Definable $\mathcal{C}^r$ quotient in a definably complete locally o-minimal structure – partial results

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概要

We introduce several results on the geometry of sets definable in definably complete locally o-minimal expansions of ordered fields such as definable $\mathcal{C}^r$ approximation theorem and the existence of definable quotient in some special case. They are the extensions of the counterparts in o-minimal expansions of ordered fields. In the appendix, we also give a complete proof for the assertion on definable topology announced in the previous RIMS report.

1 Introduction

In the main body of this paper, we consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$ in this paper. The definitions of model-theoretic structures and expansions for non-model theorists are found in [11]. We do not repeat them in this paper. Standard textbooks for model theory are [2, 19, 22, 26, 27].

We recall the basic definitions.

Definition 1.1. A densely linearly ordered structure $\mathcal{M} = (M, <, \ldots)$ is *o-minimal* if every definable subset of M is a finite union of points and open intervals [4].

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The structure $\mathcal{M}$ is *definably complete* if every definable subset of M has both a supremum and an infimum in $M \cup \{\pm\infty\}$ [23].

The structure $\mathcal{M}$ is *locally o-minimal* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals [28].

Semialgebraic algebraic geometry [1, 3] are developed by real algebraic geometers. O-minimal structure is a natural generalization of the family of semialgebraic sets. It is extensively studied both model-theoretically and geometrically [4]. Several structures relaxing the requirements of o-minimality have been introduced. One of them is local o-minimality [28, 21]. However, it is already known that local o-minimality alone does not have tame topology. In fact, the local o-minimal structure given in [21, Example 12] has a definable function which is discontinuous at every point. Therefore, definably complete locally o-minimal structures have been studied in [8, 12, 13, 18]. We have checked whether famous assertions which hold true in semialgebraic geometry or o-minimal structures still hold true in a definably complete locally o-minimal expansion of an ordered field. In this paper, we introduce some of the result given in [17]. Another result is also found in [15, 16]. In them, we study sufficient conditions for the existence of definable quotient. We consider the following problems in this paper.

(1) Is a definable closed set the zero set of a definable $\mathcal{C}^r$ function?
(2) Is a definable $\mathcal{C}^{r-1}$ map between definable $\mathcal{C}^r$ submanifolds approximated by a definable $\mathcal{C}^r$ map in the definable $\mathcal{C}^{r-1}$ topology?
(3) Is a regular definable $\mathcal{C}^r$ manifold definably $\mathcal{C}^r$ diffeomorphic to a definable $\mathcal{C}^r$ submanifold?

We treat definable $\mathcal{C}^r$ manifolds and definable $\mathcal{C}^r$ maps etc. in this paper. They are called $\mathcal{D}^r$ manifolds and $\mathcal{D}^r$ maps etc. for short.

The answers to these questions are 'yes' if we consider o-minimal structures. In fact, they are demonstrated in [5, 7, 20]. We have shown that the answers are still 'yes' even in definably complete locally o-minimal expansion of an ordered field in [17]. A locally o-minimal structure whose universe is the set of reals $\mathbb{R}$ is o-minimal. Therefore, our results are nothing new in this case.

In the o-minimal setting, definable cell decomposition theorem and definable triangulation theorem are used for the proof of the above three questions, but they are unavailable in our setting. The first author proposed the notion of special submanifolds with tubular neighborhoods and proved that each definable set is decomposed into finitely many special submanifolds with tubular neighborhoods in [13]. We used this fact in place of definable cell decomposition theorem and definable triangulation theorem. We also need to modify the definition of a '$\mathcal{D}^r$ manifold' because a $\mathcal{D}^r$ submanifold is not necessarily a $\mathcal{D}^r$ manifold in our setting if we accept the traditional definition of $\mathcal{D}^r$ manifolds used in the studies on o-minimal structures such as [20]. These results are explained in Section 2 without proofs. See [17] for more details. We have to point out a drawback in our study. A locally o-minimal expansion of the field of reals $\mathbb{R}$ is always o-minimal. Many results introduced in this paper are already known when the structure is o-minimal.

Of course, all the results in the study of o-minimal structures are not extended to the definably complete locally o-minimal case. As an appendix, we give an example in Section A. It is known that any bounded closed definable subgroup of $\mathrm{GL}(F)$ is semialgebraic when the structure is an o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ [24, Theorem]. We construct a bounded definable subgroup of $\mathrm{GL}(n, F)$ which is not semialgebraic when $\mathcal{F}$ is locally o-minimal.

Section B is another appendix. We give a complete proof of the assertion announced in the previous report [14] in this section.

2 Results

2.1 Special manifold with tubular neighborhood

We first define special submanifolds. The definition employed here is Fornasiero's [8]. It is not identical to the definition employed in [13]. However, these two coincides with each other by [13].

Definition 2.1. Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \dots)$ be an expansion of an ordered field. Let X be a definable subset of F^n of dimension d and $\pi : F^n \to F^d$ be a coordinate projection. Let σ be a permutation of $\{1, \dots, n\}$ such that the composition $\pi \circ \overline{\sigma}$ is the coordinate projection onto first d coordinates, where $\overline{\sigma} : F^n \to F^n$ is the map given by $\overline{\sigma}(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

We first consider the case in which $\overline{\sigma}(X) \subseteq F^d \times (0,1)^{n-d}$. A point $(a,b) \in F^n$ is (X,π)-*normal* if there exist a definable neighborhood A of a in F^d and a definable neighborhood B of b in F^{n-d} such that either $A \times B$ is disjoint from $\overline{\sigma}(X)$ or $(A \times B) \cap \overline{\sigma}(X)$ is the graph of a definable continuous map $f : A \to B$. We call the point $(a,b) \in F^n$ (X,π)-$\mathcal{C}^r$-*normal* if the function f given above is a $\mathcal{D}^r$ map. A point $a \in F^d$ is (X,π)-*bad* if it is the projection of a non-(X,π)-normal point; otherwise, the point a is called (X,π)-*good*. We define (X,π)-$\mathcal{C}^r$-*bad* points and (X,π)-$\mathcal{C}^r$-*good* points similarly.

If X is unbounded, let $\phi : F \to (0,1)$ be a definable C^r diffeomorphism. For simplicity, we assume that π is the projection onto the first d coordinates. We consider the other cases similarly. Consider the map $\psi := \mathrm{id}^d \times \phi^{n-d} : F^d \times F^{n-d} \to F^d \times (0,1)^{n-d}$. We say that a is (X,π)-*good* if it is $(\psi(X),\pi)$-good. We define (X,π)-bad points etc. similarly.

The definable set X is a π-*special submanifold* if every point of $\pi(X)$ is (X,π)-good. We simply call it a special submanifold when the projection π is clear from the context. A π-*special* $\mathcal{C}^r$ *submanifold* is defined similarly.

Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of F^n. A *decomposition of* F^n *into special* $\mathcal{C}^r$ *submanifolds partitioning* $\{X_i\}_{i=1}^m$ is a finite family of special $\mathcal{C}^r$ submanifolds $\{C_i\}_{i=1}^N$ such that $\bigcup_{i=1}^N C_i = F^n$, $C_i \cap C_j = \emptyset$ when $i \neq j$ and either C_i has an empty intersection with X_j or is contained in X_j for any $1 \leq i \leq m$ and $1 \leq j \leq N$. A decomposition $\{C_i\}_{i=1}^N$ into special $\mathcal{C}^r$ submanifolds satisfies the *frontier condition* if the closure of any special submanifold C_i is the union of a subfamily of the decomposition.

We next define a special $\mathcal{C}^r$ submanifold with a tubular neighborhood.

Definition 2.2 ([13])**.** Let $\mathcal{F} = (F,<,+,0,\cdot,1,\dots)$ be an expansion of an ordered field. Let $x = (x_1,\dots,x_m), y = (y_1,\dots,y_m)$ be points in F^m. The notation $\mathrm{dist}_m : F^m \times F^m \to F$ denotes the $\mathcal{D}^\infty$ function given by $\mathrm{dist}_m(x,y) = \sum_{i=1}^n (x_i - y_i)^2$. Set $\mathcal{B}_m(x,r) = \{y \in F^m \mid \mathrm{dist}_m(x,y) < r\}$ for all $x \in F^m$ and $r > 0$. Note that the definition of $\mathcal{B}_m(x,r)$ is slightly different from that in [13]. For a given coordinate projection $\pi : F^n \to F^d$, take a permutation σ of $\{1,\dots,n\}$ such that the composition $\pi \circ \overline{\sigma}$ is the coordinate projection onto first d coordinates. Set $X_u^\pi = \{x \in F^{n-d} \mid \overline{\sigma}^{-1}(u,x) \in X\}$ for all $u \in F^d$ and a definable subset X of F^n. The set

X_u^π depends on the choice of σ, but we only discuss the features of X_u^π independent of σ in this paper.

When $\dim X < n$, the tuple (X, π, T, η, ρ) is a *special $\mathcal{C}^r$ submanifold with a tubular neighborhood* if

(a) $\pi : F^n \to F^d$ is a coordinate projection, where $d = \dim X$;
(b) X is a π-special $\mathcal{C}^r$ submanifold such that $U = \pi(X)$ is a definable open set;
(c) T is a definable open subset of $\pi^{-1}(U)$;
(d) $\eta : U \to F$ is a positive bounded $\mathcal{D}^r$ function such that, for all $u \in U$, we have

$$T_u^\pi = \bigcup_{x \in X_u^\pi} \mathcal{B}_{n-d}(x, \eta(u))$$

and

$$\mathcal{B}_{n-d}(x_1, \eta(u)) \cap \mathcal{B}_{n-d}(x_2, \eta(u)) = \emptyset$$

for all $x_1, x_2 \in X_u^\pi$ with $x_1 \neq x_2$;
(e) $\rho : T \to X$ is a $\mathcal{D}^r$ retraction such that, for any $u \in U$, we have $\rho(\pi^{-1}(u) \cap T) \subseteq \pi^{-1}(u) \cap X$ and $\rho(\overline{\sigma}^{-1}(u, y))) = \overline{\sigma}^{-1}(u, x)$ for all $x \in X_u^\pi$ and $y \in \mathcal{B}_{n-d}(x, \eta(u))$.

