On a decomposition of solutions to the damped wave equation and its applications

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1. Introduction

In this paper, we consider the initial-value problem of the following second-order differential equation in a Hilbert space H:

$$\begin{cases} u''(t) + Au(t) + u'(t) = 0, & t \in (0, \infty), \\ (u, u')(0) = (u_0, u_1), \end{cases} \tag{1.1}$$

where A is a nonnegative selfadjoint operator in H endowed with domain $D(A)$. The initial data (u_0, u_1) are given and assumed to be sufficiently regular ($u_0, u_1 \in D(A^\ell)$ for sufficiently large $\ell \in \mathbb{N}$). Of course, the typical example of such a problem is the initial-boundary value problem of the usual damped wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (u, \partial_t u)(x,0) = (u_0(x), u_1(x)) & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is an open set having a smooth boundary $\partial\Omega$, $\partial_t = \frac{\partial}{\partial t}$ and $\Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$.

One can easily find that the solution u of (1.2) satisfies the energy identity

$$\|\partial_t u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\partial_t u(s)\|_{L^2(\Omega)}^2 \, ds = \|u_1\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2$$

which immediately gives the uniform boundedness of the derivative of u. In contrast, estimates for the solution u itself is not so clear. Actually, the energy functional $\|\partial_t u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$ does not have a good factor to control the L^2-norm of u. In the case $\Omega = \mathbb{R}^N$, the Fourier transform is a powerful tool to analyse the precise behavior of solution u. Even if $\Omega \neq \mathbb{R}^N$, the spectral analysis in view of selfadjointness of the Laplacian could be a good tool for the analysis of solutions to (1.2). Instead of this, aiming for generalization to the case of problems governed by some non-selfadjoint operators, we shall discuss several properties of (1.1) without such tools which force the situation to be limited. Here we focus our attention to the framework of *energy method*.

In 1961, Morawetz [**9**] suggests the following procedure for the wave equation (without damping term). Let us consider the initial-boundary value problem of the linear wave equation

$$
(1.3) \qquad \begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (u, \partial_t u)(x,0) = (u_0(x), u_1(x)), & x \in \Omega, \end{cases}
$$

where Ω is an exterior domain of a star-shaped obstacle in $\mathbb{R}^3$. She introduced the Poisson equation $\Delta h = u_1$ and the auxiliary problem

$$
(1.4) \qquad \begin{cases} \partial_t^2 \chi(x,t) - \Delta \chi(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ \chi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (\chi, \partial_t \chi)(x,0) = (h(x), u_0(x)), & x \in \Omega. \end{cases}
$$

Then one can find the relation $\partial_t \chi = u$. This relation provides that the energy identity for χ can be regarded as the L^2-estimate of u:

$$
\|u(t)\|_{L^2(\Omega)}^2 \le \|\partial_t \chi(t)\|_{L^2(\Omega)}^2 + \|\nabla \chi(t)\|_{L^2(\Omega)}^2 \le \|u_0\|_{L^2(\Omega)}^2 + \|\nabla h\|_{L^2(\Omega)}^2.
$$

This argument suggests that an L^2-estimate of solutions to the wave equation can be observed via the energy estimate for the "primitive" of the solution as the one of the wave equation.

Later, in Ikehata–Matsuyama [**4**], they developed the above "Morawetz's method" for the damped wave equation in N-dimensional exterior domain ($N \ge 2$) via the following auxiliary problem

$$
(1.5) \qquad \begin{cases} \partial_t^2 \chi(x,t) - \Delta \chi(x,t) + \partial_t \chi(x,t) = u_1(x), & (x,t) \in \Omega \times (0,\infty), \\ \chi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (\chi, \partial_t \chi)(x,0) = (0, u_0(x)), & x \in \Omega. \end{cases}
$$

The advantage of the technique in [**4**] is to avoid the analysis of the Poisson equation $\Delta h = u_1$, which depends on the structure of the fundamental solution. From this viewpoint, Ikehata–Nishihara [**5**] employ the modified version of "Morawetz's method" to the abstract Cauchy problem (1.1) to prove the diffusion phenomena. More precisely, in [**5**], the following modification is used:

$$
\begin{cases} U'(t) + AU(t) = -u'(t), & t \in (0,\infty), \\ U(0) = 0. \end{cases}
$$

In that case, one can have $u(t) = e^{-tA}(u_0 + u_1) + U'(t)$, where $(e^{-tA})_{t\ge 0}$ stands for the C_0-semigroup generated by $-A$. A suitable energy estimate for U combined with the above decomposition provides estimates for diffusion phenomena. We refer Chill–Haraux [**1**] for a further detailed discussion based on the spectral analysis.

