On the Johnson homomorphisms of the basis-conjugating automorphism groups of free groups

Joint work with Naoya Enomoto*

東京理科大学理学部第二部数学科　佐藤 隆夫 †
Satoh, Takao
Faculty of Science Division II, Tokyo University of Science

Abstract

In this short report, we exhibit our recent results for the Johnson homomorphisms and homology groups of basis-conjugating automorphism groups of free groups.

Highly motivated by the study of the Johnson homomorphisms of the mapping class groups of surfaces by pioneering works of Johnson [7], Morita [10] and Hain [6] in the 1980s and 1990s, those of the automorphism groups of free groups was initiated by several authors including Kawazumi [8] in the early 2000s. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of the automorphism groups of free groups through the works of many authors. In particular, the stable cokernel of the Johnson homomorphisms of the automorphism groups of free groups have completely determined due to Satoh [13, 14] and Darné [3]. Enomoto-Satoh [4] also gave a combinatorial description of the GL-decomposition of the stable rational cokernels of the Johnson homomorphisms. As is well-known due to a classical work of Artin, the automorphism group of a free group contains the braid group of a plane. (See also Section 1.4 in [1].) Hence we have natural problems to determine the images and the cokernels of Johnson homomorphisms of of the braid groups. However, in contrast to the case of the mapping class group of a surface, there are few studies for them. In general, it seems to be a difficult problem to attack this problem dirctly from a combinatorial group theretic and a representation theretic viewpoints. In the present paper, we forcus on the basis-conjugating automorphism groups of free groups which contains the pure braid groups as a subgroup, and study the Johsnon homomorphisms of them.

Let us fix some notations. For any $n \geq 1$, let F_n be the free group of rank n with basis $x_1, \ldots, x_n$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Set $H := H_1(F_n, \mathbf{Z})$ and

*e-address: enomoto-naoya@uec.ac.jp
†e-address: takao@rs.tus.ac.jp

$H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Denote by $\mathcal{L}_n = \bigoplus_{k\geq 1} \mathcal{L}_n(k)$ the free Lie algebra generated by H. Let $\mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$ be the Andreadakis-Johnson filtration of $\mathrm{Aut}\, F_n$. Then for each $k \geq 1$, the k-th Johnson homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is defined. Each τ_k is an injective $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism. Let

$$\mathcal{C}_n(k) := H^{\otimes k}/\langle a_1 \otimes a_2 \cdots a_k - a_k \otimes a_1 \cdots a_{k-1} \,|\, a_i \in H\rangle$$

be the quotient module of $H^{\otimes k}$ by the permutation action of the cyclic group C_k of degree k on the components. By [13, 14] and [3], it is known that the cokernel of τ_k is isomorphic to $\mathcal{C}_n(k)$ for $n \geq k+2$. In order to describe the cokernel of τ_k, we use the trace map

$$\mathrm{Tr}_k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k} \to \mathcal{C}_n(k)$$

defined as the compositions of maps including a contraction. This is a generalization of the Morita trace defined in [10]. Then we have a exact sequence

$$0 \to \mathrm{gr}^k(\mathcal{A}_n) \xrightarrow{\tau_k} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \xrightarrow{\mathrm{Tr}_k} \mathcal{C}_n(k) \to 0$$

for $n \geq k+2$. Furtheremore, Enomoto-Satoh [4] gave gave a combinatorial description of the GL-decomposition of $\mathcal{C}_n^{\mathbf{Q}}(k) := \mathcal{C}_n(k) \otimes_{\mathbf{Z}} \mathbf{Q}$. In the paper, we consider the basis-conjugating automorphism group analogue of these works.