When $\dim X = n$, the tuple (X, π, T, η, ρ) is called a *special $\mathcal{C}^r$ submanifold with a tubular neighborhood* if X is open, $T = X$, $\eta \equiv 0$, and π and ρ are the identity maps on F^n and X, respectively. A *decomposition of F^n into special $\mathcal{C}^r$ submanifolds with tubular neighborhoods* is a finite family of special $\mathcal{C}^r$ submanifolds with tubular neighborhoods $\{(X_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^N$ such that $\{(X_i, \pi_i)\}_{i=1}^N$ is a decomposition of F^n into special $\mathcal{C}^r$ submanifolds. We say that a decomposition $\{(X_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^N$ of F^n into special $\mathcal{C}^r$ submanifolds with tubular neighborhoods partitions a given finite family of definable sets and satisfies the frontier condition if so does the decomposition into special submanifolds $\{(X_i, \pi_i)\}_{i=1}^N$.

The following theorem guarantees the existence of the decomposition.

Theorem 2.3. *Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$ be a definably complete locally o-minimal expansion of an ordered field. Let r be a nonnegative integer. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of F^n. There exists a decomposition of F^n into special $\mathcal{C}^r$ submanifolds with tubular neighborhoods partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. In addition, the number of special $\mathcal{C}^r$ submanifolds with tubular* *neighborhoods is bounded by a function of m and n.*

Proof. See [13] and [17]. □

2.2 Definable closed set is the zero set of a $\mathcal{D}^r$ function

Using Theorem 2.3 instead of definable cell decomposition theorem, we get the following theorem:

Theorem 2.4. *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$. A definable closed set is the zero set of a $\mathcal{D}^r$ function for $r < \infty$.*

Strategy of the proof. We prove it employing the same strategy as the proof of [5, Theorem C.11] but using decomposition of X into special $\mathcal{C}^r$ submanifolds with tubular neighborhoods in place of definable cell decomposition. □

2.3 Approximation theorem

We next recall the definition of definable submanifolds. It is identical to the traditional definition of definable submanifolds employed in [7, 20].

Definition 2.5 (Definable submanifolds)**.** Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$ be an expansion of an ordered field. Let r be a nonnegative integer. A $\mathcal{D}^r$ *submanifold* M of dimension d in F^n is a definable subset of F^n such that, for any point $a \in M$, there exist a definable open neighborhood U of the point a, a definable open neighborhood V of the origin in F^n and a $\mathcal{D}^r$ diffeomorphism $\varphi : U \to V$ such that $\varphi(a)$ is the origin and

$$\varphi(M \cap U) = \{(x_1, \ldots, x_n) \in V \mid x_{d+1} = \cdots = x_n = 0\}.$$

The *tangent bundle* TM is the set of $(x, v) \in M \times F^n$ such that v is a tangent vector to M at x. The *normal bundle* NM the set of $(x, v) \in M \times F^n$ such that v is orthogonal to the tangent space T_xM. They are definable sets. See [3] for instance.

The proof of the following theorem is identical to the traditional proof except that we use Theorem 2.4.

Theorem 2.6 ($\mathcal{D}^{r-1}$ tubular neighborhood)**.** *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, 0, \cdot, 1, \dots)$. Let r be a positive integer. A closed $\mathcal{D}^r$ submanifold M of F^n has a $\mathcal{D}^{r-1}$-tubular neighborhood; i.e., there exists a definable open neighborhood U of the zero section of $M \times \{0\}$ in the normal bundle NM such that the restriction of the map given by $NM \ni (x, v) \mapsto x + v \in F^n$ to U is a $\mathcal{D}^{r-1}$-diffeomorphism onto a definable open neighborhood Ω of M in F^n. Moreover, we can take U of the form*

$$U = \{(x, v) \in NM \mid \|v\| < \varepsilon(x)\},$$

where ε is a positive $\mathcal{D}^r$ function on M and the notation $\|v\|$ denotes the Euclidean norm of v.

We recall the definition of $\mathcal{D}^r$ topology given in [7].

Definition 2.7. Let r be a positive integer. Consider an expansion of an ordered field $\mathcal{F} = (F, <, +, 0, \cdot, 1, \dots)$ and a $\mathcal{D}^r$ submanifolds X and Y of F^m and F^n, respectively. The notation $\mathcal{D}^r(X, Y)$ denotes the set of $\mathcal{D}^r$ maps from X to Y. We write $\mathcal{D}^r(X)$ when $Y = F$.

We define a topology on $\mathcal{D}^r(X, Y)$. We call it the $\mathcal{D}^r$ *topology* on $\mathcal{D}^r(X, Y)$. We first consider the case in which $Y = F$. Consider a $\mathcal{D}^{r-1}$ vector field V on X; that is, $V : X \to TX$ is a $\mathcal{D}^{r-1}$ map such that the composition $\pi \circ V$ is the identity map on X, where TX is the tangent bundle of X and π is the natural projection from TX to X. The notation $V(f)$ denotes the derivative of f along V for each $f \in \mathcal{D}^r(X)$. We can take a finite family $\{V_1, \dots, V_n\}$ of $\mathcal{D}^{r-1}$ vector fields which span the tangent space to X at each point $x \in X$. In fact, let $x_1, \dots, x_n$ be the coordinate functions of the ambient space F^n of Y. We can naturally define the orthogonal projection $\Pi : TF^n \to TX$. The image $\left\{\Pi\left(\frac{\partial}{\partial x_1}\right), \dots, \Pi\left(\frac{\partial}{\partial x_n}\right)\right\}$ satisfies the requirement. For each positive definable continuous function $\varepsilon : X \to F$, set

$$U_\varepsilon = \{g \in \mathcal{D}^r(X) \mid |V_{i_1} \cdots V_{i_j} g| < \varepsilon \text{ for } 1 \leq i_1, \dots, i_r \leq m, j \leq r\}.$$

The sets $\{h + U_\varepsilon\}_\varepsilon$ form a neighborhood basis of $\mathcal{D}^r(X)$ at h, which defines the $\mathcal{D}^r$ topology on $\mathcal{D}^r(X)$.

We consider the case in which Y is general. The $\mathcal{D}^r$ topology on $\mathcal{D}^r(X, F^n) = \prod_{i=1}^n \mathcal{D}^r(X)$ is the product topology. The set $\mathcal{D}^r(X, Y)$ is a subset of $\mathcal{D}^r(X, F^n)$. The

$\mathcal{D}^r$ topology on it is defined as the induced topology under the inclusion $\mathcal{D}^r(X,Y) \subseteq \mathcal{D}^r(X,F^n)$.

Theorem 2.8. *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F}=(F,<,+,0,\cdot,1,\ldots)$ and two $\mathcal{D}^r$ submanifolds X and Y. A $\mathcal{D}^{r-1}$ map $f : X \to Y$ admits a $\mathcal{D}^r$ approximation in the $\mathcal{D}^{r-1}$ topology.*

Strategy of the proof. In spirit, we follow the proof of [7, Theorem 1.1]. In it, a stratification of the ambient space F^n into cells is used. An approximation is constructed in each stratum and they are pasted in the original proof. But it is impossible in our setting. Instead, we use a decomposition of F^n into special $\mathcal{C}^{r+1}$ submanifolds with tubular neighborhoods satisfying the frontier condition given in Theorem 2.3. We also use the following key lemma in the proof.

Lemma 2.9. *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F}=(F,<,+,0,\cdot,1,\ldots)$. Let r be a nonnegative integer and (X,π,T,η,ρ) be a special $\mathcal{C}^{r+1}$ submanifold in F^n of dimension d with a tubular neighborhood, where π is the coordinate projection onto the first d-coordinates. Set $U=\pi(X)$. Let $f : \pi^{-1}(U) \to F$ be a $\mathcal{D}^r$ function which is of class $\mathcal{C}^{r+1}$ off X. Assume that the restriction $f|_X$ of f to X is of class $\mathcal{C}^{r+1}$. Let $\delta : \pi^{-1}(U) \to F$ be a positive definable continuous function. Then there exists a $\mathcal{D}^{r+1}$ function $\widetilde{f} : \pi^{-1}(U) \to F$ such that*

$$\left| \frac{\partial^{|a|+|b|}}{\partial x^a \partial y^b}(f-\widetilde{f}) \right| < \delta$$

for all sequences of nonnegative integers a and b with $|a|+|b| \leq r$.

Sketch of the proof. By consider the Taylor expansion of f, we construct a $\mathcal{D}^{r+1}$ function P sufficiently close to f on X. We also construct a $\mathcal{D}^{r+1}$ function λ which is one on X and zero out of tubular neighborhood. The $\mathcal{D}^{r+1}$ function $\widetilde{f}=\lambda P+(1-\lambda)f$ satisfies the requirement. □

□

2.4 Imbedding of $\mathcal{D}^r$ manifolds

Definition 2.10. Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$ be a definably complete locally o-minimal expansion of an ordered field. Suppose that $1 \leq r < \infty$.

(1) A pair $(M, \{\varphi_i : U_i \to U_i'\}_{i \in I})$ of a topological space and a finite family of homeomorphisms is called a *definable* $\mathcal{C}^r$ *manifold* or a $\mathcal{D}^r$ *manifold* if

- $\{U_i\}_{i \in I}$ is a finite open cover of M,
- U_i' is a $\mathcal{D}^r$ submanifold of M^{m_i} for any $i \in I$ and,
- the composition $(\varphi_j|_{U_i \cap U_j}) \circ (\varphi_i|_{U_i \cap U_j})^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a $\mathcal{D}^r$ diffeomorphism whenever $U_i \cap U_j \neq \emptyset$.

Here, the notation $\varphi_i|_{U_i \cap U_j}$ denotes the restriction of φ_i to $U_i \cap U_j$. We use similar notations throughout the rest of this paper. The family $\{\varphi_i : U_i \to U_i'\}_{i \in I}$ is called a $\mathcal{D}^r$ *atlas* on M. We often write M instead of $(M, \{\varphi_i : U_i \to U_i'\}_{i \in I})$ for short. Note that a $\mathcal{D}^r$ submanifold is naturally a $\mathcal{D}^r$ manifold.