The asymptotic expansion of solutions to (1.2) with $\Omega = \mathbb{R}^N$ has been dealt with in [**11**]. The strategy in [**11**] heavily depends on the knowledge of the Fourier analysis, but (one of) the expansion is written in term of the heat semigroup $e^{t\Delta}$ as

$$
\begin{aligned}
u(t) \sim \frac{1}{2} \sum_{j=0}^{[M/2]} \sum_{k=0}^{[M/2]-j} \alpha_{j,k}(-t)^j(-\Delta)^{2j+k}\Delta^{t\Delta}u_0 \\
- \sum_{j=0}^{[M/2]} \sum_{k=0}^{[M/2]-j} \sum_{\ell=0}^{[M/2]-j-k} \alpha_{j,k}\beta_\ell(-t)^j(-\Delta)^{2j+k+\ell}e^{t\Delta}\left(\frac{1}{2}u_0+u_1\right)
\end{aligned}
$$

where $\alpha_{j,k}$ and β_ℓ are the appropriate constants (determined through the Taylor expansion for the Fourier symbol of the solution map). This expansion seems to be reasonable from the viewpoint of an abstract framework in Hilbert spaces.

In the present paper, an alternative framework for the asymptotic expansion for (1.1) via energy methods with a decomposition is proposed. This part is based on [**10**]. Incidentally, we have found a technique applicable to the following singular limit problem of the abstract Cauchy problem

$$
\begin{cases} \varepsilon u_\varepsilon''(t) + Au_\varepsilon(t) + u_\varepsilon'(t) = 0, & t \in (0,\infty), \\ (u_\varepsilon, u_\varepsilon')(0) = (u_0, u_1) \end{cases} \tag{1.6}
$$

with the parameter $\varepsilon > 0$. This part is based on the joint work [**6**] with Professor Ryo Ikehata (Hiroshima University). The problem is to analyse the behavior of solution u_ε when ε tends to 0. An expected limit problem can be seen as

$$
\begin{cases} Au_\varepsilon(t) + u_\varepsilon'(t) = 0, & t \in (0,\infty), \\ u(0) = u_0 \end{cases} \tag{1.7}
$$

which seems to be not reasonable if $Au_0 + u_1 \neq 0$. This kind of problem has been dealt with in Kisynski [**7**] via the spectral analysis. In [**7**], it is proved the following.

Theorem 1.1 (Kisynski [**7**]). *Let u_ε be the solution of* (1.7). *Then*

$$
\begin{cases} \|u_\varepsilon(t) - e^{-tA}u_0\|_H = O(\varepsilon^{1/2}) & \text{if } (u_0,u_1) \in D(A^{1/2}) \times H, \\ \|u_\varepsilon'(t) + Ae^{-tA}u_0 - e^{-t/\varepsilon}(Au_0+u_1)\|_H = O(\varepsilon^{1/2}) & \text{if } (u_0,u_1) \in D(A^{3/2}) \times D(A^{1/2}) \end{cases}
$$

as $\varepsilon \to +0$.

The factor $e^{-t/\varepsilon}(Au_0+u_1)$ is so-called the initial-layer which bridges the gap between (1.6) and the limit problem (1.7) (for the general theory for singular limit problems with boundary layer, see e.g., the book of Lions [**8**]). Ikehata discussed in [**3**] the singular limit problem (1.6) from the viewpoint of the (modified) Morawetz's method explained above. Although, in Chill–Haraux [**2**] the analysis of the singular limit problem has been developed by the spectral analysis, in the connection explained above, we shall explain how to apply the decomposition in the idea of asymptotic expansion to the singular limit problem (1.6).

