Note on spaces of non-resultant systems of bounded multiplicity

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Abstract

For each pair (m,n) of positive integers with $(m,n)\neq(1,1)$ and an arbitrary field $\mathbb{F}$ with its algebraic closure $\overline{\mathbb{F}}$, let $\mathrm{Poly}^{d,m}_n(\mathbb{F})$ denote the space of m-tuples $(f_1(z),\cdots,f_m(z))\in\mathbb{F}[z]^m$ of $\mathbb{F}$-coefficients monic polynomials of the same degree d such that the polynomials $\{f_k(z)\}_{k=1}^m$ have no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$. The space $\mathrm{Poly}^{d,m}_n(\mathbb{F})$ was first defined and studied by B. Farb and J. Wolfson [8] for investigating the homological densities of algebraic cycles in a manifold ([9]). In this note, we shall report about the recent results concerning the homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{F})$ for $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. These results are based on the joint works with A. Kozlowski ([15], [19], [20]).

1 Introduction

Let $\mathbb{N}$ be a set of all positive integers. For connected spaces X and Y, let $\mathrm{Map}(X,Y)$ denote the space consisting of all continuous maps $f:X\to Y$ with the compact open topology. Let $\mathrm{Map}^*(X,Y)\subset\mathrm{Map}(X,Y)$ be the subspace of all base point preserving maps $f:(X,*)\to(Y,*)$. For a based homotopy class $D\in\pi_0(\mathrm{Map}^*(X,Y))=[X,Y]$, we denote by $\mathrm{Map}^*_D(X,Y)\subset\mathrm{Map}^*(X,Y)$ the path component containing the homotopy class D.

When X and Y are complex manifolds, let $\mathrm{Hol}^*_D(X,Y)\subset\mathrm{Map}^*_D(X,Y)$ denote the subspace consisting of all based holomorphic maps $f\in\mathrm{Map}^*_D(X,Y)$. Then we have the natural inclusion

$$i_D:\mathrm{Hol}^*_D(X,Y)\xrightarrow{\subset}\mathrm{Map}^*_D(X,Y). \tag{1.1}$$

Definition 1.1. Let $f:X\to Y$ be a based continuous map, and let $N_0\in\mathbb{N}$ be a fixed positive integer.

(i) The map f is called *a homology* (*resp. homotopy*) *equivalence through dimension* N_0 if the induced homomorphism

$$f_*:H_k(X;\mathbb{Z})\to H_k(Y;\mathbb{Z})\quad(\text{resp. } f_*:\pi_k(X)\to\pi_k(Y)) \tag{1.2}$$

is an isomorphism for any $k\leq N_0$.

(ii) Similarly, the map f is called *a homology* (*resp. homotopy*) *equivalence up to dimension* N_0 if the induced homomorphism f_* (given by (1.2)) is an isomorphism for any $k < N_0$ and an epimorphism for $k = N_0$.

Definition 1.2. Let $N_0 \in \mathbb{N}$ be a fixed positive integer and let G be a group. Let $f : X \to Y$ be a G-equivariant map between G-spaces X and Y.

(i) The map f is called *a G-equivariant homology (resp. homotopy) equivalence through dimension* N_0 if the map f^H is a homology (resp. homotopy) equivalence through dimension N_0 for any subgroup $H \subset G$, where $f^H = f|X^H$ and $X^H \subset X$ denotes the H-fixed subspace defined by

$$X^H = \{x \in X : h \cdot x = x \text{ for any } h \in H\}. \tag{1.3}$$

(ii) Similarly, the map f is called *a G-equivariant homology (resp. homotopy) equivalence up to dimension* N_0 if the map f^H is a homology (resp. homotopy) equivalence up to dimension N_0 for any subgroup $H \subset G$.

Definition 1.3. From now on, let $d \in \mathbb{N}$, let $(m, n) \in \mathbb{N}^2$ be a pair of positive integers such that $(m, n) \neq (1, 1)$, and let $\mathbb{F}$ be a field with its algebraic closure $\overline{\mathbb{F}}$.

(i) Let $\mathrm{P}_d(\mathbb{F})$ denote the space of all $\mathbb{F}$-coefficients monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1}z + a_d \in \mathbb{F}[z]$ of degree d.

