Higher codimensional birational equivalences and counter-examples of the integral Hodge conjecture

南 範彦

Norihiko Minami

名古屋工業大学

Nagoya Institute of Technology *

Abstract

Consider the naively formulated "weak integral Hodge conjecture"without making use of the Hodge structure, whose validity is equivalent to the validity of the usual integral conjecture when the corresponding Hodge conjecture holds. Then I shall report my recent theorem which derives some algebro-geometric property from being a counter-example of the weak integral Hodge conjecture, and outline its proof.

Together with a theorem of Totaro, this theorem actually deduce the above algebro-geometric property from some purely homotopy theoretical chromatric cohomological condition.

1 Background

1.1 Weak integral Hodge conjecture

For a complex projective algebraic manifold X, we have the cycle map

$$cycl^i : \mathrm{CH}^i(X) \ \to \ \mathrm{Hdg}^{2i}(X,\mathbb{Z}) := H^{2i}(X,\mathbb{Z}) \cap H^{i,i}(X,\mathbb{C}) \ \big(\subseteq H^{2i}(X,\mathbb{Z})\big)$$

whose cokernel is commonly denoted by $Z^{2i}(X)$:

$$Z^{2i}(X) \ := \ \mathrm{Coker}\left(cycl^i : \mathrm{CH}^i(X) \to \mathrm{Hdg}^{2i}(X,\mathbb{Z}) := H^{2i}(X,\mathbb{Z}) \cap H^{i,i}(X,\mathbb{C}) \right).$$

While the traditional (codimension i) Hodge conjecture predicts $Z^{2i}(X)_{\mathbb{Q}} = 0$, the (codimension i) integral Hodge conjecture asks $Z^{2i}(X) = 0$, for which many counter examples have been presented by Atiyah-Hirzebruch [AH62], Kollár [K90], Totaro [T97, **?**], and many others.

Here, I consider geometric implication of being a conter example of the codimension i weak integral Hodge conjecture, which asks where the torsion part $Z^{2i}(X)\{tors\}$ of the finitely generated abelian group $Z^{2i}(X)$ vanishes:

$$\boxed{Z^{2i}(X)\{tors\} \overset{?}{=} 0}$$

I am more interested in the weak integral Hodge conjecture than the integral Hodge conjecture here, because of the following reasons:

*nori@nitech.ac.jp

- Assuming the validity of the codimension i Hodge conjecture $Z^{2i}(X)_{\mathbb{Q}} = 0$, the validity of the codimension i weak integral Hodge conjecture is clearly equivalent to the validity of the codimension i integral Hodge conjecture:

$$Z^{2i}(X)\{tors\} = 0 \quad \Longleftrightarrow \quad Z^{2i}(X) = 0$$

  Not surprisingly, all the known counter examples of the integral Hodge conjecture are also counter examples of the weak integral Hodge conjecture.

- The weak integral Hodge conjecture can be formulated much easily than the the integral Hodge conjecture, without using the Hodge theory. In fact, by the structure theorem of finite genrated abelian groups,

$$Z^{2i}(X)\{tors\} = Z^{2i}_{top}(X)\{tors\} = \oplus_{\ell:\text{prime}} \overbrace{Z^{2i}_{top}(X)\{\ell^\infty\}}^{\ell\text{-primary torsion part}} \quad ,$$

  where

$$\begin{aligned} Z^{2i}_{top}(X) &:= \text{Coker}\left(\iota \circ cycl^i : \text{CH}^i(X) \xrightarrow{cycl} \text{Hdg}^{2i}(X, \mathbf{Z}) := H^{2i}(X, \mathbf{Z}) \cap H^{i,i}(X, \mathbf{C}) \xhookrightarrow{\iota} H^{2i}(X, \mathbf{Z})\right) \\ &= \text{Coker}\left(cycl_{top} : \text{CH}^i(X) \ \rightarrow \ H^{2i}(X, \mathbf{Z}) \right). \end{aligned}$$

  And, for each prime ℓ, we may further rewrite $Z^{2i}_{top}(X)\{\ell^\infty\}$ as:

$$Z^{2i}_{top}(X)\{\ell^\infty\} = Z^{2i}_{et}(X)\{\ell^\infty\},$$

  where $Z^{2i}_{et}(X)$ is defined using the étale cycle map $cycl_{et}$:

$$Z^{2i}_{et}(X) \ := \ \text{Coker}\left(cycl_{et} : \text{CH}^i(X) \otimes \mathbb{Z}_\ell \ \rightarrow \ H^{2i}_{et}\left(X, \mathbb{Z}_\ell(i)\right) := \varprojlim_n H^{2i}_{et}\left(X, \mu_{\ell^n}^{\otimes i}\right) \right).$$