Remark 2.11. In the o-minimal setting, a $\mathcal{D}^r$ manifold is defined as the object obtained by pasting finitely many definable open sets. $\mathcal{D}^r$ submanifolds are pasted in our definition. If we adopt the same definition of $\mathcal{D}^r$ manifolds as in the o-minimal setting, $\mathcal{D}^r$ manifolds of dimension zero should be a finite set because F^0 is a singleton. A $\mathcal{D}^r$ submanifold of dimension zero is not necessarily a $\mathcal{D}^r$ manifold in this definition. It seems to be strange, so we employed our definition of $\mathcal{D}^r$ manifolds.

Definition 2.12. Given a $\mathcal{D}^r$ manifold M, two $\mathcal{D}^r$ atlases $\{\varphi_i : U_i \to U_i'\}_{i \in I}$ and $\{\psi_j : V_j \to V_j'\}_{j \in J}$ on M are *equivalent* if, for all $i \in I$ and $j \in J$,

- the images $\varphi_i(U_i \cap V_j)$ and $\psi_j(U_i \cap V_j)$ are open definable subsets of U_i' and V_j', respectively, and
- the $\mathcal{D}^r$ diffeomorphism $(\psi_j|_{U_i \cap V_j}) \circ (\varphi_j|_{U_i \cap V_j})^{-1} : \varphi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$ are definable whenever $U_i \cap U_j \neq \emptyset$.

The above relation is obviously an equivalence relation.

A subset X of the $\mathcal{D}^r$ manifold M is *definable* when $\varphi_i(X \cap U_i)$ are definable for all $i \in I$. When two atlases $\{\varphi_i : U_i \to U_i'\}_{i \in I}$ and $\{\psi_j : V_j \to V_j'\}_{j \in J}$ of a $\mathcal{D}^r$ manifold

M is equivalent, it is obvious that a subset of the $\mathcal{D}^r$ manifold $(S, \{\varphi_i\}_{i\in I})$ is definable if and only if it is definable as a subset of the $\mathcal{D}^r$ manifold $(M, \{\psi_j\}_{j\in J})$.

The Cartesian product of two $\mathcal{D}^r$ manifold is naturally defined. A map $f : S \to T$ between $\mathcal{D}^r$ manifolds is *definable* if its graph is definable in $S \times T$.

(2) A definable subset Z of X is called a *k-dimensional $\mathcal{D}^r$ submanifold* of X if each point $x \in Z$ there exist an open box U_x of x in X and a $\mathcal{D}^r$ diffeomorphism ϕ_x from U_x to some open box V_x of F^d such that $\phi_x(x) = 0$ and $U_x \cap Y = \phi_x^{-1}(F^k \cap V_x)$.

(3) Let X and Y be $\mathcal{D}^r$ manifolds with $\mathcal{D}^r$ charts $\{\phi_i U_i \to V_i\}_{i\in A}$ and $\{\psi_j : U'_j \to V'_j\}_{j\in B}$, respectively. A continuous map $f : X \to Y$ is said to be a *definable C^r map* or a *$\mathcal{D}^r$ map* if for any $i \in A$ and $j \in B$, the image $\phi_i(f^{-1}(V'_j) \cap U_i)$ is definable and open in F^n and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to F^m$ is a $\mathcal{D}^r$ map.

(4) Let X and Y be $\mathcal{D}^r$ manifolds. We say that X is *definably C^r diffeomorphic to Y* or $\mathcal{D}^r$ diffeomorphic to Y if there exist $\mathcal{D}^r$ maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$.

The following is the $\mathcal{D}^r$ imbedding theorem of $\mathcal{D}^r$ manifolds:

Theorem 2.13. *Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$ be a definably complete locally o-minimal expansion of an ordered field. Every regular $\mathcal{D}^r$ manifold is definably imbeddable into some F^n, and its image is a $\mathcal{D}^r$ submanifold of F^n.*

Strategy of the proof. A classical proof using partition of unity works also in this case. For the constriction of a partition of unity, we used Theorem 2.4. □

2.5 Application to $\mathcal{D}^r$ group

Definition 2.14. Consider an expansion of a dense linear order without endpoints. A *definable group* is a group $(G, \cdot, e)$ such that G is definable and both the multiplication $(a, b) \mapsto a \cdot b$ and the inverse $a \mapsto a^{-1}$ are definable maps. We define a *definable subgroup* of a definable group naturally. A definable group is called a *definable $\mathcal{C}^r$ group* or a *$\mathcal{D}^r$ group* if both the multiplication and the inverse are of class $\mathcal{C}^r$.

A *definable equivalence relation* E on a definable set X is a definable subset of $X \times X$ such that the relation $\sim$ defined by $a \sim b \Leftrightarrow (a, b) \in E$ is an equivalence relation. Let G be a definable group and H be its definable subgroup. The relation

E_H given by $E_H = \{(g, hg) \in G \times G \mid g \in G, h \in H\}$ is a definable equivalence relation.

Definition 2.15. Consider an expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $X \subseteq M^m$ and $Y \subseteq M^n$ be definable sets and $f : X \to Y$ be a definable continuous map. The map f is *definably identifying* if it is surjective and, for each definable subset K in Y, K is closed in Y whenever $f^{-1}(K)$ is closed in X.

Consider a structure $\mathcal{M} = (M, \ldots)$, a definable set X and a definable equivalence relation E on X. A *definable quotient of X by E* is a definably identifying definable surjective continuous map $f : X \to Y$ such that $f(x) = f(x')$ if and only if $(x, x') \in E$. In addition, when both X and Y are $\mathcal{D}^r$ submanifolds and f is a $\mathcal{D}^r$ map, we call it a *definable $\mathcal{C}^r$ quotient of X by E* or a *$\mathcal{D}^r$ quotient of X by E*

We consider the case in which a definable group G acts on a definable set X. Assume that the action $G \times X \to X$ is a $\mathcal{D}^r$ map. A $\mathcal{D}^r$ quotient of X by G is defined as the $\mathcal{D}^r$ quotient of X by the definable equivalence relation $E_{G,X} := \{(x, gx) \in X \times X \mid x \in X, g \in G\}$.

Theorem 2.16. *Consider a definably complete locally o-minimal expansion of an ordered field. Let $0 \leq r < \infty$. Let G be a $\mathcal{D}^r$ group and H be a definable subgroup of G. There exist a $\mathcal{D}^r$ submanifold X of dimension $\dim G - \dim H$ and a $\mathcal{D}^r$ quotient $\iota : G \to X$ of G by H.*

In addition, if H is a normal subgroup of G, there exists a $\mathcal{D}^r$ maps $\mathrm{mult} : X \times X \to X$ and $\mathrm{inv} : X \to X$ such that $\mathrm{mult}_{G/H}(\iota(g_1), \iota(g_2)) = \iota(g_1 g_2)$ and $\mathrm{inv}_{G/H}(\iota(g)) = \iota(g^{-1})$ for $g, g_1, g_2 \in G$. In other word, the definable set X is a $\mathcal{D}^r$ group and it is isomorphic to the quotient group G/H as a group.

Strategy of the proof. We first demonstrate that both G and H are $\mathcal{D}^r$ submanifolds. Using this fact, we construct a $\mathcal{D}^r$ manifold X. We then apply Theorem 2.13. □

We also get the following basic results:

Proposition 2.17. *Consider a definably complete locally o-minimal expansion of an ordered field. A $\mathcal{D}^r$ group is a $\mathcal{D}^r$ manifold.*

Proposition 2.18. *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$. Let G be a $\mathcal{D}^r$ group and H be a definable* *subgroup of G. Then, H is a closed in G and $\mathcal{D}^r$ submanifold of G.*

Definition 2.19. Let G be a definable group. A $\mathcal{D}^r$ *group structure* on G is a pair of a $\mathcal{D}^r$ group H and a definable group isomorphism $\iota : G \to H$. Two $\mathcal{D}^r$ group structures (H_1, ι_1) and (H_2, ι_2) are *equivalent* if the composition $\iota_2 \circ \iota_1^{-1}$ is a $\mathcal{D}^r$ diffeomorphism.

Theorem 2.20. *Consider a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let G be a definable group. There exists a $\mathcal{D}^r$ group structure on G and it is unique up to equivalence.*

A Definably complete locally o-minimal expansion of an ordered field which has a non-semialgebraic bounded closed definable subgroup of the general linear group

We demonstrated that several properties of o-minimal structures still hold true in definably complete locally o-minimal structures in the main body of this paper. We provides an opposite example in this section. When an expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ is o-minimal, any bounded closed definable subgroup of $\mathrm{GL}(n, F)$ is semialgebraic [24, Theorem]. We construct a definably complete locally o-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ which has a bounded closed definable non-semialgebraic subgroup of $\mathrm{GL}(2, F)$.

The following construction of a structure is standard in model theory and it is called an ‘ultraproduct’. We give a detailed explanation for non-model theorists here in a concrete case. See standard textbooks of model theory for more information in more general case. The authors recommend [22, Exercise 2.5.18, Exercise 2.5.19] for instance.

Definition and Facts A.1. Consider a subfamily I of the power set of $\mathbb{N}$. The family I is called a *filter* if the following conditions are satisfied:

(i) $\emptyset \notin I$ and $\mathbb{N} \in I$;
(ii) $A, B \in I \Rightarrow A \cap B \in I$;
(iii) $A \in I,\ A \subseteq B \subseteq \mathbb{N} \Rightarrow B \in I$.

An *ultrafilter* I is a filter such that, for each subset A of $\mathbb{N}$, we have either $A \in I$ or $\mathbb{N} \setminus A \in I$.

It is already known that, for any filter I, there exists an ultrafilter J such that $I \subseteq J \subseteq 2^{\mathbb{N}}$.