(ii) For each m-tuple $D = (d_1, \cdots, d_m) \in \mathbb{N}^m$ of positive integers, we denote by $\mathrm{Poly}_n^{D;m}(\mathbb{F}) = \mathrm{Poly}_n^{d_1,\cdots,d_m;m}(\mathbb{F})$ the space consisting of all m-tuples $(f_1(z), \cdots, f_m(z)) \in \mathrm{P}_{d_1}(\mathbb{F}) \times \mathrm{P}_{d_2}(\mathbb{F}) \times \cdots \times \mathrm{P}_{d_m}(\mathbb{F})$ of monic polynomials such that the polynomials $\{f_j(z)\}_{j=1}^m$ have no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$. This space $\mathrm{Poly}_n^{D;m}(\mathbb{F})$ is usually called *the space of non-resultant system of bounded multiplicity n with coefficients in* $\mathbb{F}$. In particular, when $D_m = (d, d, \cdots, d) \in \mathbb{N}^m$ (m-times), we write

$$\mathrm{Poly}_n^{d,m}(\mathbb{F}) = \mathrm{Poly}_n^{D_m;m}(\mathbb{F}) = \mathrm{Poly}_n^{d,d,\cdots,d;m}(\mathbb{F}). \tag{1.4}$$

Remark 1.4. (i) The space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ may be also regarded as one of generalizations of spaces first studied by Arnold, Vassiliev and Segal and others in several different contexts (e.g. [2], [4], [5], [6], [11], [12], [25], [28]).

(ii) Recall that the classical resultant of a systems of polynomials vanishes if and only if they have a common solution in an algebraically closed field containing the coefficients. Systems which have no common roots are called "non-resultant". This is the intuition behind our choice of the term "non-resultant system of bounded multiplicity." □

Definition 1.5. From now on, let us suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(i) For a monic polynomial $f(z) \in \mathrm{P}_d(\mathbb{K})$, we define the n-tuple $F_n(f) = F_n(f)(z) \in \mathrm{P}_d(\mathbb{K})^n$ of the monic polynomials of the same degree d by

$$F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \cdots, f(z) + f^{(n-1)}(z)). \tag{1.5}$$

Note that $f(z) \in \mathrm{P}_d(\mathbb{K})$ is not divisible by $(z-\alpha)^n$ for some $\alpha \in \mathbb{K}$ if and only if $F_n(f)(\alpha) \neq \mathbf{0}_n$, where we set $\mathbf{0}_n = (0, 0, \cdots, 0) \in \mathbb{K}^n$.

(ii) When $\mathbb{K}=\mathbb{C}$, by identifying $S^2=\mathbb{C}\cup\infty$ we define *the natural map*

$$i^{d,m}_{n,\mathbb{C}}:\mathrm{Poly}^{d,m}_n(\mathbb{C})\to\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1}\simeq\Omega^2S^{2mn-1}\quad\text{by}$$

$$i^{d,m}_{n,\mathbb{C}}(f)(\alpha)=\begin{cases}[F_n(f_1)(\alpha):F_n(f_2)(\alpha):\cdots:F_n(f_m)(\alpha)] & \text{if } \alpha\in\mathbb{C}\\ [1:1:\cdots:1] & \text{if } \alpha=\infty\end{cases}\tag{1.6}$$

for $f=(f_1(z),\cdots,f_m(z))\in\mathrm{Poly}^{d,m}_n(\mathbb{C})$ and $\alpha\in\mathbb{C}\cup\infty=S^2$, where we choose the points ∞ and $*=[1:1:\cdots:1]$ as the base-points of S^2 and $\mathbb{C}\mathrm{P}^{mn-1}$, respectively.

Definition 1.6. Let $\mathbb{Z}_2=\{\pm 1\}$ denote the (multiplicative) cyclic group of order 2, and we will regard the three spaces $S^2=\mathbb{C}\cup\infty$, $\mathbb{C}\mathrm{P}^{mn-1}$ and $\mathrm{Poly}^{d,m}_n(\mathbb{C})$ as $\mathbb{Z}_2$-spaces with actions induced by the complex conjugation on $\mathbb{C}$.