This paper is organized as follows. In Section 2, a decomposition of solutions to the abstract second order differential equations is explained. In Section 3, we give the idea of asymptotic expansion of solutions to (1.1) and the (successive) construction of each asymptotic profiles via the decomposition. The verification is done by the energy method. In Section 4, we treat (1.6) from the viewpoint of our decomposition.

2 A decomposition lemma

Here we introduce the following abstract Cauchy problem with inhomogeneous term:

$$\begin{cases} \varepsilon w''(t) + Aw(t) + w'(t) = F(t), \quad t \in (0,\infty), \\ (w, w')(0) = (w_0, w_1), \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, $(w_0, w_1) \in D(A^{1/2}) \times H$ and $F \in C([0,\infty); H)$. Then we can prove the following decomposition for w.

Lemma 1. *Let w be the solution of (2.1) for $\varepsilon > 0$. Set v and U as the respective solutions to the following problems:*

$$\begin{cases} v'(t) + Av(t) = F(t), \quad t \in (0,\infty), \\ v(0) = v_0. \end{cases} \tag{2.2}$$

$$\begin{cases} \varepsilon U''(t) + AU(t) + U'(t) = -v'(t), \quad t \in (0,\infty), \\ (U, U')(0) = (U_0, U_1), \end{cases} \tag{2.3}$$

where $v_0 \in D(A^{1/2})$ and $(U_0, U_1) \in D(A) \times D(A^{1/2})$. If

$$v_0 + \varepsilon U_1 = w_0, \quad U_1 + AU_0 = -w_1,$$

then one has

$$w(t) = v(t) + \varepsilon U'(t).$$

Proof. By a suitable approximation, we can assume without loss of generality that v_0, U_0 and U_1 are regular enough.

Put $\tilde{w} = v + \varepsilon U'$. Then we easily have $\tilde{w}(0) = v_0 + \varepsilon U_1 = w_0$. The equation in (2.3) gives

$$\tilde{w}' = v' + \varepsilon U'' = -(AU + U')$$

which yields $\tilde{w}'(0) = -(AU_0 + U_1) = w_1$. Moreover, one can show by the equation in (2.2) that

$$(\varepsilon w' + w)' = -(\varepsilon AU - v)' = -\varepsilon AU' - Av + F = -Aw + F.$$

The uniqueness of solutions to (2.1) provides $\tilde{w} = w$. □

Remark 2.1. In Ikehata–Nishihara [**5**], the decomposition $w = v + U'$ as also used as explained in Introduction. In [**5**], U was regarded as the solution of the first order equation $U' + AU = -u'(t)$ which includes u itself in the inhomogeneous term. From this viewpoint, the possibility of a further decomposition cannot seen.

The merit of this decomposition is the following. The problem of U has the same structure as the one of w. Therefore we can successively apply Lemma 1 to the remainder term in the following way:

$$w = v + \varepsilon U' = v + \varepsilon(v_* + \varepsilon U_*')' = v + \varepsilon(v_* + \varepsilon(v_{**} + \varepsilon U_{**}')')' = \cdots$$

which suggests the expansion

$$w = v + \varepsilon v'_* + \varepsilon^2 v''_{**} + \cdots$$

in some sense. This consideration will be used in the proof of asymptotic expansion ($\varepsilon = 1$) and also in the proof of singular limit problem ($0 < \varepsilon < 1$).

3 Asymptotic expansion

Here we give a successive derivation of arbitrary order of the asymptotic profiles of the solution u to (1.1). To shorten the notation we set $u_0^* = u_0 + u_1$. Applying Lemma 1 with $(v_0, U_0, U_1) = (u_0^*, 0, -u_1)$ to (1.1), we first have $u = V_0 + U_1'$ with

$$\begin{cases} V_0'(t) + AV_0(t) = 0, \quad t \in (0,\infty), \\ V_0(0) = u_0^*. \end{cases} \tag{3.1}$$

$$\begin{cases} U_1''(t) + AU_1(t) + U_1'(t) = -V_0'(t), \quad t \in (0,\infty), \\ (U_1, U_1')(0) = (0, -u_1). \end{cases} \tag{3.2}$$

