BOREL-HIRZEBRUCH TYPE FORMULA FOR THE GRAPH EQUIVARIANT COHOMOLOGY OF A PROJECTIVE BUNDLE OVER A GKM-GRAPH

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This is a short summary of the preprint [KS23] that is a joint work with S. Kuroki.

The notion of a GKM-graph was introduced by Guillemin and Zara ([GZ01]) by following the work of Goresky, Kottwitz and MacPherson ([GKM98]). Consider any GKM-graph Γ with the underlying n-valent graph $G=(V,E)$, axial function $\alpha\colon E\to\mathbb{Z}^k$ and a connection $\nabla=\{\nabla_e\}_{e\in E}$ (called an (n,k)-type GKM-graph, to indicate the values of n and k). The corresponding graph equivariant cohomology ring is an $\mathrm{Sym}(\mathbb{Z}^k)$-algebra (where $\mathrm{Sym}(\mathbb{Z}^k)=\mathbb{Z}[x_1,\dots,x_k]$, $\deg x_i=2$) defined by

$$H^*(\Gamma):=\{f\colon\mathcal{V}\to\mathrm{Sym}(\mathbb{Z}^k)\mid f(p)-f(q)\equiv 0 \mod \alpha(e),\ pq\in E\}.$$

By [GKM98], the equivariant cohomology ring $H^*_T(M;\mathbb{Z})$ is isomorphic to $H^*(\Gamma)$ as an $H^*(BT)$-algebra, where Γ is the associated GKM-graph of any GKM-manifold M with a T-action.

For any T-equivariant rank r vector bundle $\xi_{\mathbb{C}}$ over a GKM T-manifold, there is the corresponding projectivization $\mathbb{P}(\xi_C)\to M$. The T-equivariant cohomology of the projectivization $\mathbb{P}(\xi_{\mathbb{C}})$ over a GKM-manifold M is a free $H^*_T(M)$-module by Leray-Hirsch theorem. This result was proved in [GSZ12] in a different way, by using combinatorial notions of GKM-theory (with $\mathbb{R}$ coefficients). The classical theorem of Borel and Hirzebruch [GH78] determines the $H^*_T(M)$-algebra structure of $\mathbb{P}(\xi_{\mathbb{C}})$ in terms of equivariant Chern classes for $\xi_{\mathbb{C}}$. In what follows we give a certain combinatorial version of this theorem.

Define a *leg bundle* $\xi\to\Gamma$ of rank r to be the triple consisting of the graph $G\times[r]$ with noncompact edges, where $[r]$ has a unique vertex and r distinct noncompact edges emanating from it, a collection $\xi^i_p\in\mathbb{Z}^k$ of vectors labelling the noncompact edge pi ($p\in V$, $i=1,\dots,r$), and a collection of permutations $\{\sigma_e\}_{e\in E}$ from the permutation group S_r on $[r]$ satisfying the congruence relation

$$(1)\qquad \xi^i_p-\xi^{\sigma_e i}_q=c^i_{pq}\alpha(e),\ e=pq\in E,\ i=1,\dots,r,\ c^i_{pq}\in\mathbb{Z}.$$

(Here by a slight abuse of the notation $[r]=\{1,2,\dots,r\}$.) For $\xi_{\mathbb{C}}\to M$ from above, the corresponding leg bundle ξ over the GKM-graph Γ of M is given by the weights of the tangential representation on $\xi_{\mathbb{C}}$.

To any leg bundle $\xi\to\Gamma$ associate the triple $\Pi(\xi)=(P(\xi),\alpha^{\Pi(\xi)},\nabla^{\Pi(\xi)})$ called the *projectivization* of the leg bundle ξ. The graph $P(\xi)$ on the vertex set $V_{P(\xi)}=V\times[r]$ has edges of two different kinds. The *vertical* edges are of the form pij, $p\in V$, $i\neq j\in[r]$.

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The *horizontal* edges are of the form (pi, qj), where $pq \in E$, $j = \sigma_{pq}(i)$. Using this notation, define $\alpha^{\Pi(\xi)}\colon E_{P(\xi)} \to \mathbb{Z}^k$ by

$$\alpha^{\Pi(\xi)}(pi, qj) := \alpha(pq),\ \alpha^{\Pi(\xi)}(pij) := \xi_p^i - \xi_p^j.$$

In addition, there exists a canonical connection $\{\nabla^{\Pi(\xi)}\}_{e\in E_{P(\xi)}}$ [KS23], such that the congruence relation is satisfied for $\alpha^{\Pi(\xi)}$. If $\alpha^{\Pi(\xi)}$ satisfies 2-independence condition then $\Pi(\xi)$ is a GKM-graph. In the above notation, the projectivization $\mathbb{P}(\xi_{\mathbb{C}})$ of the complex vector bundle $\xi_{\mathbb{C}}$ is a GKM-manifold having the GKM-graph $\Pi(\xi)$ (if it is 2-independent). The i-th *equivariant Chern class* $c_i^T(\xi)$ of a leg bundle ξ is the element of $H^*(\Gamma)$ given by

$$c_i^T(\xi)_p := \mathfrak{S}_i(\xi_p^1, \dots, \xi_p^r),$$

where $\mathfrak{S}_i$ denotes the i-th elementary symmetric polynomial in r variables. By the localization formula, this agrees with the standard definition of equivariant Chern classes in the case of a toric vector bundle $\xi_{\mathbb{C}}$, e.g. see [P08]. Define the *tautological class* $c_\xi \in H^2(\Pi(\xi))$ of ξ by $(c_\xi)_{pi} := \xi_p^i$.