Let $\mathcal{L}$ be a language and $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ be a family of $\mathcal{L}$-structures. Let M_i be the universe of $\mathcal{M}_i$. Fix an ultrafilter I of $\mathbb{N}$. Set $X = \prod_{i\in\mathbb{N}} M_i := \{(x_i)_{i\in\mathbb{N}} \mid x_i \in M_i\}$. We define the equivalence relation $\sim$ on X by $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \Leftrightarrow \{i \in \mathbb{N} \mid x_i = y_i\} \in I$. Set $M = X/\sim$. We denote the equivalence class of $(x_i)_{i\in\mathbb{N}}$ by $[(x_i)]$. We can naturally interpret $\mathcal{L}$ in M and define an $\mathcal{L}$-structure $\mathcal{M}$ whose universe is M. For any $\mathcal{L}$-formula $\phi(\overline{x})$ and any point $([(x_{1i})], \dots, [(x_{ni})]) \in M^n$, the formula $\phi([(x_{1i})], \dots, [(x_{ni})])$ holds true in the structure $\mathcal{M}$ if and only if $\{i \in \mathbb{N} \mid \phi(x_{1i}, \dots, x_{ni})$ holds true in $\mathcal{M}_i\} \in I$. This fact implies that any property described by a first-order logic which holds true in $\mathcal{M}_i$ for all $i \in \mathbb{N}$ is still true in $\mathcal{M}$. The structure $\mathcal{M}$ is the ultraproduct of $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ by I.

Example A.1. Consider the language $\mathcal{L} = \{<, +, \cdot, 0, 1, G\}$. For each $i \in \mathbb{N}$, consider the $\mathcal{L}$-structure $\mathcal{F}_i$ whose universe is the set of reals $\mathbb{R}$. The symbols in $\mathcal{L}$ other than G is interpreted naturally and we interpret G as

$$G = \left\{ \left(\begin{array}{cc} \cos 2(k/i)\pi & -\sin 2(k/i)\pi \\ \sin 2(k/i)\pi & \cos 2(k/i)\pi \end{array} \right) \middle| \; 0 \leq k < i \right\} \subseteq \mathrm{GL}(2, \mathbb{R})$$

in $\mathcal{F}_i$. Remark that G is a finite set and semialgebraic. Therefore, any set definable in $\mathcal{F}_i$ is semialgebraic, and $\mathcal{F}_i$ is o-minimal. Since an o-minimal structure is always locally o-minimal and definably complete, $\mathcal{F}_i$ is locally o-minimal and definably complete.

We next consider the filter $I \subseteq 2^{\mathbb{N}}$ defined by

$$A \in I \Leftrightarrow \exists m \in \mathbb{N},\ \forall n \in \mathbb{N}, (m \leq n) \to n \in A.$$

There exists an ultrafilter J with $I \subseteq J$. Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, G)$ be the ultraproduct of $\{F_i\}_{i\in\mathbb{N}}$ by J. Since $\mathcal{F}_i$ is locally o-minimal and definably complete, $\mathcal{F}$ is also definably complete and locally o-minimal. Since G is bounded, discrete and closed in the structure $\mathcal{F}_i$ for each i, G is bounded, discrete and closed also in $\mathcal{F}$.

Fix an arbitrary $m \in \mathbb{N}$. The sentence "$\exists A \in G$ such that $A^m \neq \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$" holds true in the structure $\mathcal{F}_i$ for $i > m$. We have

$$\left\{ i \in \mathbb{N} \;\middle|\; \exists A \in G \text{ such that } A^m \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \in I \subseteq J.$$

It implies that the sentence "$\exists A \in G$ such that $A^m \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$" holds true in the ultraproduct $\mathcal{F}$. Since m is arbitrary, G is not a finite group in $\mathcal{F}$.

The definable set G is bounded, closed, discrete and infinite group in $\mathcal{F}$. It implies that G is not semialgebraic in $\mathcal{F}$.

B Affiness of topological space definable in a definably complete uniformly locally o-minimal structure of the second kind

B.1 Introduction

In this section, we give a necessary and sufficient condition for a one-dimensional regular and Hausdorff topological space definable in a definably complete uniformly locally o-minimal structure of the second kind. It was announced in [14].

We recall a basic definition.

Definition B.1. A locally o-minimal structure $\mathcal{M} = (M, <, \ldots)$ is a *uniformly locally o-minimal structure of the second kind* if, for any positive integer n, any definable set $X \subset M^{n+1}$, $a \in M$ and $b \in M^n$, there exist an open interval I containing the point a and an open box B containing b such that the definable sets $X_y \cap I$ are finite unions of points and open intervals for all $y \in B$.

Here is the definition of definable topology.

Definition B.2. Consider an expansion of a dense linear order and a definable set X. A topology τ on X is *definable* when τ has a basis of the form $\{B_y \subseteq X\}_{y \in Y}$, where Y is a definable set and $\bigcup_{y \in Y} \{y\} \times B_y$ is definable. We call the family $\{B_y\}_{y \in Y}$ a *definable basis* of τ. The pair (X, τ) of a definable set and a definable topology on it is called a *definable topological space*.

Since X is a subset of a Cartesian product M^n, X has the topology induced from the product topology of M^n. It is definable and called the *affine topology.* The notation τ^{af} denotes the affine topology.

A Definable Complete Uniformly Locally O-minimal expansion of ordered Abelian group of the Second kind is called a DCULOAS structure in short. We are now ready to introduce our main result. Our target in this paper is to demonstrate the following theorem:

Theorem B.3. *Consider a DCULOAS structure $\mathcal{M} = (M; <, +, 0, \ldots)$ such that every automorphism of $(M, +, 0)$ is definable. Let X be a definable bounded subset of M^n of dimension one. Let τ be a definable topology on X which is Hausdorff and regular. Assume further that, for any $0 < u < v$, there exist $0 < u' < u < v < v'$ and a definable increasing τ^{af}-homeomorphism between $[0, u']$ and $[0, v']$.*

The following are equivalent:

(1) The definable topological space (X, τ) is definably homeomorphic to a definable subset of M^k with its affine topology for some k.

(2) There is a definable τ-closed and τ-discrete subset G of X at most of dimension zero satisfying the following conditions:

(i) The restriction of τ to $X \setminus G$ coincides with the affine topology on $X \setminus G$;

(ii) There exists a positive integer K such that, for any $x \in G$ and a definable τ-open neighborhood U of x, we can find a definable τ-open neighborhood V of x contained in U such that $V \setminus \{x\}$ has at most K τ^{af}-definably connected components.

B.2 Preliminary

We prove four lemmas in this section. We crucially use the local definable cell decomposition theorem in proving the second lemma.

Lemma B.4. *Consider a DCULOAS structure $\mathcal{M} = (M; <, +, 0, \ldots)$. Let $\gamma : (0, u] \to M^n$ be a definable continuous map having a bounded image. There exists a unique point $x \in M^n$ such that, for any open box B containing the point x, the intersection $\gamma((0, u]) \cap B$ is not empty. We denote this point by $\lim_{t \to 0} \gamma(t)$.*

In particular, if C is a bounded definable cell of dimension one, the frontier of C consists of two points.

Proof. The image $D = \gamma((0,u])$ is not closed because γ is continuous and $(0,u]$ is not closed. The frontier ∂D of D is not empty. We have only to demonstrate that ∂D is a singleton. Since $\gamma([t,u])$ is closed by [23, Proposition 1.10] for any $0 < t < u$, we have $(*) : \partial D \subseteq \bigcap_{0<t<u} \mathrm{cl}(\gamma((0,t)))$. Assume for contradiction that ∂D contains two points, say x and y. Take $\varepsilon > 0$ so that $2\varepsilon < \max\{|x_i - y_i| \mid 1 \leq i \leq n\}$, where x_i and y_i are the i-th coordinates of x and y, respectively. Let B_x and B_y be the open boxes whose length are 2ε centered at x and y, respectively. There exists $\delta_x > 0$ such that either $(0,\delta_x) \subseteq \gamma^{-1}(B_x)$ or $(0,\delta_x) \cap \gamma^{-1}(B_x) = \emptyset$ by local o-minimality. The latter equality contradicts the inclusion (*). We have $(0,\delta_x) \subseteq \gamma^{-1}(B_x)$. We can also take $\delta_y > 0$ so that $(0,\delta_y) \subseteq \gamma^{-1}(B_y)$. Set $\delta = \min\{\delta_x, \delta_y\}$. The open interval $(0,\delta)$ is contained in $\gamma^{-1}(B_x \cap B_y)$, which contradicts the fact that $B_x \cap B_y = \emptyset$.

When C is a bounded definable cell of dimension one, it is the image of a bounded open interval (a,b) under a definable continuous map $\gamma : (a,b) \to M^n$ having the bounded image. Therefore, the frontier of C consists of two points $\lim_{t\to a} \gamma(t)$ and $\lim_{t\to b} \gamma(t)$. □

Notation B.5. When $\mathcal{M} = (M;+,0,\ldots)$ is an expansion of an abelian group, M^n is naturally an abelian group. For any definable subset C of M^n and a point $a \in M^n$, the notation $a + C$ denotes the set $\{x \in M^n \mid x - a \in C\}$.

Lemma B.6. *Consider a DCULOAS structure $\mathcal{M} = (M;<,+,0,\ldots)$. Let X and Z be definable subsets of M^n with $\dim X = 1$ and $\dim Z = 0$. Let R be a positive element in M. There exist finitely many bounded definable subsets $C_1,\ldots,C_N$ of M^n of dimension one satisfying the following conditions:*

- *(a) For any $z \in Z$ and $1 \leq i \leq N$, the intersection $X \cap (z + C_i)$ either coincides with $z + C_i$ or is an empty set;*
- *(b) There exist $0 < u < R$ and definable continuous injective maps $\gamma_i : (0,u) \to M^n$ such that*
  - *the limits $\lim_{t\to 0} \gamma_i(t)$ are the origin,*
  - *the limits $z + \lim_{t\to u} \gamma_i(t)$ are not in Z for all $z \in Z$ and*
  - *the images $\gamma_i((0,u))$ coincide with C_i for all $1 \leq i \leq N$;*

(c) *The closure of* $X \setminus \left(\{z\} \cup \bigcup_{i=1}^N (z+C_i)\right)$ *intersects with* $z+\mathrm{cl}(C_i)$ *possibly only at the point* $z + \lim_{t \to u} \gamma_i(t)$ *for any* $z \in Z$.