1.2 The codimension $i = 1$ case

The codimension 1 integral Hodge conjecture is known to hold by Lefschetz and Hodge [KS53, p.876, Theorem 4]. The essense may be summarized by the following commutative diagram arising from the short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}^*_{X^{an}} \to 0$ of sheaves on the corresponding complex analytic manifold X^{an} :

$$\begin{array}{ccccc} H^1\left(X, \mathcal{O}_X^*\right) & = & CH^1(X) & & \\ \text{Serre's GAGA} \Vert & & \downarrow cycl^1 & & \\ H^1\left(X^{an}, \mathcal{O}^*_{X^{an}}\right) & \longrightarrow & H^2\left(X^{an}, \mathbb{Z}\right) & \longrightarrow & H^2\left(X^{an}, \mathcal{O}_{X^{an}}\right) \end{array}$$

1.3 The codimension $i = 2$ case

In this case, we have the following fantastic theorem of Colliot-Thélène and Voisin:

Theorem 1.1. (i) [CTV12, Theorem 3.7] *For a complex projective algebraic manifold X, we have a short exact sequence:*

$$H^3_{nr}\left(X, \mathbb{Q}(2)\right) \to H^3_{nr}\left(X, \mathbb{Q}/\mathbb{Z}(2)\right) \to Z^4(X)\{tors\} \to 0,$$

(ii) [CTV12, Proposition 3.3, Theorem 3.9] *Suppose further X is (-2) rationally connected in the sense of* [M21, Definition 2.1.(v)], *then $H^3_{nr}(X, \mathbb{Q}(2)) = 0$, and so*

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \;=\; Z^4(X)\{tors\},$$

which is actually equal to $Z^4(X)$ under the (-2) rationally connected assumption [BS83, Theorem 1 (iv)]. □

Together with [V19, Lemma 1.14], the above Theorem 1.1 immediately implies the following:

Corollary 1.2.
- *$Z^4(X)\{tors\}$ is a stable birational invariant.*
- *If $Z^4(X)\{tors\} \neq 0$, then X is not a rational retract of a smooth projective Y with trivial $H^3_{nr}(Y, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*
- *In particular, X is not (-2) retract rational in the sense of Definition 2.1 (v).*

In fact, from Theorem 1.1 (i), we immediately find

$$Z^4(X)\{\text{tors}\} \neq 0 \quad \Longrightarrow \quad H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0,$$

from which, the claim is proved by use of the motivic property of the unramified cohomology. □

Remark 1.3. Looking at the above proof, Corollary 1.2 might appear too crude. However, when restricted to the case of (-2) rationally connected smooth projective complex varieties like the setting of the Lüroth problem, Theorem 1.1 (ii) suggests that would not be the case.

The main result I shall state in the next section generalizes this Corollary 1.2.

2 Main Theorem

To state my main theorem, I have to preprare some terminologies:

2.1 Double Hierarchy in algebraic geometry

Definition 2.1. *For all of the definitions below, if $c = 0$ we shall usually omit "$> c(= 0)$ from their terminologies:*
(i) *Let us say a rational map $f : X - - > Y$ a <u>codimension $> c$ rational map</u> and denote it by*

$$f : X \overset{(>c)}{- -} > Y,$$

if its indeterminacy locus $I_f \subset X$ has codimension larger than c :

$$\mathrm{codim}_X I_f > c.$$

(ii) *Let us say codimension $> c$ rational maps $f : X - - > Y,\; g : X - - > Y$ <u>codimension $> c$ equivalent</u> and denote this situation by*

$$f \overset{(>c)}{=} g,$$

if there is a closed subscheme I such that $I_f \cup I_g \subset I \subset X$ of codimension $> c$: $\mathrm{codim}_X I > c$, such that

$$f\big|_{X\setminus I} \;=\; g\big|_{X\setminus I}$$

(iii) *Let us say two equi-dimensional k-schemes X, Y codimension $> c$ birational equivalent and denote this situation by*

$$X \overset{(>c)}{\approx} Y,$$

if there are codimension $> c$ rational maps $f : X \overset{(>c)}{--\!>} Y$, $g : Y \overset{(>c)}{--\!>} X$ such that both $g \circ f$ and $f \circ g$ are also codimension $> c$ rational maps such that

$$g \circ f \overset{(>c)}{=} Id_X, \qquad f \circ g \overset{(>c)}{=} Id_Y$$

(iv) *Let us say X is codimension $> c$ rational retract of Y, if there are codimension $> c$ rational maps $f : X \overset{(>c)}{--\!>} Y$, $g : Y \overset{(>c)}{--\!>} X$ such that $g \circ f$ is also a codimension $> c$ rational map such that*

$$g \circ f \overset{(>c)}{=} Id_X.$$

(v) *Let us say a smooth proper X $(-i)$ codimension $> c$ rational, if there exists an i-dimensional smooth proper Z^i such that*

$$X \overset{(>c)}{\approx} \mathbb{P}^{\dim X - i} \times Z^i.$$

(vi) *Let us say a smooth proper X $(-i)$ codimension $> c$ stable rational, if there exists some $N \in \mathbb{Z}_{\geq 0}$ such that $\mathbb{P}^N \times X$ is $(-i)$ codimension $> c$ rational.*
(vii) *Let us say a smooth proper X $(-i)$ codimension $> c$ retract rational if there exist an i-dimensional smooth proper Z^i, $N \in \mathbb{Z}_{\geq n}$, such that X is codimension $> c$ rational retract of $\mathbb{P}^N \times Z^i$.*

Remark 2.2. (i) We have the following impliations of double hierarchies:

$$\{\ (-i) \text{ codimension } > c \text{ rational }\}_{i,c\in\mathbb{Z}_{\geq 0}} \implies$$
$$\{\ (-i) \text{ codimension } > c \text{ stable rational }\}_{i,c\in\mathbb{Z}_{\geq 0}} \implies \{\ (-i) \text{ codimension } > c \text{ retract rational }\}_{i,c\in\mathbb{Z}_{\geq 0}},$$

which induces a web of implications induced by $(-i) \implies \big(-(i+1)\big)$ and $> (c+1) \implies > c$.
(ii) The cases $i = 0$ essentially correspond to the hierarchies considered in [M21, M22].