We can find the well-known representation of the first asymptotic profile $V_0(t) = e^{-tA}u_0^*$. Then we apply Lemma 1 with $(-u_1, 0, u_1)$ to (3.2), we secondly have $U_1 = V_1 + U_2'$ with

$$\begin{cases} V_1'(t) + AV_1(t) = -V_0'(t) = Ae^{-tA}u_0^*, \quad t \in (0,\infty), \\ V_1(0) = -u_1. \end{cases} \tag{3.3}$$

$$\begin{cases} U_2''(t) + AU_2(t) + U_2'(t) = -V_1'(t), \quad t \in (0,\infty), \\ (U_2, U_2')(0) = (0, u_1). \end{cases} \tag{3.4}$$

The problem (3.3) gives $V_1(t) = -e^{-tA}u_1 + tAe^{-tA}u_0^*$ which suggests that the second asymptotic profile is given by

$$V_1'(t) = Ae^{-tA}u_1 + Ae^{-tA}u_0^* - tA^2e^{-tA}u_0^*.$$

As in the same way, we successively determine V_m ($m \geq 2$) by the respective solutions of the following problems

$$\begin{cases} V_m'(t) + AV_m(t) = -V_{m-1}'(t), \quad t \in (0,\infty), \\ V_m(0) = (-1)^m u_1. \end{cases} \tag{3.5}$$

Then by induction, we can verify the following representation of V_m for $m \in \mathbb{N}$.

Definition 1. For $m \in \mathbb{N}$, define

$$V_m(t) = (-1)^m \left[\sum_{j=1}^{m} \binom{m-1}{j-1} \frac{(-tA)^j}{j!} e^{-tA}u_0^* + (-1)^m \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-tA)^k}{k!} e^{-tA}u_1 \right];$$

note that V_m satisfies (3.5).

Moreover, by the direct calculation we reach the representations of $\frac{d^m V_m}{dt^m}$ which are nothing but the representation of all asymptotic profiles of the solution u to the problem (1.1).

Definition 2. For $m \in \mathbb{N} \cup \{0\}$, define $\overline{u}_m$ as

$$\overline{u}_0(t) = e^{-tA} u_0^*,$$
$$\overline{u}_m(t) = A^m \left[\sum_{j=0}^{m} \binom{2m-1}{m+j-1} \frac{(-tA)^j}{j!} e^{-tA} u_0^* + \sum_{k=0}^{m-1} \binom{2m-1}{m+k} \frac{(-tA)^k}{k!} e^{-tA} u_1 \right].$$

Then the following assertion holds, which describes the asymptotic expansion of solutions to the damped wave equations.

Theorem 3.1 ([**10**])**.** *Assume* $(u_0, u_1) \in [D(A^{n+\frac{1}{2}})]^2$ *for some* $n \in \mathbb{N}$. *Let* u *be the solution of* (1.1) *and let* $(\overline{u}_m)_{m \in \mathbb{N} \cup \{0\}}$ *be given in definition 2. Then there exists a positive constant* $C_n > 0$ *such that for every* $t \geq 0$,

$$\left\| u(t) - \sum_{m=0}^{n} \overline{u}_m(t) \right\|_H \leq C_m (1+t)^{-n-\frac{1}{2}} \Big(\|u_0\|_{D(A^{n+\frac{1}{2}})} + \|u_1\|_{D(A^{n+\frac{1}{2}})} \Big).$$

Since for every $m = 0, 1, \ldots, n$, one can find the following estimate for $\overline{u}_m$ as

$$\|\overline{u}_m(t)\|_H \leq \widetilde{C}_m (1+t)^{-m} \left(\|u_0^*\|_{D(A^m)} + \|u_1\|_{D(A^m)} \right)$$

for some $\widetilde{C}_m$, we can say that $\overline{u}_m$ is surely the m-th order asymptotic profile of the solution u to (1.1).

Here we already have found the decomposition

$$u(t) = \sum_{m=0}^{n} \overline{u}_m(t) + \frac{d^{n+1} U_{n+1}}{dt^{n+1}}(t)$$

where U_{n+1} is the solution of

$$\begin{cases} U_{n+1}''(t) + AU_{n+1}(t) + U_{n+1}'(t) = -V_n'(t), & t \in (0, \infty), \\ (U_{n+1}, U_{n+1}')(0) = (0, (-1)^{n+1} u_1). \end{cases} \tag{3.6}$$

Via the energy method, we can prove Theorem 3.1. To achieve this, we need to control the inhomogeneous term $-V_n'$ (given by the operator A and the C_0-semigroup e^{-tA}). The following basic estimate for the analytic semigroup e^{-tA} is crucial.