Proof. We first construct a bounded open box B with $Z \cap \mathrm{cl}(z+B) = \{z\}$ for any $z \in Z$. Let $p_i : M^n \to M$ be the projection onto the i-th coordinate for every $1 \leq i \leq n$. The image $p_i(Z)$ is closed and discrete by [10, Theorem 1.1] and [12, Lemma 2.3]. We assume that $p_i(Z)$ has at least three points for every $1 \leq i \leq n$. We can construct B in the same manner in the other cases. Define the definable functions $g_i : p_i(Z) \setminus \{\sup p_i(Z)\} \to M$ and $h_i : p_i(Z) \setminus \{\inf p_i(Z)\} \to M$ by

$$g_i(z) = \inf\{x \in Z \mid x > z\} - z \text{ and}$$
$$h_i(z) = z - \sup\{x \in Z \mid x < z\}.$$

Here, we set $p_i(Z) \setminus \{\sup p_i(Z)\} = p_i(Z)$ when $\sup(Z) = \infty$. We define $p_i(Z) \setminus \{\inf p_i(Z)\}$ in the same manner. The images of g_i and h_i are of dimension zero by [10, Theorem 1.1]. They are discrete and closed by [12, Lemma 2.3]. We can take a positive $d_i \in M$ such that $d_i < \inf g_i(p_i(Z) \setminus \{\sup p_i(Z)\})$ and $d_i < \inf h_i(p_i(Z) \setminus \{\inf p_i(Z)\})$. Set $B = \prod_{i=1}^N (-d_i, d_i)$. We obviously have $Z \cap \mathrm{cl}(z+B) = \{z\}$ for any $z \in Z$.

Consider the definable set $Y = \bigcup_{z \in Z} \{z\} \times ((-z)+X)$. It is of dimension one by [12, Proposition 2.13, Theorem 3.14]. Let $\pi : M^{2n} \to M^n$ be the coordinate projection forgetting the first n coordinates. We have $\dim \pi(Y) = 1$ by [9, Lemma 5.1] and [10, Theorem 1.1] because $\pi(Y)$ contains the one-dimensional definable set $(-z)+X$.

Shrink the open box B if necessary. Then, there exists a definable cell decomposition $\mathcal{D} = \{D_i\}_{i=1}^L$ of B partitioning the singleton consisting of the origin and $\pi(Y) \cap B$ by [9, Theorem 4.2]. Let $\mathcal{E}$ be the family of the cells in $\mathcal{D}$ of dimension one contained in $\pi(Y)$ whose closure contains the origin. Let $E_1, \ldots, E_N$ be the enumeration of the elements in $\mathcal{E}$. For all $1 \leq i \leq N$, we can take definable homeomorphisms $\gamma_i : I_i \to E_i$, where I_i is bounded open intervals, for all $1 \leq i \leq N$ by the definition of cells. We may assume that $I_i = (0, r_i)$ for some positive r_i and the limit $\lim_{t \to 0} \gamma_i(t)$ is the origin without loss of generality. The number of cells are finite and there are only finitely many points contained in the frontier of 1-dimensional cell by Lemma B.4. Taking smaller r_i if necessary, we may assume that $\mathrm{cl}(D) \cap \gamma_i((0, r_i)) = \emptyset$ for any cell $D \in \mathcal{D}$ of dimension one with $D \neq E_i$. Furthermore, we may assume that $r_i < R$.

Fix $1 \leq i \leq N$. Set $Z_i = \{z \in Z \mid X \cap (z+E_i) \neq \emptyset\}$. It is a definable set of dimension zero. For any $z \in Z_i$, one of the following condition is satisfied because of local o-minimality.

- There exists a positive $r < r_i$ such that $z + \gamma_i(t) \in X$ for all $0 < t < r$;
- There exists a positive $r < r_i$ such that $z + \gamma_i(t) \notin X$ for all $0 < t < r$.

Let Z_{i1} be the set of points in Z_i satisfying the former condition. The set of points in Z_i satisfying the latter condition is denoted by Z_{i2}. They are obviously definable subsets of Z_i at most of dimension zero. In this proof, we construct C_i only when both Z_{i1} and Z_{i2} are not empty. We can construct C_i similarly in the other cases. We define definable maps $f_{i1} : Z_{i1} \to M$ and $f_{i2} : Z_{i2} \to M$ by

$$f_{i1}(z) = \sup\{t \in M \mid 0 < t < r_i \text{ and } z + \gamma_i(s) \in X \text{ for all } 0 < s < t\} \text{ and}$$
$$f_{i2}(z) = \sup\{t \in M \mid 0 < t < r_i \text{ and } z + \gamma_i(s) \notin X \text{ for all } 0 < s < t\}.$$

The images $f_{ij}(Z_{ij})$ are of dimension zero by [10, Theorem 1.1] for $j = 1, 2$. They are closed and discrete by [12, Lemma 2.3]. The infimums of $f_{ij}(Z_{ij})$ are elements in $f_{ij}(Z_{ij})$, and they are positive. Set $u = \min\{\inf f_{ij}(Z_{ij}) \mid 1 \leq i \leq N, 1 \leq j \leq 2\}$. We put $C_i = \gamma_i((0, u))$. Three conditions (a) through (c) in the lemma are obviously satisfied. □

Lemma B.7. *Consider a DCULOAS structure $\mathcal{M} = (M; <, +, 0, \ldots)$. Let Z and U be definable subsets of M^n such that $\dim Z = 0$. Let $\gamma : (0, u) \to M^n$ be a definable map such that, for any $z \in Z$, there exists $t > 0$ satisfying the inclusion $z + \gamma((0, t)) \subseteq U$. There exists a definable map $r : Z \to (0, \infty)$ such that $z + \gamma((0, r(z))) \subseteq U$ for all $z \in Z$.*

Proof. Consider the definable function $r : Z \to (0, \infty)$ given by $r(z) = \sup\{t > 0 \mid z + \gamma((0, t)) \subseteq U\}$. It satisfies the requirement of the lemma. □

B.3 Proof of main theorem

We begin to demonstrate the main theorem. Consider a DCULOAS structure $\mathcal{M} = (M; <, +, 0, \ldots)$. Let X be a definable subset of M^n. For any definable subset U of X, the notation $\text{cl}^{\text{af}}(U)$ means the closure of U in M^n under the affine topology. Consider a definable topology τ on X. The notation $\mathcal{B}_a$ denotes the definable basis of neighborhoods of $a \in X$.

Definition B.8. We define the *set of shadows* of a point $a \in X$ to be

$$\mathbf{S}_\tau(a) := \bigcap_{U \in \mathcal{B}_a} \mathrm{cl}^{\mathrm{af}}(U).$$

The set $\mathbf{S}_\tau(a)$ is a definable closed subset of M^n. We call a point in $\mathbf{S}_\tau(a)$ a *shadow* of a. We simply write $\mathbf{S}(a)$ when the definable topology τ is clear from the context.

Proof of Theorem B.3. We first demonstrate that the condition (1) implies the condition (2). Let $f : (X, \tau) \to (Y, \tau_Y^{\mathrm{af}})$ be a definable homeomorphism, where Y is a definable subset of M^k. Here, the notation τ_Y^{af} denotes the affine topology on Y. Consider the affine topology τ_X^{af} on X. Let $f' : X \to Y$ be a definable map defined by $f'(x) = f(x)$. Let G be the set of points at which f' is discontinuous with respect to τ_X^{af}. We have $\dim G < \dim X = 1$ by [10, Corollary 1.2]. We get $\dim f(G) \leq 0$ by [10, Theorem 1.1]. The set $f(G)$ is τ^{af}-closed and τ^{af}-discrete by [12, Lemma 2.3]. The set G is τ-closed and τ-discrete because f is a homeomorphism.

The restriction $f'|_{X \setminus G}$ of f' to $X \setminus G$ induces a homeomorphism between $(X \setminus G, \tau_{X \setminus G}^{\mathrm{af}})$ and $(Y \setminus f(G), \tau_{Y \setminus f(G)}^{\mathrm{af}})$. We call it g. The composition $h = g^{-1} \circ f|_{X \setminus G} : (X \setminus G, \tau|_{X \setminus G}) \to (X \setminus G, \tau_{X \setminus G}^{\mathrm{af}})$ is a definable homeomorphism given by $h(x) = x$. Here, the notation $\tau|_{X \setminus G}$ denotes the topology on $X \setminus G$ induced from τ. In particular, it implies the condition (2)(i).

Apply Lemma B.6 to Y and $f(G)$. There are finitely many definable subsets $D_1, \ldots, D_K$ of M^k such that the conditions (a) through (c) in Lemma B.6 are satisfied. Let $\eta_i : (0, u') \to M^n$ be the definable continuous maps given in the condition (b) for all $1 \leq i \leq K$. Take $x \in G$. Set $z = f(x)$. Permuting the sequence $D_1, \ldots, D_K$ if necessary, we may assume that $Y \cap (z + D_i) = z + D_i$ for $1 \leq i \leq L$ and $Y \cap (z + D_i) = \emptyset$ for $L < i \leq K$ by the condition (a). For any definable τ-open neighborhood U of x, the image $U' = f(U)$ is a τ^{af}-open neighborhood of z. We can take a τ^{af}-open neighborhood V' of z contained in U' such that V' does not intersect with $Y \setminus (\{z\} \cup \bigcup_{i=1}^K z + D_i)$ and $\eta_i^{-1}(V')$ is of the form $(0, r)$, where r is a sufficiently small positive element. The inverse image $V = f^{-1}(V')$ is a τ-open neighborhood of x. The τ^{af}-definably connected components of $V' \setminus \{z\}$ are of the form $\eta_i((0, r))$. Their inverse images $f^{-1}(\eta_i((0, r)))$ are also τ^{af}-definably connected by the condition (2)(i). Therefore, $V \setminus \{x\}$ has at most K τ^{af}-definably connected components. We have demonstrated that the condition (2)(ii) is satisfied.

We next prove the opposite implication. There is nothing to prove when G is an empty set. So, we assume that G is not empty. The frontier of ∂X is at most of dimension zero by [9, Theorem 5.6] and it is discrete and closed by [12, Proposition 2.13, Proposition 3.2]. Set $Z = \partial X \cup G$. Note that Z is discrete and closed in the affine topology. We demonstrate two claims.

Claim 1. $\mathbf{S}(x) \subseteq Z$ for all $x \in G$.