(i) Let $(\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}$ denote the space consisting of all $\mathbb{Z}_2$-equivariant based maps $f:(S^2,\infty)\to(\mathbb{C}\mathrm{P}^{mn-1},*)$. Since $\mathrm{Poly}^{d,m}_n(\mathbb{R})\subset\mathrm{Poly}^{d,m}_n(\mathbb{C})$ and $i^{d,m}_{n,\mathbb{C}}(\mathrm{Poly}^{d,m}_n(\mathbb{R}))\subset(\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}$, we also define *the natural map*

$$i^{d,m}_{n,\mathbb{R}}:\mathrm{Poly}^{d,m}_n(\mathbb{R})\to(\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}\tag{1.7}$$

by the restriction

$$i^{d,m}_{n,\mathbb{R}}=i^{d,m}_{n,\mathbb{C}}|\mathrm{Poly}^{d,m}_n(\mathbb{R}):\mathrm{Poly}^{d,m}_n(\mathbb{R})\to(\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}.\tag{1.8}$$

(ii) For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let

$$s^{d,m}_{n,\mathbb{K}}:\mathrm{Poly}^{d,m}_n(\mathbb{K})\to\mathrm{Poly}^{d+1,m}_n(\mathbb{K})\tag{1.9}$$

denote *the stabilization map* given by adding the points from the infinity as in [19, (5.8)]. Note that one can define the map $s^{d,m}_{n,\mathbb{C}}$ which satisfies the condition

$$s^{d,m}_{n,\mathbb{R}}=s^{d,m}_{n,\mathbb{C}}|\mathrm{Poly}^{d,m}_n(\mathbb{R})=(s^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2}.\tag{1.10}$$

Definition 1.7. Let X be a connected space.

(i) Let $F(X,d)$ denote *the ordered configuration space* of distinct d points of X defined by

$$F(X,d)=\{(x_1,\cdots,x_d)\in X^d : x_i\neq x_j \text{ if } i\neq j\}.\tag{1.11}$$

(ii) Let S_d denote the symmetric group of d-letters. Then the group S_d acts on $F(X,d)$ by the coordinate permutation and let $C_d(X)$ denote *the unordered configuration space of d-distinct points* of X defined by the orbit space

$$C_d(X)=F(X,d)/S_d.\tag{1.12}$$

Remark that the space $C_d(X)$ can be identified with the space of all finite subset of X with cardinal d.

(iii) For connected space X, let $D_j(X)$ denote *the equivariant half-smash product of* X defined by

$$D_j(X) = F(X,j)_+ \wedge_{S_j} X^{\wedge j}, \tag{1.13}$$

where we set $F(X,j)_+ = F(X,j) \cup \{*\}$ (disjoint union), $X^{\wedge j} = X \wedge X \wedge \cdots \wedge X$ (j-times) and the j-th symmetric group S_j acts on $X^{\wedge j}$ by the coordinate permutation. In particular, for $X = S^1$, we set

$$D_j = D_j(S^1) = F(\mathbb{C},j)_+ \wedge_{S_j} (S^1)^{\wedge j}. \tag{1.14}$$

(iv) Let $m,n,d \geq 1$ be positive integers such that $(m,n) \neq (1,1)$. Define the integers $D(d;m,n;\mathbb{C})$ and $D(d;m,n)$ by

$$\begin{cases} D(d;m,n;\mathbb{C}) & = (2mn-3)(\lfloor d/n \rfloor + 1) - 1, \\ D(d;m,n) & = (mn-2)(\lfloor d/n \rfloor + 1) - 1, \end{cases} \tag{1.15}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x.

2 The space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$

Consider the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{C}$. Recall that $mn = 2 \Leftrightarrow (m,n) = (1,2)$ or $(m,n) = (2,1)$.

