UNIQUE TORIC STRUCTURE ON A FANO BOTT MANIFOLD

YUNHYUNG CHO

To each symplectic manifold (M, ω), the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$ is one of the most fundamental objects for understanding (M, ω). The group $\mathrm{Ham}(M, \omega)$ is infinite dimensional and non-compact in general, and it might possess more than one maximal tori with distinct conjugacy classes unlike the case of finite dimensional Lie groups. McDuff and Borisov proved that the number of conjugacy classes of maximal tori in $\mathrm{Ham}(M, \omega)$, allowing conjugations by elements of $\mathrm{Symp}(M, \omega)$, is finite. See [KKP07], [Pin08], and [McD11, Proposition 3.1].

In this article, we are interested in *monotone* symplectic manifolds, that is, a symplectic manifold (M, ω) equipped with a *monotone* symplectic form defined by

$$c_1(M) := c_1(TM, J) = \lambda \cdot [\omega]$$

for some $\lambda > 0$ and an ω-compatible almost complex structure J on M. Since the scaling factor λ does not contribute to the Hamiltonian diffeomorphism group, we always assume that $\lambda = 1$. In [McD11], McDuff posed the following question.

Question 0.1 (McDuff)**.** [McD11, Question 1.11] *Let (M, ω) be a $2n$-dimensional closed monotone symplectic manifold. If T_1 and T_2 are n-tori in* $\mathrm{Ham}(M, \omega)$*, are (M, ω, T_1) and (M, ω, T_2) equivariantly symplectomorphic? Equivalently, is the number of conjugacy classes of n-tori in* $\mathrm{Ham}(M, \omega)$ *precisely one?*

When a symplectic form is monotone, it is equivariantly symplectomorphic to a smooth Fano toric variety by the Kleiman's ampleness criterion [Kle66, Theorem 1 in Section III-1] and its moment polytope becomes *reflexive*, i.e., it is integral and has a unique interior lattice point such that the affine distance from the point to each facet is equal to one. McDuff [McD11] gave an affirmative answer to Question 0.1 when $M = \mathbb{C}P^k \times \mathbb{C}P^m$, and Fanoe [Fan14] generalized McDuff's result to the case of $\mathbb{C}P^k$-bundle over $\mathbb{C}P^m$.

This paper is addressed to Question 0.1 in case that M is a Bott manifold. A *Bott tower*, first introduced by Grossberg and Karshon [GK94], is an iterated $\mathbb{C}P^1$-bundle starting with a point

$$\mathcal{B}_n \xrightarrow{\pi_n} \mathcal{B}_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow \mathcal{B}_1 = \mathbb{C}P^1 \xrightarrow{\pi_1} \mathcal{B}_0 = \{\text{a point}\}$$

where each $\mathcal{B}_i$ is obtained by projectivizing the direct sum of the trivial line bundle $\underline{\mathbb{C}}$ and a complex line bundle ξ_i over $\mathcal{B}_{i-1}$, i.e., $\mathcal{B}_i = P(\underline{\mathbb{C}} \oplus \xi_i)$. The total space $\mathcal{B}_n$ is called a *Bott manifold*.

We may equip a natural complex structure on a Bott manifold by taking each ξ_i as a holomorphic line bundle so that $\mathcal{B}_n$ becomes a complex manifold with a natural $(\mathbb{C}^*)^n$-action constructed in an iterative way using a toric structure of a base space and a $\mathbb{C}^*$-action on a fiber at each stage, see [Oda78, Section I-7.6′]. Indeed, $\mathcal{B}_n$ is a smooth projective toric variety.

We note that not all Bott manifolds are Fano. Recently, Suyama [Suy19] classified all Fano Bott manifolds in terms of Bott matrices, see Section **??** for the detail. We prove the following.

Theorem 0.2. *Let X and Y be Fano Bott manifolds. If there exists a c_1-preserving graded ring isomorphism*

$$\varphi\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z}),$$

then X and Y are isomorphic as toric varieties, i.e., the fans associated to X and Y are unimodularly equivalent.

Using Theorem 0.2, we obtain a positive answer to Question 0.1.

Corollary 0.3. *Any monotone Bott manifold has a unique toric structure.*

We note that 0.2 is closely related to a problem posed by the third author and Suh [MS08, Problems 1 and 4] which asks whether two smooth complete toric varieties having isomorphic cohomology rings (as graded rings) are diffeomorphic or not. This problem is now called the *cohomological rigidity* for toric varieties. There are many partial affirmative answers to the problem. For instance, two smooth complete toric varieties with Picard number 2 are diffeomorphic if and only if their integral cohomology rings are isomorphic as graded rings, see [CMS10]. We also refer the reader to [CMS11, BEM$^+$17] and references therein for recent accounts of this problem.