Take a point $y \notin Z$. Since τ is Hausdorff, we take a τ-open definable subset U of X such that $x \in U$ and $y \notin U$. Since τ is regular, we get a τ-open definable subset V of X such that $x \in V \subseteq \mathrm{cl}^\tau(V) \subseteq U$. In particular, we have $y \notin \mathrm{cl}^\tau(V)$. By the assumption, we get $\mathrm{cl}^\tau(V) \cap (X \setminus G) = \mathrm{cl}^{\mathrm{af}}(V) \cap (X \setminus G)$. In particular, $y \notin \mathrm{cl}^{\mathrm{af}}(V)$. It implies that $y \notin \mathbf{S}(x)$. The proof of Claim 1 has finished.

Claim 2. Let A be a bounded definable subset such that $\mathrm{cl}^{\mathrm{af}}(A) \subseteq X \setminus G$. We have $\mathrm{cl}^{\mathrm{af}}(A) = \mathrm{cl}^\tau(A)$.

The condition (2)(i) implies the inclusion $\mathrm{cl}^{\mathrm{af}}(A) \subseteq \mathrm{cl}^\tau(A) \subseteq \mathrm{cl}^{\mathrm{af}}(A) \cup G$. We have only to demonstrate that $\mathrm{cl}^\tau(A) \cap G$ is an empty set. Assume for contradiction that we can take a point $x \in \mathrm{cl}^\tau(A) \cap G$. Consider the definable family $\{\mathrm{cl}^{\mathrm{af}}(A) \cap \mathrm{cl}^{\mathrm{af}}(U)\}_{U \in \mathcal{B}_x}$ of nonempty definable τ^{af}-closed sets. This family is obviously a definable filtered collection, which is defined in [18, Definition 5.5]. Since $\mathrm{cl}^{\mathrm{af}}(A)$ is definably compact by [18, Remark 5.6], the intersection $\bigcap_{U \in \mathcal{B}_x} (\mathrm{cl}^{\mathrm{af}}(A) \cap \mathrm{cl}^{\mathrm{af}}(U))$ is not an empty set. We take a point y in this intersection. The relation $y \in \mathbf{S}(x)$ immediately follows from the definition of the set of shadow points. It contradicts Claim 1 because $y \in \mathrm{cl}^{\mathrm{af}}(A) \subseteq X \setminus G$. We have demonstrated Claim 2.

We have $\dim Z = 0$ by [9, Corollary 5.4(ii), Theorem 5.6]. Let M^n be the ambient space of X. Shifting X if necessary, we may assume that $\mathrm{cl}^{\mathrm{af}}(X)$ is contained in $(0, R)^n$ for some $R > 0$. Applying Lemma B.6 to X and Z, we obtain an open box B in M^n containing the origin and finitely many bounded definable subsets $C_1, \dots, C_N$ of B of dimension one satisfying the following conditions:

(a) For any $z \in Z$ and $1 \leq i \leq N$, the intersection $X \cap (z + C_i)$ either coincides with $z + C_i$ or is an empty set;
(b) There exist $0 < u < R$ and definable continuous injective maps $\gamma_i : (0, u) \to M^n$ such that

- the limits $\lim_{t\to 0}\gamma_i(t)$ are the origin,
- the limits $z+\lim_{t\to u}\gamma_i(t)$ are not in Z for all $z\in Z$ and
- the images $\gamma_i((0,u))$ coincide with C_i for all $1\leq i\leq N$;

(c) The closure of $X\setminus\left(\{z\}\cup\bigcup_{i=1}^N(z+C_i)\right)$ intersects with $z+\mathrm{cl}^{\mathrm{af}}(C_i)$ possibly only at the point $z+\lim_{t\to u}\gamma_i(t)$ for any $z\in Z$.

By the assumption, there exists a definable increasing τ^{af}-homeomorphism $\Phi:[0,u]\to[0,3nR]$ taking a smaller u and a larger R if necessary.

The notation $C_{i,z}$ denotes the set $z+C_i$ for every $1\leq i\leq N$ and $z\in Z$. We say that $C_{i,z}$ is *inclusive* if $X\cap C_{i,z}=C_{i,z}$, and we call it *exclusive* if $X\cap C_{i,z}=\emptyset$. The definable set $C_{i,z}$ is either inclusive or exclusive by the condition (a). By the condition (b) and Lemma B.4, the frontier $\partial C_{i,z}$ of $C_{i,z}$ under the affine topology consists of two points. One is the point z. Another point is called the *non-trivial endpoint* of $C_{i,z}$. The map

$$p_i:Z\to M^n$$

is the definable map so that $p_i(z)$ is the non-trivial endpoint of $C_{i,z}$ for all $1\leq i\leq N$. Note that the map p_i is injective. By the condition (b), the non-trivial endpoint of inclusive $C_{i,z}$ lies in $X\setminus G$.

Consider an inclusive $C_{i,z}$. The notation $\partial^\tau C_{i,z}$ denotes the τ-frontier $\mathrm{cl}^\tau(C_{i,z})\setminus C_{i,z}$. We investigate the τ-frontier $\partial^\tau C_{i,z}$ of $C_{i,z}$.

Claim 3. The intersection $(\partial^\tau C_{i,z})\cap G$ has at most one point.

Consider the case in which $(\partial^\tau C_{i,z})\cap G$ is not empty. Take a point x in this set. We have only to demonstrate that $y\notin\mathrm{cl}^\tau C_{i,z}$ for $y\in G$ with $y\neq x$. Since τ is Hausdorff, we can take a τ-open definable subset U of X with $x\in U$ and $y\notin U$. Since τ is regular, we can take a τ-open definable subset V of X such that $x\in V\subseteq\mathrm{cl}^\tau(V)\subseteq U$.

By local o-minimality, we have either $z+\gamma_i((0,r))\subseteq V$ or $(z+\gamma_i((0,r)))\cap V=\emptyset$ for any sufficiently small $r>0$. If the latter holds true, we have $x\in\mathrm{cl}^\tau(z+\gamma_i([r,u)))$. On the other hand, we get $\mathrm{cl}^\tau(z+\gamma_i([r,u)))=\mathrm{cl}^{\mathrm{af}}(z+\gamma_i([r,u)))$ by Claim 2. It implies $x\in\mathrm{cl}^{\mathrm{af}}(z+\gamma_i([r,u)))$, which is a contradiction to the condition (b).

Take $r>0$ so that $z+\gamma_i((0,r))\subseteq V$. We finally get $\mathrm{cl}^\tau(C_{i,z})=\mathrm{cl}^\tau(z+\gamma_i((0,r)))\cup\mathrm{cl}^\tau(z+\gamma_i([r,u)))\subseteq\mathrm{cl}^\tau(V)\cup\mathrm{cl}^{\mathrm{af}}(z+\gamma_i([r,u)))$ using Claim 2 again. It implies that $y\notin\mathrm{cl}^\tau C_{i,z}$ because $y\in G$ and $G\cap\mathrm{cl}^{\mathrm{af}}(z+\gamma_i([r,u)))=\emptyset$. The proof of Claim 3 has been completed.

Thanks to Claim 3, $(\partial^\tau C_{i,z}) \cap G$ has a unique point when it is not empty. This unique point is called the *τ-connection point* of $C_{i,z}$. We set

$$\begin{aligned} \mathcal{Z}_{\text{no},i} &= \{z \in Z \mid C_{i,z} \text{ is inclusive and does not have its } \tau\text{-connection point}\}, \\ \mathcal{Z}_{\text{self},i} &= \{z \in Z \mid C_{i,z} \text{ is inclusive, and it has its } \tau\text{-connection point} \\ &\qquad \text{and its } \tau\text{-connection point} = z\} \\ \mathcal{Z}_i &= \{z \in Z \mid C_{i,z} \text{ is inclusive, and it has its } \tau\text{-connection point} \\ &\qquad \text{and its } \tau\text{-connection point} \neq z\} \end{aligned}$$

for $1 \leq i \leq N$. The sets $\mathcal{Z}_{\text{no},i}$, $\mathcal{Z}_{\text{self},i}$ and $\mathcal{Z}_i$ are definable. We also define the definable maps

$$\zeta_i : \mathcal{Z}_i \cup \mathcal{Z}_{\text{self},i} \to G$$

so that $\zeta_i(z)$ is the τ-connection point of $C_{i,z}$.

Fix $1 \leq i \leq N$ and $z \in \mathcal{Z}_i$. Let $\overline{\gamma}_{i,z} : [0, u] \to M^n$ be the τ^{af}-continuous extension of the definable τ^{af}-continuous curve $t \mapsto z + \gamma_i(t)$ for all $1 \leq i \leq N$ and $z \in \mathcal{Z}_i$. Such extension exists by Lemma B.4. On the other hand, for any two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we set

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

We set $d'_{i,z}(t) = d(\zeta_i(z), \overline{\gamma}_{i,z}(t)) + 2d(\overline{0}_n, \overline{\gamma}_{i,z}(t))$ for simplicity, where $\overline{0}_n$ denotes the origin of M^n. The function $d'_{i,z}$ is definable and τ^{af}-continuous. We get $\Phi(0) = 0 < d'_{i,z}(0) = d(\zeta_i(z), z) + 2d(\overline{0}_n, z)$ and $\Phi(u) = 3nR > d(\zeta_i(z), p_i(z)) + 2d(\overline{0}_n, p_i(z)) = d'_{i,z}(u)$. By the intermediate value theorem, we obtain $\Phi(v) = d'_{i,z}(v)$ for some $0 < v < u$. Therefore, the map $v_i : \mathcal{Z}_i \to (0, u)$ given by

$$v_i(z) = \inf\{0 < t < u \mid \Phi(t) = d'_{i,z}(t)\}$$

is a well-defined definable function. The definable map $q_i : \mathcal{Z}_i \to \bigcup_{z \in \mathcal{Z}_i} C_{i,z}$ is given by $q_i(z) = z + \gamma_i(v_i(z))$. We get $q_i(z) \in C_{i,z}$ and

$$d(\zeta_i(z), q_i(z)) + 2d(\overline{0}_n, q_i(z)) = \Phi(v_i(z)) \tag{1}$$

because both Φ and $d'_{i,z}$ are continuous. We put

$$C'_{i,z} = z + \gamma((0, v_i(z))).$$

We also need the following claim.

Claim 4. The inverse images $\zeta_i^{-1}(x)$ have at most K points for all $1 \leq i \leq N$ and $x \in G$.