The case $(m,n) = (1,2)$**.** If $f(z) \in \mathrm{Poly}_2^{d,1}(\mathbb{C})$, it is a monic polynomial without multiple root. Thus, it is represented as

$$f(z) = \prod_{k=1}^{d} (z - \alpha_k) \quad \text{for some } \{\alpha_k\}_{k=1}^d \in C_d(\mathbb{C}). \tag{2.1}$$

Using the above representation, we easily obtain the homeomorphism given by

$$\begin{array}{ccc} C_d(\mathbb{C}) & \xrightarrow[\cong]{\Phi_d} & \mathrm{Poly}_2^{d,1}(\mathbb{C}) \\ c = \{\alpha_k\}_{k=1}^d & \longrightarrow & \prod_{k=1}^d (z - \alpha_k). \end{array} \tag{2.2}$$

Now recall the electric field map

$$E_d : C_d(\mathbb{C}) \to \Omega_d^2 S^2 \tag{2.3}$$

given by

$$E_d(c)(\alpha) = \begin{cases} 1 + \displaystyle\sum_{k=1}^d \frac{1}{\alpha - \alpha_k} & \text{if } \alpha \notin c \\ \infty & \text{otherwise} \end{cases} \tag{2.4}$$

for $c = \{\alpha_k\}_{k=1}^d \in C_d(\mathbb{C})$ and $\alpha \in S^2 = \mathbb{C} \cup \infty$.

If $f(z) \in \mathrm{Poly}_2^{d,1}(\mathbb{C})$ is represented by (2.1), we see that

$$\frac{f(z)+f'(z)}{f(z)} = 1 + \sum_{k=1} \frac{1}{z-\alpha_k} \tag{2.5}$$

Hence, if we identify $\mathbb{C}\mathrm{P}^1 \cong S^2 = \mathbb{C} \cup \infty$ by the identification $[x:y] \leftrightarrow \frac{y}{x}$, it follows from [25, page 42], (2.4) and (2.5) that the natural map $i_{2,\mathbb{C}}^{d,1} : \mathrm{Poly}_2^{d,1}(\mathbb{C}) \to \Omega_d^2 S^2 \simeq \Omega^2 S^3$ can be identified with the electric field map E_d. Hence, by [25, page 41-42] we have the following result.

Theorem 2.1 ([25], [28]). *The natural map* $i_{2,\mathbb{C}}^{d,1} : \mathrm{Poly}_2^{d,1}(\mathbb{C}) \to \Omega_d^2 S^2 \simeq \Omega^2 S^3$ *is a homology equivalence up to dimension* $\lfloor d/2 \rfloor$. □

Remark 2.2. Let β_d denote the Artin's classical braid group of d-strings. Since there is a homotopy equivalence $\mathrm{Poly}_2^{d,1}(\mathbb{C}) \cong C_d(\mathbb{C}) \simeq K(\beta_d, 1)$, $\pi_k(\mathrm{Poly}_2^{d,1}(\mathbb{C})) = 0$ for any $k \geq 2$. Hence, the natural map $i_{2,\mathbb{C}}^{d,1}$ is not a homotopy equivalence in any range. □

The case $(m,n) = (2,1)$**.** Note that there is a homeomorphism

$$\mathrm{Poly}_1^{d,2}(\mathbb{C}) \cong \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^1). \tag{2.6}$$

Hence, the homotopy type of the space $\mathrm{Poly}_1^{d,2}(\mathbb{C})$ can be easily seen by using the classical results ([5], [6], [12], [14], [25], [28]).

Remark 2.3. If $(m,n) = (2,1)$, the homotopy stability holds for the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$. However, if $(m,n) = (1,2)$, the homotopy stability does not hold for the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$, although the homology stability holds (c.f. Theorem 2.1 and Remark 2.2). □

The case $mn \geq 3$**.** Now we assume that $mn \geq 3$. Then we obtain the following results from [15].

Theorem 2.4 ([15]). *Let* $m, n \geq 1$ *be positive integers such that* $mn \geq 3$.

(i) *The natural map*

$$i_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n;\mathbb{C})$.

(ii) *The stabilization map*

$$s_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \mathrm{Poly}_n^{d+1,m}(\mathbb{C})$$

is a homotopy equivalence if $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$ *and it is a homotopy equivalence through dimension* $D(d;m,n;\mathbb{C})$ *otherwise.*

(iii) *There is a homotopy equivalence* $\mathrm{Poly}_n^{d,m}(\mathbb{C}) \simeq \mathrm{Poly}_1^{\lfloor d/n \rfloor, mn}(\mathbb{C})$.