An automorphism σ of F_n such that x_i^σ is conjugate to x_i for each $1 \leq i \leq n$ is called a basis-conjugating automorphism of F_n. Let $\mathrm{P}\Sigma_n$ be the subgroup of $\mathrm{Aut}\, F_n$ consisting of all basis-conjugating automorphisms of F_n. The group $\mathrm{P}\Sigma_n$ is called the basis-conjugating automorphism group of F_n. Since McCool [9] gave the first finite presentation for $\mathrm{P}\Sigma_n$, it is also called the McCool group. For any $k \geq 1$, set $\mathcal{P}_n(k) := \mathrm{P}\Sigma_n \cap \mathcal{A}_n(k)$. Then we have an injective $\mathfrak{S}_n$-equivariant homomorphism

$$\tau_k^P : \mathrm{gr}^k(\mathcal{P}_n) := \mathcal{P}_n(k)/\mathcal{P}_n(k+1) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

which we call the k-th Johnson homomorphism of $\mathrm{P}\Sigma_n$. Here we consider $\mathfrak{S}_n$ as a subgroup of $\mathrm{GL}(n, \mathbf{Z})$ consisting of all of monomial matricies. Satoh [16] showed that for any $n \geq 2$ and $k \geq 1$, the image of τ_k^P is contained in a certain submodule $\mathfrak{p}_n(k)$ of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$. Let $\mathrm{Tr}_k^P : \mathfrak{p}_n(k) \to \mathcal{C}_n(k)$ be the restriction of Tr_k to $\mathfrak{p}_n(k)$. By the definition of $\mathcal{P}_n(k)$, it is easy to see that for any $k \geq 2$, $\langle x_1^k, \ldots, x_n^k\rangle \cap \mathrm{Im}(\mathrm{Tr}_k^P) = \{0\}$. Let $\overline{\mathrm{Tr}}_k^P : \mathfrak{p}_n(k) \to \overline{\mathcal{C}}_n(k)$ be the composition of Tr_k^P and the natural projection $\mathcal{C}_n(k) \to \overline{\mathcal{C}}_n(k) := \mathcal{C}_n(k)/\langle x_1^k, \ldots, x_n^k\rangle$. Then we have

Theorem 1. *For any $n \geq 3$, we have an $\mathfrak{S}_n$-equivariant exact sequences*

$$0 \to \mathrm{gr}^k(\mathcal{P}_n) \xrightarrow{\tau_k^P} \mathfrak{p}_n(k) \xrightarrow{\overline{\mathrm{Tr}}_k^P} \overline{\mathcal{C}}_n(k) \to 0$$

for $2 \leq k \leq 3$, and

$$0 \to \mathrm{gr}^4(\mathcal{P}_n) \xrightarrow{\tau_4^P} \mathfrak{p}_n(4) \xrightarrow{\overline{\mathrm{Tr}}_4^P} \overline{\mathcal{C}}_n(4) \to \langle x_a x_b x_a x_b \,|\, 1 \leq a < b \leq n\rangle \to 0.$$

This shows that the trace map $\overline{\mathrm{Tr}}_k^P$ is not surjective in general. In order to detect non-trivial $\mathfrak{S}_n$-irreducible components in the cokernel, we introduce a reduced version of the trace map. For any $k \geq 2$, let $N(k)$ be the $\mathfrak{S}_n$-invariant submodule of $\mathcal{C}_n(k)$ generated by $x_{j_1}x_{j_2}\cdots x_{j_k}$ for all $1 \leq j_1, \ldots, j_k \leq n$ such that each j_l $(1 \leq l \leq k)$ appears at least twice in $j_1, j_2, \ldots, j_k$. Let $\widetilde{\mathrm{Tr}}_k^P : \mathfrak{p}_n(k) \to \widetilde{\mathcal{C}}_n(k) := \mathcal{C}_n(k)/N(k)$ be the composition of Tr_k^P and the natural projection $\mathcal{C}_n(k) \to \widetilde{\mathcal{C}}_n(k)$. Then we have

$$\mathrm{Im}(\tau_k^P) \subset \mathrm{Ker}(\mathrm{Tr}_k^P) = \mathrm{Ker}(\overline{\mathrm{Tr}}_k^P) \subset \mathrm{Ker}(\widetilde{\mathrm{Tr}}_k^P) \subset \mathfrak{p}_n(k).$$

Theorem 2. *For any $n \geq 2$ and $k \geq 2$, the map $\widetilde{\mathrm{Tr}}_k^P$ is surjective. Hence $\widetilde{\mathcal{C}}_n(k) \otimes_{\mathbf{Z}} \mathbf{Q}$ injects into* $\mathrm{Coker}(\tau_k^P \otimes \mathrm{id}_{\mathbf{Q}})$ *as an $\mathfrak{S}_n$-module.*