Assume for contradiction that $\zeta_i^{-1}(x)$ has $K+1$ points $z_1, \ldots, z_{K+1}$. Consider the definable τ^{af}-closed subsets $z_j + \gamma_i([u/3, 2u/3])$ for $1 \leq j \leq K+1$. It is also τ-closed by Claim 2. There exists a definable τ-open neighborhood U of x in X such that $U \cap (z_j + \gamma_i([u/3, 2u/3])) = \emptyset$ for all $0 \leq j \leq K+1$ and $U \setminus \{x\}$ has at most K τ^{af}-definably connected components by the condition (2)(ii). On the other hand, the set $z_j + \gamma_i((0, r_j))$ is a τ^{af}-definably connected component of $U \setminus \{x\}$ for some sufficiently small r_j for any $1 \leq j \leq K+1$ because of the definition of τ-connection points. Contradiction. We have proved Claim 4.

The inverse image $\zeta_i^{-1}(x)$ consists of at most K points by Claim 4. We set

$$\begin{aligned} \mathcal{Z}_{i,k} = \{z \in \mathcal{Z}_i \mid\ & z \text{ is the } k\text{-th smallest element in } \zeta_i^{-1}(\zeta_i(z)) \\ & \text{in the lexicographic order}\} \end{aligned}$$

for $1 \leq i \leq N$ and $1 \leq k \leq K$. It is also definable. We have $\mathcal{Z}_i = \bigcup_{k=1}^K \mathcal{Z}_{i,k}$. We also put

$$\mathcal{C}_{\mathrm{no},i} = \bigcup_{z \in \mathcal{Z}_{\mathrm{no},i}} C_{i,z}, \qquad \mathcal{C}_{\mathrm{self},i} = \bigcup_{z \in \mathcal{Z}_{\mathrm{self},i}} C_{i,z}, \qquad \mathcal{C}'_{i,k} = \bigcup_{z \in \mathcal{Z}_{i,k}} C'_{i,z} \text{ and}$$
$$\mathcal{C}_i = \mathcal{C}_{\mathrm{no},i} \cup \bigcup_{k=1}^K \mathcal{C}'_{i,k}.$$

We can define the definable map $\rho_i : \mathcal{C}_i \to Z$ so that $\rho_i(x) = z$ if and only if $x \in C_{i,z}$.

We need an extra preparation. For given points $p, q \in M^n$, the *standard connection between p and q* is the union of n segments connecting the $n+1$ points $p = v_0, v_1, \ldots, v_n = q$ in the given order, where v_j is the point whose last j coordinates equal the last j coordinates of q and whose first $n-j$ coordinates are the first $n-j$ coordinates of p. We denote it by $\mathrm{Std}(p,q)$. We can easily construct a definable map $\Psi : \{(t,p,q) \in M \times M^n \times M^n \mid 0 \leq t \leq d(p,q)\} \to M^n$ such that, for fixed $p, q \in M^n$, the restriction $\Psi(\cdot, p, q) : [0, d(p,q)] \to M^n$ has the image $\mathrm{Std}(p,q)$ and this restriction is a τ^{af}-homeomorphism onto its image.

We have now finished long preparation. We construct a definable subset Y of $M^{n(1+KN)}$ and a definable homeomorphism $f:(X,\tau)\to(Y,\tau^{\mathrm{af}})$. The notation $\overline{0}_m$ denotes the origin of M^m for any positive integer m.

We define that Y is the union of the following subsets of $M^{n(1+KN)}$:

- $(X\setminus\bigcup_{i=1}^N\mathcal{C}_i)\times\{\overline{0}_{nKN}\}$;
- For any $1\leq i\leq N$ and $z\in Z$ such that $C_{i,z}$ is inclusive,
  - $\{p_i(z)\}\times\{\overline{0}_{(i-1)nK}\}\times(0,u)\times\{\overline{0}_{nK(N-i+1)+n-1}\}$ for $z\in\mathcal{Z}_{\mathrm{no},i}$;
  - when $z\in\mathcal{Z}_{i,k}$, the union of the following three definable sets:
    - $\{\zeta_i(z)\}\times\{\overline{0}_{m_1}\}\times\mathrm{Std}(\overline{0}_n,q_i(z))\times\{\overline{0}_{m_2}\}$,
    - $\mathrm{Std}(\zeta_i(z),q_i(z))\times\{\overline{0}_{m_1}\}\times\{q_i(z)\}\times\{\overline{0}_{m_2}\}$ and
    - $\{q_i(z)\}\times\{\overline{0}_{m_1}\}\times\mathrm{Std}(q_i(z),\overline{0}_n)\times\{\overline{0}_{m_2}\}$,

    where $m_1=n((i-1)K+(k-1))$ and $m_2=n(KN-iK+K-k)$.

Roughly speaking, when $z\in\mathcal{Z}_i$, we connect the point $q_i(z)$ and the τ-connection point $\zeta_i(z)$ with a curve for any $1\leq i\leq N$ and $z\in Z$. Since at most KN curves whose τ^{af}-closures do not contain the point z gather at any point z in G under the topology τ, we can connect them so that two curves do not intersect each other. We can construct a first order formula defining the set Y using q_i, ζ_i and C_i. It implies that Y is definable.

We next construct the definable homeomorphism $f:(X,\tau)\to(Y,\tau^{\mathrm{af}})$. We define $f(x)$ as follows:

- $f(x)=(x,\overline{0}_{nKN})$ when x is in $X\setminus\bigcup_{i=1}^N\mathcal{C}_i$;
- When x is contained in $\mathcal{C}_i$, we define as follows:
  - When $x\in\mathcal{C}_{\mathrm{no},i}$, we set $f(x)=(p_i(\rho_i(x)),\overline{0}_{m_1'},u-\gamma_i^{-1}(x-\rho_i(x)),\overline{0}_{m_2'})$, where $m_1'=(i-1)nK$ and $m_2'=nK(N-i+1)+n-1$.
  - We consider the case in which $x\in\mathcal{C}_{i,k}$, set $z=\rho_{i,k}(x)$, $t=\gamma_i^{-1}(x-z)$, $\mathfrak{q}=q_i(z)$ and $\zeta=\zeta_i(z)$. We define as follows:
    - $f(x)=(\zeta,\overline{0}_{m_1},\Psi(t,\overline{0}_n,\mathfrak{q}),\overline{0}_{m_2})$ when $0<t\leq d(\overline{0}_n,\mathfrak{q})$;
    - $f(x)=(\Psi(t-d(\overline{0}_n,\mathfrak{q}),\zeta,\mathfrak{p}),\overline{0}_{m_1},\mathfrak{q},\overline{0}_{m_2})$ when $d(\overline{0}_n,\mathfrak{q})<t\leq d(\overline{0}_n,\mathfrak{q})+d(\zeta,\mathfrak{q})$;
    - $f(x)=(\mathfrak{q},\overline{0}_{m_1},\Psi(t-d(\overline{0}_n,\mathfrak{q})-d(\zeta,\mathfrak{q}),\mathfrak{q},0_n),\overline{0}_{m_2})$ when $d(\overline{0}_n,\mathfrak{q})+d(\zeta,\mathfrak{q})<t<\Phi(v_i(z))$,

where $m_1 = n((i-1)K + (k-1))$ and $m_2 = n(KN - iK + K - k)$.

The map f defined above is a definable map. It is not difficult to prove that f is a bijection. The proof is left to the readers. The remaining task is to show that f is a homeomorphism. Note that the restriction of f to $X \setminus G$ induces a definable homeomorphism between $(X \setminus G, \tau_{\text{af}})$ and $(Y \setminus f(G), \tau_{\text{af}})$ by the equality (1).

We prove that f is an open map. Take a τ-open definable set U of X. Set $V = f(U)$. Take a point $y \in V$ and set $x = f^{-1}(y)$. We construct a definable open neighborhood W of y contained in V. The case in which $x \notin G$ is easy. The set $U \setminus G$ is τ^{af}-open by the assumption. Take a τ^{af}-open neighborhood U_1 of x in X contained in $U \setminus G$. It is also τ-open because G is τ-closed. Since the restriction of f to $X \setminus G$ induces a definable homeomorphism, the image $W = f(U_1)$ is an τ^{af}-open neighborhood of y.

When $x \in G$, take all $C_{i,z}$ whose τ-connection point is x. We can take finitely many such sets by Claim 4. Set $\mathcal{D}(x) = \{(i,z) \in \mathbb{Z} \times Z \mid 1 \leq i \leq N,\ z \in \mathcal{Z}_i \cup \mathcal{Z}_{\text{self},i},\ \zeta_i(z) = x\}$. For any $(i,z) \in \mathcal{D}(x)$, there exists $u_{i,z} > 0$ such that $z + \gamma_i((0, u_{i,z}))$ is contained in $U \setminus G$ by the definition of τ-connection points. We can choose $u_{i,z}$ so that the map $z \mapsto u_{i,z}$ is definable by Lemma B.7. By the definition of Y and f,

$$W = \{y\} \cup \bigcup_{(i,z) \in \mathcal{D}(x)} f(\gamma_i((0, u_{i,z})))$$

is a definable τ^{af}-open subset of V containing the point y.

We next prove that the inverse f^{-1} of f is an open map. Take a τ^{af}-open definable set V of Y. Set $U = f^{-1}(V)$. Take a point $x \in U$. We construct a definable open neighborhood U_1 of x contained in U. The case in which $x \notin G$ is easy. We omit the proof.

We consider the case in which $x \in G$. Since G is τ-closed and τ-discrete, the set $G \setminus \{x\}$ is τ-closed. We can take disjoint definable τ-open subsets U_1 and U_2 of X such that $x \in U_1$ and $G \setminus \{x\} \subseteq U_2$ because τ is regular. We show that $(z + \gamma_i((0,t))) \cap U_1 = \emptyset$ for all sufficiently small $t > 0$ when $1 \leq i \leq N$, $z \in \mathcal{Z}_i$ and $\zeta_i(z) \neq x$. Assume the contrary. There exists $1 \leq i \leq N$ and $z \in \mathcal{Z}_i$ such that $\zeta_i(z) \neq x$ and $z + \gamma_i((0,t)) \subseteq U_1$ for some $t > 0$ by local o-minimality. The set $z + \gamma_i((0,t))$ has a nonempty intersection with any τ-neighborhood of $\zeta_i(z)$ by the definition of a τ-connection point. In particular, we have $U_1 \cap U_2 \neq \emptyset$, which is a contradiction.