Lemma 2. *If* $f \in H$, *then for every* $n \in \mathbb{N} \cup \{0\}$ *and* $t \geq 0$,

$$\|e^{-tA} f\|^2 + \frac{2^{n+1}}{n!} \int_0^t s^n \|A^{\frac{n+1}{2}} e^{-sA} f\|^2 \, ds = \|f\|^2. \tag{3.7}$$

Moreover, if $f \in D(A^{n/2})$, *then there exists a positive constant* C *(depending only* n*) such that*

$$\int_0^t (1+s)^n \|A^{\frac{n+1}{2}} e^{-sA} f\|^2 \, ds \leq C(\|f\|^2 + \|A^{n/2} f\|^2).$$

Sketch of the proof of Theorem 3.1. We only demonstrate the usual energy estimate for U_{n+1}. To simplify the notation, we use $U = U_{n+1}$ and $V = V_n$. The computation is the following:

$$\begin{aligned}
\frac{d}{dt}\Big[\|A^{1/2}U\|_H^2 + \|U' + U\|_H^2\Big] &= 2\big(U', AU\big)_H + 2\big(U'' + U', U' + U\big)_H \\
&= 2\big(U', AU\big)_H + 2\big(- AU - V', U' + U\big)_H \\
&= -2\|A^{1/2}U\|_H^2 - 2\big(V', U'\big)_H - 2\big(V', U\big)_H \\
&\leq -\|A^{1/2}U\|_H^2 + \|U'\|_H^2 + \|A\widetilde{V}\|_H^2 + \|A^{1/2}\widetilde{V}\|_H^2, \\
\frac{d}{dt}\Big[\|A^{1/2}U\|_H^2 + \|U'\|_H^2\Big] &= 2\big(U', U'' + AU\big)_H \\
&= -2\|U'\|_H^2 - 2\big(U', V'\big)_H \\
&= -\|U'\|_H^2 + \|A\widetilde{V}\|_H^2,
\end{aligned}$$

where we have used the equation in (3.6) and the fact $V' = A\widetilde{V}$ for some $\widetilde{V}$. These inequalities imply that

$$\begin{aligned}
&\frac{d}{dt}\Big[(4+t)\Big(\|A^{1/2}U\|_H^2 + \|U'\|_H^2\Big) + 2\|A^{1/2}U\|_H^2 + 2\|U' + U\|_H^2\Big] \\
&\leq -(1+t)\|U'\|_H^2 - \|A^{1/2}U\|_H^2 + (6+t)\|A\widetilde{V}\|_H^2 + 2\|A^{1/2}\widetilde{V}\|_H^2.
\end{aligned}$$

The integrability of $(1+t)\|A\widetilde{V}\|_H^2$ and $\|A^{1/2}\widetilde{V}\|_H^2$ (by Lemma 2) shows the estimate

$$\sup_{t\geq 0}\Big((1+t)\|U'(t)\|_H^2\Big) + \int_0^\infty (1+t)\|U'(t)\|_H^2\,dt < +\infty,$$

which is the end of estimate for U'. To reach the estimate for $\frac{d^{n+1}U}{dt^{n+1}}$, we argue by induction with a similar energy method for $\frac{d^\ell U}{dt^\ell}$ $(\ell = 1, \ldots n)$ by computing the derivative of

$$\begin{gathered}
(a_\ell + t)^{2\ell+1}\Big(\Big\|A^{1/2}\frac{d^\ell U}{dt^\ell}\Big\|_H^2 + \Big\|\frac{d^{\ell+1}U}{dt^{\ell+1}}\Big\|_H^2\Big), \\
(a_\ell + t)^{2\ell}\Big(\Big\|A^{1/2}\frac{d^\ell U}{dt^\ell}\Big\|_H^2 + \Big\|\frac{d^{\ell+1}U}{dt^{\ell+1}} + \frac{d^\ell U}{dt^\ell}\Big\|_H^2\Big)
\end{gathered}$$

with suitable positive constants a_ℓ. As a result, we obtain

$$\sup_{t\geq 0}\Big((1+t)^{2n+1}\Big\|\frac{d^{n+1}U}{dt^{n+1}}(t)\Big\|_H^2\Big) < +\infty$$

which is the desired estimate. □

4 Singular limit problem

The content of this section is based on [**6**]. The problem in this section is the singular limit problem (1.6). From the viewpoint of the decomposition in Lemma 1, we reprove