Inspired by Theorem 0.2, we pose the following conjecture.

Conjecture 0.4. *Suppose that M and N are smooth toric Fano varieties. If there exists a c_1-preserving graded ring isomorphism between their integral cohomology rings, then M and N are isomorphic as toric varieties.*

Conjecture 0.4 was verified for some other classes of smooth toric Fano varieties. Indeed, the authors confirmed Conjecture 0.4 for smooth toric Fano varieties with Picard number 2, whose proof will be provided in an upcoming manuscript [CLMP21a]. Also the third author together with Higashitani and Kurimoto [HKM20] proved Conjecture 0.4 for smooth toric Fano varieties with small dimension ($\dim X_{\mathbb{C}} \leq 4$) or with large Picard number.)

Note that if Conjecture 0.4 is true, then the answer to Question 0.1 is positive. More precisely, if (M,ω,T_1) and (M,ω,T_2) are two toric structures over the same monotone symplectic manifold (M,ω), then the identity map on $H^*(M;\mathbb{Z})$ satisfies the hypothesis in Conjecture 0.4. Therefore, Conjecture 0.4 can be thought as a stronger version of Question 0.1.

References

[BEM$^+$17] Victor M. Buchstaber, Nikolay Yu. Erokhovets, Mikiya Masuda, Taras E. Panov, and Seonjeong Park, *Cohomological rigidity of manifolds defined by 3-dimensional polytopes*, Uspekhi Mat. Nauk **72** (2017), no. 2(434), 3–66.

[BK] Gavin Brown and Alexander Kasprzyk, *Graded ring database*, http://www.grdb.co.uk/forms/toricsmooth.

[BP15] Victor M. Buchstaber and Taras E. Panov, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015.

[CLMP21a] Yunhyung Cho, Eunjeong Lee, Mikiya Masuda, and Seonjeong Park, *On Fano generalized Bott manifolds*, in preparation.

[CLMP21b] Yunhyung Cho, Eunjeong Lee, Mikiya Masuda, and Seonjeong Park, *On the number of Fano Bott manifolds*, in preparation.

[CMM15] Suyoung Choi, Mikiya Masuda, and Satoshi Murai, *Invariance of Pontrjagin classes for Bott manifolds*, Algebr. Geom. Topol. **15** (2015), no. 2, 965–986.

[CMS10] Suyoung Choi, Mikiya Masuda, and Dong Youp Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 1097–1112.

[CMS11] ———, *Rigidity problems in toric topology: a survey*, Tr. Mat. Inst. Steklova **275** (2011), 188–201.

[Del88] Thomas Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339.

[Fan14] Andrew Fanoe, *Toric structures on bundles of projective spaces*, J. Symplectic Geom. **12** (2014), no. 4, 685–724.

[GK94] Michael Grossberg and Yael Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. **76** (1994), no. 1, 23–58.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[HKM20] Akihiro Higashitani, Kazuki Kurimoto, and Mikiya Masuda, *Cohomological rigidity for toric Fano manifolds of small dimensions or large Picard numbers*, to appear in Osaka J. Math.

[KKP07] Yael Karshon, Liat Kessler, and Martin Pinsonnault, *A compact symplectic four-manifold admits only finitely many inequivalent toric actions*, J. Symplectic Geom. **5** (2007), no. 2, 139–166.

[Kle66] Steven L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344.

[McD11] Dusa McDuff, *The topology of toric symplectic manifolds*, Geom. Topol. **15** (2011), no. 1, 145–190.

[MP08] Mikiya Masuda and Taras E. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, Mat. Sb. **199** (2008), no. 8, 95–122.

[MS08] Mikiya Masuda and Dong Youp Suh, *Classification problems of toric manifolds via topology*, Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 273–286.

[Oda78] Tadao Oda, *Torus embeddings and applications*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 57, Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin-New York, 1978, Based on joint work with Katsuya Miyake.

[Pin08] Martin Pinsonnault, *Maximal compact tori in the Hamiltonian group of 4-dimensional symplectic manifolds*, J. Mod. Dyn. **2** (2008), no. 3, 431–455.

[Suy19] Yusuke Suyama, *Fano generalized Bott manifolds*, Manuscripta Mathematica **163** (2020), no. 3–4, 427–435.

Department of Mathematics Education, Sungkyunkwan University, Seoul, Republic of Korea

Email address: `yunhyung@skku.edu`