(iv) *There is a stable homotopy equivalence* $\mathrm{Poly}_n^{d,m}(\mathbb{C}) \simeq_s \bigvee_{j=1}^{\lfloor d/n \rfloor} \Sigma^{2(mn-2)j} D_j$*, where* Σ^j *denotes the* j*-fold reduced suspension.* □

3 The space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$.

Next, consider the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$.

The case $mn = 2$**.** If $(m, n) = (2, 1)$, the homotopy type of the space $\mathrm{Poly}_1^{d,2}(\mathbb{R})$ was well studied by G. Segal in [25, Proposition 1.4]. In particular, the homotopy stability holds in this case.

If $(m, n) = (1, 2)$, we can easily see that there is a homeomorphism

$$\mathrm{Poly}_2^{d,1}(\mathbb{R}) \cong \coprod_{j=0}^{\lfloor d/2 \rfloor} C_j(\mathbb{C}) \quad \text{(disjoint union)}. \tag{3.1}$$

Thus, if $(m, n) = (1, 2)$, the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ can be easily understood by using Theorem 2.1. In this case, the homology stability holds, but the homotopy stability does not hold (as explained in Remark 2.2).

The case $mn \geq 3$**.** Now, consider the case $mn \geq 3$. Recall the following homotopy equivalence:

$$(\Omega_d^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \simeq \Omega^2 S^{2N+1} \times \Omega S^N \qquad \text{for } N \geq 2. \tag{3.2}$$

Moreover, note that $mn = 3 \Leftrightarrow (m, n) = (3, 1)$ or $(m, n) = (1, 3)$, and

$$D(d; m, n) = \lfloor d/n \rfloor = \begin{cases} d & \text{if } (m, n) = (3, 1), \\ \lfloor d/3 \rfloor & \text{if } (m, n) = (1, 3). \end{cases} \tag{3.3}$$

First, recall the following two results:

Lemma 3.1 ([19])**.** (i) *If* $mn \geq 4$*, the space* $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ *is simply connected.*

(ii) *If* $mn = 3$ *an* $d \geq n$*,* $\pi_1(\mathrm{Poly}_n^{d,m}(\mathbb{R})) = \mathbb{Z}$*.* □

Theorem 3.2 ([20])**.** *The space* $\mathrm{Poly}_1^{d,3}(\mathbb{R})$ *is simple if* $d \equiv 1 \pmod 2$*, and it is simple up to dimension* d *if* $d \equiv 0 \pmod 2$*.* □

Then we obtain the following results:

Theorem 3.3 ([19], [20]). (i) *If $mn \geq 4$, the natural map*

$$i_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n)$.

(ii) *Let $(m,n) = (3,1)$. Then the natural map*

$$i_{1,\mathbb{R}}^{d,3} : \mathrm{Poly}_1^{d,3}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2} \simeq \Omega^2 S^5 \times \Omega S^2$$

is a homotopy equivalence through dimension d if $d \equiv 1 \pmod 2$, and it is a homotopy equivalence up to dimension d if $d \equiv 0 \pmod 2$.

(iii) *Let $(m,n) = (1,3)$. Then the natural map*

$$i_{3,\mathbb{R}}^{d,1} : \mathrm{Poly}_3^{d,1}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2} \simeq \Omega^2 S^5 \times \Omega S^2$$

is a homology equivalence through dimension $\lfloor d/3 \rfloor$. □

Theorem 3.4 ([19], [20]). (i) *Let $mn \geq 4$. Then the stabilization map*

$$s_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to \mathrm{Poly}_n^{d+1,m}(\mathbb{R})$$

is a homotopy equivalence if $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$, and it is a homotopy equivalence through dimension $D(d;m,n)$ if $\lfloor \frac{d}{n} \rfloor < \lfloor \frac{d+1}{n} \rfloor$.

(ii) *Let $(m,n) = (3,1)$. Then the stabilization map*

$$s_{1,\mathbb{R}}^{d,3} : \mathrm{Poly}_1^{d,3}(\mathbb{R}) \to \mathrm{Poly}_1^{d+1,3}(\mathbb{R})$$

is a homotopy equivalence through dimension d if $d \equiv 1 \pmod 2$, and it is a homotopy equivalence up to dimension d if $d \equiv 0 \pmod 2$.