We next consider the intersection of U_1 with $C_{i,z}$ for $z \in \mathcal{Z}_{\text{no},i}$. Shrinking U_1 if necessary, we may assume that U_1 has at most K τ^{af}-definably connected components by the condition (2)(ii). In particular, we have only finitely many $z \in \mathcal{Z}_{\text{no},i}$ such that U_1 contains $z + \gamma_i((0,t))$ for some $t > 0$. Since $z + \gamma_i((0,t/2])$ is τ-closed by Claim 3, we may assume that, for any $z \in \mathcal{Z}_{\text{no},i}$, $(z + \gamma_i((0,s))) \cap U_1 = \emptyset$ for all sufficiently small $s > 0$ by removing $z + \gamma_i((0,t/2])$ from U_1 if necessary.

We set $\mathcal{Z}_{\text{in},i}(x) = \{z \in \mathcal{Z}_i \cup \mathcal{Z}_{\text{self},i} \mid \zeta_i(z) = x\}$ and $\mathcal{Z}_{\text{out},i}(x) = \mathcal{Z}_{\text{no},i} \cup \{z \in \mathcal{Z}_i \cup \mathcal{Z}_{\text{self},i} \mid \zeta_i(z) \neq x\}$. We also consider the definable map $\xi_i : \mathcal{Z}_{\text{out},i}(x) \to M$ given by

$$\xi_i(z) = \sup\{0 < t < u \mid (z + \gamma_i((0,t))) \cap U = \emptyset\}.$$

Consider the definable set

$$\Pi = X \setminus \left(D \cup \bigcup_{i=1}^{N} \left(\bigcup_{z \in \mathcal{Z}_{\text{out},i}(x)} (z + \gamma((0, \xi_i(z)))) \cup \bigcup_{z \in \mathcal{Z}_{\text{in},i}(x)} C_{i,z} \right) \right).$$

It is a definable τ^{af}-closed subset of $X \setminus D$. It is also τ-closed by Claim 2. Removing Π from U_1, we may assume that U_1 is contained in $D \cup \bigcup_{i=1}^N \bigcup_{z \in \mathcal{Z}_{\text{in},i}(x)} C_{i,z}$. As we demonstrated previously, $D \setminus \{x\}$ is τ-closed. Removing it from U_1, we may assume that U_1 is contained in $\{x\} \cup \bigcup_{i=1}^N \bigcup_{z \in \mathcal{Z}_{\text{in},i}(x)} C_{i,z}$.

Take a sufficiently small $r_{i,z} > 0$ so that $z + \gamma_i((0, r_{i,z})) \subseteq U$ for any $1 \leq i \leq N$ and $z \in \mathcal{Z}_{\text{in},i}(x)$. We can choose $r_{i,z}$ so that the map $z \mapsto r_{i,z}$ is definable by Lemma B.7. The definable set $\bigcup_{i=1}^N \bigcup_{z \in \mathcal{Z}_{\text{in},i}(x)} (z + \gamma([r_{i,z}, u)) \cup p_i(z))$ is τ^{af}-closed. It is also τ-closed by Claim 2. Removing this set from U_1, we may assume that U_1 is contained in U. We have finally constructed a definable τ-open neighborhood U_1 of x contained in U. □

Remark B.9. Consider the following condition:

- There exists a positive integer K such that $|\{y \in X \mid x \in \mathbf{S}(y)\}| \leq K$ for all $x \in G$.

The condition (2)(ii) implies this condition as demonstrated in Claim 4, but the converse is not true at least when X is not bounded. Consider the locally o-minimal structure $(\mathbb{R}; <, +, 0, \mathbb{Z})$ given in [21, Example 20]. Consider the definable topology τ on $\mathbb{R}$ whose open basis $\mathcal{B}$ is given below: For any $x \neq 0$, we set $\mathcal{B}_x = \{(x - r, x +$

$r) \mid 0 < r \in \mathbb{R}\}$. We also set $\mathcal{B}_0 = \{(-r,r) \cup (\mathbb{Z}_{>n}) \mid 0 < r \in \mathbb{R},\ n \in \mathbb{Z}\}$, where $\mathbb{Z}_{>n} := \{x \in \mathbb{Z} \mid x > n\}$. We have $\mathbf{S}(x) = \{x\}$ for all $x \in \mathbb{R}$, and the above condition is satisfied. However, the condition (2)(ii) fails when $x = 0$ because any definable τ-open neighborhood of the origin is of the form $(-r,r) \cup (\mathbb{Z}_{>n})$, and it has infinitely many τ^{af}-definably connected components of the form $\{m\}$ with $m \in \mathbb{Z}$.

B.4 Extension of Peterzil and Rosel's result to non-bounded case

Y. Peterzil and A. Rosel gave necessary and sufficient conditions for a one-dimensional topological space with a topology definable in an o-minimal structure to be affine in [25] when the definable set in consideration is bounded. We extend this result to the non-bounded case in this appendix. We first review their main theorem.

Theorem B.10 ([25, Main theorem]). *Let $\mathcal{M} = (M;<,0,+,\dots)$ be an o-minimal expansion of an ordered group. Assume that arbitrary two closed bounded intervals are definably homeomorphic. Let $X \subseteq M^n$ be a definable bounded set with $\dim X = 1$, and let τ be a definable Hausdorff topology on X. Then the following are equivalent:*

(1) (X,τ) is definably homeomorphic to a definable subset of M^k for some k, with its affine topology.

(2) There is a finite set $G \subseteq X$ such that every τ-open subset of $X \setminus G$ is open with respect to the affine topology on $X \setminus G$.

(3) Every definable subset of X has finitely many definably connected components, with respect to τ.

(4) τ is regular and X has finitely many definably connected components with respect to τ.

The assumption that X is bounded could be omitted when there exists a definable bijection between a bounded interval and an unbounded interval. The structure not satisfying the above condition is investigated in [6]. It is called a *semi-bounded* o-minimal structure. Theorem B.10 is not true if we omit the assumption that X is bounded as in the example in [25, Section 4.3]. In the non-bounded case, we get the following proposition:

Proposition B.11. *Let $\mathcal{M} = (M;< .0,+,\dots)$ be a semi-bounded o-minimal expan-*

sion of an ordered group. Assume that arbitrary two closed bounded intervals are definably homeomorphic. Let $X \subseteq M^n$ be a definable set with $\dim X = 1$, and let τ be a definable Hausdorff topology on X. Then the following are equivalent:

(1) (X, τ) is definably homeomorphic to a definable subset of M^k for some k, with its affine topology.

(2) There is a finite set $G \subseteq X$ such that the restriction of τ to $X \setminus G$ coincides with the affine topology on $X \setminus G$.

Proof. When X is bounded, the proposition follows from Theorem B.10. Therefore, we only treat the case in which X is not bounded.

We first demonstrate that the condition (1) implies the condition (2). Let $f : (X, \tau) \to (Y, \tau_Y^{\mathrm{af}})$ be a definable homeomorphism, where Y is a definable subset of M^k. Here, the notation τ_Y^{af} denotes the affine topology on Y. Consider the affine topology τ_X^{af} on X. Let $f' : (X, \tau_X^{\mathrm{af}}) \to (Y, \tau_Y^{\mathrm{af}})$ be a definable map defined by $f(x) = f'(x)$. Let G be the set of points at which f' is discontinuous with respect to τ_X^{af}. It is well-known that $\dim G < \dim X = 1$, so G is a finite set. The restriction $f'|_{X \setminus G}$ of f' to $X \setminus G$ induces a homeomorphism between $(X \setminus G, \tau_{X \setminus G}^{\mathrm{af}})$ and $(Y \setminus f(G), \tau_{Y \setminus f(G)}^{\mathrm{af}})$. We call it g. The composition $h = g^{-1} \circ f|_{X \setminus G} : (X \setminus G, \tau|_{X \setminus G}) \to (X \setminus G, \tau_{X \setminus G}^{\mathrm{af}})$ is a definable homeomorphism given by $h(x) = x$. Here, the notation $\tau|_{X \setminus G}$ denotes the topology on $X \setminus G$ induced from τ. In particular, it implies the condition (2).

We next prove the opposite implication. Since G is finite, there exists a bounded open box B in M^n containing the set G. We get a stratification of $\mathrm{cl}(X)$ partitioning X, G and $X \cap B$ by [4, Chapter 4, Proposition 1.13]. Let X' be the union of bounded cells in the stratification contained in X. It is a bounded definable set. There exist a definable subset Y' of M^l for some l and a definable homeomorphism $f' : (X', \tau|_{X'}) \to (Y', \tau_{Y'}^{\mathrm{af}})$ by Theorem B.10.

Let $C_1, \ldots, C_N$ be the unbounded cells of the stratification contained in X. Note that the topology on C_i induced from τ coincides with the affine topology by the definition of the stratification. The cells C_i are the graphs of definable continuous maps $\varphi_i : I_i \to M^n$, where I_i are open intervals, and the frontier ∂C_i is either a finite set or an empty set for all $1 \leq i \leq N$. We may assume that either $I_i = (0, \infty)$ or $I_i = M$ without loss of generality. Since C_i is unbounded $X \cap \partial C_i$ is at most one point. We may assume that $X \cap \partial C_i \neq \emptyset$ for all $1 \leq i \leq L$ and $X \cap \partial C_i = \emptyset$ for all

$L < i \leq N$. Let x_i be the unique points in $X \cap \partial C_i$ for all $1 \leq i \leq L$. For $L < i \leq N$, take distinct points x_i in M^l out of $\mathrm{cl}(B)$.

Set $k = l+1$. We construct a definable subset Y of M^k and a definable homeomorphism $f : (X, \tau) \to (Y, \tau_Y^{\mathrm{af}})$. We set

$$Y = (Y' \times \{0\}) \cup \bigcup_{i=1}^{N} \{x_i\} \times I_i.$$

The map $f : X \to Y$ is defined as follows. We set $f(x) = (f'(x), 0)$ when $x \in X'$. We set $f(x) = (x_i, \varphi_i^{-1}(x))$ when $x \in C_i$ for some $1 \leq i \leq N$. It is a routine to demonstrate that f is a homeomorphism. $\square$

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