2.2 Schreieder's higher unramified cohomologies

For a separated scheme X of finite type over a field k, Schreieder [S23] consider the increasing filtration by pro-schemes

$$X^{(\leq 0)} \subset X^{(\leq 1)} \subset \cdots \subset X^{(\leq \dim X)} = X,$$

where, by setting $\mathrm{codim}(x) := \dim X - \dim(\overline{\{x\}})$, each $X^{(\leq j)}$ is defined as follows: [1)]

$$X^{(\leq j)} := \{x \in X \mid \mathrm{codim}(x) \leq j\} \approx \varprojlim_{\substack{U \subset X,\ \text{open subset} \\ \text{s.t. } \mathrm{codim}_X(X\setminus U) > j.}} U.$$

[1)] In [S23], $X^{(\leq j)}$ is denoted by $F_j X$.

If we set $H^*(-,n) := H^*_{et}\left(-,\mu_{\ell^r}^{\otimes n}\right)$ for each $1 \leq r \leq +\infty$, then by working as in [BS15], we have

$$H^*(X^{(\leq j)}, n) = \varinjlim_{\substack{U \subset X,\ \text{open subset} \\ \text{s.t. } \operatorname{codim}_X(X \setminus U) > j.}} H^*(U,n).$$

Then, Schreieder [S23] defined his higher unramified cohomologies [2] by

$$\boxed{H^*_{j,nr}(X,n)} := \operatorname{Im}\left(H^*(X^{(\leq j+1)},n) \to H^*(X^{(\leq j)},n)\right),$$

which generalizes the usual unramified cohomology: $H^*_{0,nr}(X,n) = H^*_{nr}\left(X,\mu_{\ell^r}^{\otimes n}\right)$. Now the following theorem, which generalizes the theorem of Colliot-Thélène and Voisin recalled in Theorem 1.1, explains why Schreieder's higher unramified cohomologies are so important for my purpose:

Theorem 2.3. [S23, Theorem 1.6] *For a complex projective algebraic manifold X, we have a short exact sequence:*

$$H^{2i-1}_{i-2,nr}(X,\mathbb{Q}(i)) \to H^{2i-1}_{i-2,nr}(X,\mathbb{Q}/\mathbb{Z}(i)) \to Z^{2i}(X)\{tors\} \to 0,$$

□

2.3 Statement of my Main Theorem

Now, I am ready to state my main theorem:

Theorem 2.4. • *$Z^{2i}(X)\{tors\}$ is a codimension $> (i-2)$ stable birational invariant.*

- *If $Z^{2i}(X)\{tors\} \neq 0$, then X is not a $(-i)$ codimension $> (i-2)$ retract rational of a smooth projective Y with trivial $H^{2i-1}_{i-2,nr}(Y,\mathbb{Q}/\mathbb{Z}(i)) = 0$.*
- *In particular, X is not $(-i)$ codimension $> (i-2)$ retract rational in the sense of Definition 2.1 (v).*

This is proved just like Corollary 1.2: By Theorem 2.3, note that

$$Z^{2i}(X)\{\text{tors}\} \neq 0 \quad \Longrightarrow \quad H^{2i-1}_{i-2,nr}(X,\mathbb{Q}/\mathbb{Z}(i)) \neq 0,$$

from which, the claim is proved by use of some basic properties of the higher unramified cohomologies proved in [S23] and the crucial motivic property of the higher unramified cohomologies proved in [S22]. □

Remark 2.5. Let us recall Totaro's theorem [T97] on the factorization of the cycle map through the Thom reduction:

$$\begin{array}{ccccccccccc}
MU^*(X) \otimes_{MU^*} H^* & \longrightarrow & & & & & & & & & H^*(X,\mathbb{Z}) \\
\uparrow & & & & & & & & & & \downarrow \\
MU^*(X) & \longrightarrow & BP^*(X) & \longrightarrow & \cdots & \longrightarrow & BP\langle n+1\rangle^*(X) & \longrightarrow & BP\langle n\rangle^*(X) & \longrightarrow & \cdots BP\langle 0\rangle^*(X) = H^*(X,\mathbb{Z})_{(l)}
\end{array}$$

Then we find the conclusions of Theorem 2.4 are implied by some purely homotopy theoretical chromatic condition.

The detail will appear elsewhere.

[2] In [S23], Schreieder's higher unramified cohomologies are called refined unramified cohomologies.

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