(iii) *Let $(m,n) = (1,3)$. Then the stabilization map*

$$s_{3,\mathbb{R}}^{d,1} : \mathrm{Poly}_3^{d,1}(\mathbb{R}) \to \mathrm{Poly}_3^{d+1,1}(\mathbb{R})$$

is a homology equivalence if $\lfloor \frac{d}{3} \rfloor = \lfloor \frac{d+1}{3} \rfloor$, and it is a homology equivalence through dimension $\lfloor \frac{d}{3} \rfloor$ if $\lfloor \frac{d}{3} \rfloor < \lfloor \frac{d+1}{3} \rfloor$. □

Theorem 3.5 ([19]). *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \Big(\bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i} \Big) \vee \Big(\bigvee_{i \geq 0, j \geq 1, i+2j \leq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j \Big). \qquad \square$$

Corollary 3.6 ([19]). *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \mathrm{Poly}_1^{\lfloor d/n \rfloor, mn}(\mathbb{R}). \qquad \square$$

Since $\mathrm{Poly}^{d,m}_n(\mathbb{C})^{\mathbb{Z}_2} = \mathrm{Poly}^{d,m}_n(\mathbb{R})$, $(i^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2} = i^{d,m}_{n,\mathbb{R}}$ and $(s^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2} = s^{d,m}_{n,\mathbb{R}}$, by using Theorems 2.4, 3.3 and 3.4, we also obtain the following results:

Corollary 3.7 ([19], [20]). (i) *If $mn \geq 4$, the natural map*

$$i^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension $D(d;m,n)$.

(ii) *Let $(m,n) = (3,1)$. Then the natural map*

$$i^{d,3}_{1,\mathbb{C}} : \mathrm{Poly}^{d,3}_1(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^2 \simeq \Omega^2 S^5$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension d if $d \equiv 1 \pmod 2$, and it is a $\mathbb{Z}_2$-equivariant homotopy equivalence up to dimension d if $d \equiv 0 \pmod 2$.

(iii) *Let $(m,n) = (1,3)$. Then the natural map*

$$i^{d,1}_{3,\mathbb{C}} : \mathrm{Poly}^{d,1}_3(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^2 \simeq \Omega^2 S^5$$

is a $\mathbb{Z}_2$-equivariant homology equivalence through dimension $\lfloor d/3 \rfloor$. □

Corollary 3.8 ([19], [20]). (i) *Let $mn \geq 4$. Then the stabilization map*

$$s^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{C})$$

is a $\mathbb{Z}_2$-homotopy equivalence if $\lfloor \frac{d}{n} \rfloor = \lfloor \frac{d+1}{n} \rfloor$, and it is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension $D(d;m,n)$ if $\lfloor \frac{d}{n} \rfloor < \lfloor \frac{d+1}{n} \rfloor$.

(ii) *Let $(m,n) = (3,1)$. Then the stabilization map*

$$s^{d,3}_{1,\mathbb{C}} : \mathrm{Poly}^{d,3}_1(\mathbb{C}) \to \mathrm{Poly}^{d+1,3}_1(\mathbb{C})$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension d if $d \equiv 1 \pmod 2$, and it is a $\mathbb{Z}_2$-equivariant homotopy equivalence up to dimension d if $d \equiv 0 \pmod 2$.

(iii) *Let $(m,n) = (1,3)$. Then the stabilization map*

$$s^{d,1}_{3,\mathbb{C}} : \mathrm{Poly}^{d,1}_3(\mathbb{C}) \to \mathrm{Poly}^{d+1,1}_3(\mathbb{C})$$

is a $\mathbb{Z}_2$-equivariant homology equivalence if $\lfloor \frac{d}{3} \rfloor = \lfloor \frac{d+1}{3} \rfloor$, and it is a $\mathbb{Z}_2$-equivariant homology equivalence through dimension $\lfloor d/3 \rfloor$ if $\lfloor \frac{d}{3} \rfloor < \lfloor \frac{d+1}{3} \rfloor$ □

Remark 3.9. (i) Although we suppose that the homotopy stability also holds for the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ for the case $(m,n) = (1,3)$, we cannot prove this at the moment.

(ii) We can consider the space of non-resultant systems determined by a toric variety as one of generalizations of the space $\mathrm{Poly}^{d,m}_n(\mathbb{F})$ (see [16] and [18] for further details). □

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