* Obviously, if X is doubling, then it is s-doubling for all $s > 0$.

Remark 2. Fix any positive number $r > 0$. In the case where X is bounded, the following are equivalent:

(1) $\inf\{B(x, r) \mid x \in X\} > 0$;

(2) X is s-doubling for each $s \geq r$.

On the other hand, when X is not bounded, the above equivalence does not hold. Indeed, the space $X = [1, \infty)$ as in (2) of Example 1 is doubling, and hence s-doubling for every $s > 0$, but $\inf\{B(x, s) \mid x \in X\} = 0$.

Due to the same argument as [5, Lemma 2.1], we have the following:

Proposition 4.1. *Let any $s \in (0, \infty)$. If X is an s-doubling metric measure space with the s-doubling constant $\gamma(s)$, then $\|A_s\|_p \leq \gamma(s)^{1/p}$ for all $p \in [1, \infty)$, that is, the operator A_s is bounded on $L^p(X)$.*

Set the space

$$L^\infty(X) = \{f \in L^0(X) \mid \inf\{t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\} < \infty\}$$

with the following norm

$$\|f\|_\infty = \inf\{t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\}.$$

For every positive number $s > 0$, define a function $\alpha_s : X \to \mathbb{R}$ by

$$\alpha_s(x) = \int_X \frac{\chi_{B(x,s)}(y)}{\mu(B(y,s))} d\mu(y).$$

The boundedness of averaging operators on $L^p(X)$ can be characterized as follows, refer to [1, Theorem 3.3][3].

Theorem 4.2. *For each $p \in [1, \infty)$ and each $s \in (0, \infty)$, the operator A_s is bounded on $L^p(X)$ if and only if the function $\alpha_s \in L^\infty(X)$. Then $\|A_s\|_p = (\|\alpha_s\|_\infty)^{1/p}$.*

A metric measure space X has *the Vitali covering property* provided that the following is satisfied:

- Let $A \subset X$ and $\mathcal{B}$ be any colllection of closed balls whose centers are in A and whose radii are uniformly bounded. Moreover, for every point $x \in A$, there is some closed ball belonging to $\mathcal{B}$ centered at x and

$$\inf\{s > 0 \mid B(x, s) \in \mathcal{B}\} = 0.$$

  Then there exists a subcollection $\mathcal{B}' \subset \mathcal{B}$ consisting of pairwise disjoint countable balls such that $\mu(A \setminus \bigcup \mathcal{B}') = 0$.

Recall that if X is doubling, then it has the Vitali covering property, see the Vitali covering theorem [8, Theorem 6.20]. It seems that the compactness of averaging operators have not been well studied. Recently, P. Górka and the author research on it and have given some characterizations of it, which will be shown in a paper in preparation for submission. We shall introduced it partly as follows:

Theorem 4.3. *Let any $p > 1$ and any $r > 0$. Suppose that X is an s-doubling metric measure space having the Vitali covering property for all $s \geq r$ such that for each $x \in X$, $\mu(B(x, r) \triangle B(y, r)) \to 0$ as $y \to x$. Then $A_r : L^p(X) \to L^p(X)$ is compact if and only if X is bounded.*

[3] In Theorem 3.3 of [1], the characterization is shown only in the case that $p = 1$, but it is valid in the other case.

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