As a corollary to Theorem 2, we see that for any odd $k \geq 3$ and $n \geq k$, the module $\wedge^k H_{\mathbf{Q}}$ injects into $\mathrm{Coker}(\tau_k^P \otimes \mathrm{id}_{\mathbf{Q}})$ as an $\mathfrak{S}_n$-module. Furthermore, for any even $k \geq 4$ and $n \geq k+2$, the GL-irreducible module $[2, 1^{k-2}]$ injects into $\mathrm{Coker}(\tau_k^P \otimes \mathrm{id}_{\mathbf{Q}})$ as an $\mathfrak{S}_n$-module.

The study of the Johnson homomorphisms gives us several applications. Here we consider three applications. The first one is to apply to the Andreadakis problem for $\mathrm{P}\Sigma_n$. Let $\mathrm{P}\Sigma_n = \mathrm{P}\Sigma_n(1) \supset \mathrm{P}\Sigma_n(2) \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n$. We have $\mathrm{P}\Sigma_n(k) \subset \mathcal{P}_n(k)$ for any $k \geq 1$. The Andreadakis problem for $\mathrm{P}\Sigma_n$ asks whether $\mathrm{P}\Sigma_n(k) = \mathcal{P}_n(k)$ or not. In general, except for some low degree or unstable cases, the Andreadakis problems for $\mathrm{P}\Sigma_n$ and $\mathrm{Aut}\,F_n$ are still open. Satoh [15] showed that $\mathrm{P}\Sigma_n(k) = \mathcal{P}_n(k)$ for $k \leq 3$. By using the thrid Johnson homomorphism, we obtain the following.

Theorem 3. *For $n \geq 3$, we have $\mathcal{P}_n(4) = \mathrm{P}\Sigma_n(4)$.*

The second one is to apply to the second homology and cohomology groups of $\mathrm{P}\Sigma_n$. By using the second Johnson homomorphism and the McCool presentation for $\mathrm{P}\Sigma_n$, we obtain the following.

Theorem 4. *For $n \geq 3$,*
(1) $H_2(\mathrm{P}\Sigma_n, \mathbf{Z}) \cong \mathbf{Z}^{\frac{1}{2}n^2(n-1)(n-2)}$.
(2) *The cup product map* $\cup : \wedge^2 H^1(\mathrm{P}\Sigma_n, \mathbf{Z}) \to H^2(\mathrm{P}\Sigma_n, \mathbf{Z})$ *is surjective.*

Finally, we consider twisted first homology groups of the braid-permutation automorphism groups of free groups. An element $\sigma \in \mathrm{Aut}\,F_n$ a braid-permutation automorphism of F_n if there exists some $\mu \in \mathfrak{S}_n$ and some $a_i \in F_n$ for any $1 \leq i \leq n$ such that

$$x_i^\sigma = a_i^{-1} x_{\mu i} a_i \quad (1 \leq i \leq n).$$

Let BP_n be the subgroup of $\mathrm{Aut}\,F_n$ consisting of all braid-permutation automorphisms of F_n. According to [5], we call BP_n the braid-permutation automorphism group of F_n. We remark that BP_n is isomorphic to the Loop braid group on n components. (For details, see Sections 3 and 4 in [2] for example.) We see $\mathrm{BP}_n \cap \mathcal{A}_n(1) = \mathrm{P}\Sigma_n$. Let V be the standard irreducible representation of $\mathfrak{S}_n$. By observing the decomposition of the image of $\tau_1 \otimes \mathrm{id}_{\mathbf{Q}}$, we see that V appears in the decomposition of $H_1(\mathrm{P}\Sigma_n, \mathbf{Q})$ By using a finite presentation for BP_n obtained by Fenn-Rimányi-Rourke [5], as an analogue of calculations in [12], we obtain the following.

Theorem 5. *For $n \geq 3$, $H^1(\mathrm{BP}_n, V) \cong \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}/4\mathbf{Z}$.*

As a corollary to our calculations, we see that for $n \geq 3$,

$$H^1(B_n, V) = \mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}, \quad H^1(\mathfrak{S}_n, V) = \mathbf{Z}/4\mathbf{Z}.$$

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