Theorem 1 ([KS23])**.** *Let ξ be a leg bundle of rank $r+1$ over a GKM graph Γ. Assume that its projectivization $\Pi(\xi)$ is again a GKM graph. Then there is the following isomorphism of $H^*(\Gamma)$-algebras:*

$$H^*(\Pi(\xi)) \cong H^*(\Gamma)[\kappa]/\Big(\sum_{k=0}^{r+1}(-1)^k c_k^T(\xi)\cdot\kappa^{r+1-k}\Big),\ c_\xi \mapsto \kappa. \tag{2}$$

Example 2. *The projectivization $\mathbb{P}\mathbb{C}P^2$ of the tangent bundle to $\mathbb{C}P^2$ is a GKM-variety with respect to the natural $T = (\mathbb{C}^\times)^2$-action. The equivariant cohomology of the base is well-known (e.g. by applying the result of Masuda and Panov* [MP06]*) to be*

$$H_T^*(\mathbb{C}P^2) \cong \mathbb{Z}[\tau_1, \tau_2, \tau_3]/(\tau_1\tau_2\tau_3),$$

where $\tau_i \in H_T^2(\mathbb{C}P^2)$ are equivariant Thom classes of the T-invariant rational lines $\mathbb{C}P_i^1$, $i = 1, 2, 3$. As an element of the GKM-ring, τ_i is supported on the respective edge in the GKM-graph of $\mathbb{P}^2$ and is equal to the label of the transversal edge at a vertex. This defines explicitly the $H^*(BT)$-algebra structure on the right hand side of the above formula. The equivariant Chern class of $T\mathbb{C}P^2$ is computed from the tangential representation data as

$$c^T(T\mathbb{C}P^2) = 1 + (\tau_1 + \tau_2 + \tau_3) + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3).$$

Therefore, by Theorem 1, there is the following isomorphism of $H^*(BT)$-algebras

$$H_T^*(\mathbb{P}\mathbb{C}P^2) \cong \mathbb{Z}[\tau_1, \tau_2, \tau_3, \kappa]/(\kappa^2 - (\tau_1 + \tau_2 + \tau_3)\kappa + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3),\ \tau_1\tau_2\tau_3).$$

Any locally trivial fiber bundle $P \to M$ with fiber $\mathbb{C}P^r$ over a complex GKM-manifold M is a projectivization of some vector bundle. Indeed, the corresponding obstruction in $H^2(M; \mathbb{C}^\times) \cong H^3(M; \mathbb{Z}) = 0$ vanishes. Furthermore, the equivariant version of this statement holds for any T-equivariant projective fiber bundle. In what follows we consider the relation between projective and projectivization fiber bundles in the combinatorial setting.

A *projective GKM fiber bundle* $\pi\colon \Pi \to \Gamma$ is by definition a morphism of GKM-graphs (where both axial functions take value in $\mathbb{Z}^k$), such that the preimage of any point in

V_Γ is a complete graph K_{r+1} equipped with an axial function satisfying

$$\alpha^\Pi(pij) = \alpha^\Pi(pil) - \alpha^\Pi(pjl),\ i, j, l \in [r+1],\ p \in V_\Gamma.$$

In addition, the connection of Π preserves horizontal and vertical edges, and the identity

$$\alpha^\Pi(pi, qj) = \alpha^\Gamma(pq),$$

holds. The projectivization of any leg bundle is a projective GKM fiber bundle (if it is 2-independent). Any leg bundle and any projective GKM fiber bundle are GKM fiber bundles in sense of [GSZ12].

Proposition 3 ([KS23]). *Let $\Pi \to \Gamma$ be a projective GKM fiber bundle over a GKM graph $\mathcal{G}$. Then the following are equivalent:*

(1) $\Pi = \Pi(\xi)$ for some leg bundle $\xi \to \Gamma$;
(2) there exists a line leg bundle $\zeta \to \Pi$ such that $c^1_{pq} = 1$ holds for any $pq \in E_\Gamma$ (see (1)).

By replacing $\mathbb{Z}$ with $\mathbb{Q}$ in the above definitions, one obtains the notions of a $\mathbb{Q}$-GKM graph, a $\mathbb{Q}$-leg bundle, etc.

Theorem 4 ([KS23]). *Any projective GKM fiber bundle $\Pi \to \Gamma$ with fiber K_{r+1} is the projectivization $\Pi(\xi) \to \Gamma$ of the $\mathbb{Q}$-leg bundle $\xi = \xi(\Pi)$ given by*

$$\xi^i_p := \frac{1}{r+1} \sum_{j \neq i} \alpha(pij), \tag{3}$$

with the permutations σ^i_e described uniquely by the horizontal edges of Π.

Corollary 5 ([KS23]). *Let $\Pi \to \Gamma$ be a projective GKM fiber bundle over a GKM graph $\mathcal{G}$. Then $H^*(\Pi)$ is isomorphic to the right-hand side of (2) as $H^*(\Gamma)$-algebra with $\mathbb{Q}$-coefficients, where $\xi = \xi(\Pi)$ is given by (3).*

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