Theorem 1.1. It should be noticed that if we proceed the same strategy in Section 3 for the singular limit problem (1.7), then we will apply Lemma 1 with $(u_0 + \varepsilon u_1, 0, -u_1)$. This implies

$$\|u_\varepsilon(t) - e^{-tA}(u_0 + \varepsilon u_1)\|_H \le C\varepsilon^{1/2}\big(\|u_0\|_{D(A^{1/2})} + \|u_1\|_{D(A^{1/2})}\big)$$

which is properly weaker than the first assertion in Theorem 1.1. In the other words, we cannot deal with the initial data (u_0, u_1) in the energy space $D(A^{1/2}) \times H$ in this treatment.

To fill the gap, we introduce the resolvent operator $J_\varepsilon = (1+\varepsilon A)^{-1}$, that is, $h = J_\varepsilon g$ is the solution of the equation $h+\varepsilon Ah = g$. Then we apply Lemma 1 with $(v_0, U_0, U_1) = (u_0 + \varepsilon J_\varepsilon u_1, -\varepsilon J_\varepsilon u_1, -J_\varepsilon u_1)$. This enables us to find the relation $u_\varepsilon = v_\varepsilon + \partial_t U_\varepsilon$ with

$$\begin{cases} v_\varepsilon'(t) + Av_\varepsilon(t) = 0, & t \in (0,\infty), \\ v_\varepsilon(0) = u_0 + \varepsilon J_\varepsilon u_1, \end{cases} \tag{4.1}$$

$$\begin{cases} \varepsilon U_\varepsilon''(t) + AU_\varepsilon(t) + U_\varepsilon'(t) = -v_\varepsilon'(t) = Av_\varepsilon(t), & t \in (0,\infty), \\ (U_\varepsilon, U_\varepsilon')(0) = (-\varepsilon J_\varepsilon u_1, -J_\varepsilon u_1). \end{cases} \tag{4.2}$$

Since the resolvent equation shows the estimate

$$\|J_\varepsilon g\|_H^2 + \varepsilon\|A^{1/2}J_\varepsilon g\|_H^2 \le \|g\|_H^2,$$

the following estimate for the solution $v_\varepsilon = e^{-tA}(u_0 + \varepsilon J_\varepsilon u_1)$ of (4.1) holds via Lemma 2:

$$2\int_0^\infty \|Av_\varepsilon(t)\|_H^2\,dt \le \|A^{1/2}v_\varepsilon(0)\|_H^2 \le \Big(\|A^{1/2}u_0\|_H + \varepsilon^{1/2}\|u_1\|_H\Big)^2.$$

Moreover, by the equation in (4.2) we have

$$\begin{aligned} \frac{d}{dt}\Big(\varepsilon\|U'(t)\|_H^2 + \|A^{1/2}U(t)\|_H^2\Big) &= 2\big(U'(t), \varepsilon U''(t) + AU(t)\big)_H \\ &= 2\big(U'(t), -U'(t) + Av_\varepsilon(t)\big)_H \\ &\le \frac{1}{2}\|Av_\varepsilon(t)\|_H^2 \end{aligned}$$

which implies

$$\begin{aligned} \varepsilon\|U'(t)\|_H^2 + \|A^{1/2}U(t)\|_H^2 &\le \varepsilon\|J_\varepsilon u_1\|_H^2 + \varepsilon^2\|A^{1/2}J_\varepsilon u_1\|_H^2 + \frac{1}{4}\Big(\|A^{1/2}u_0\|_H + \varepsilon^{1/2}\|u_1\|_H\Big)^2 \\ &\le \frac{9}{4}\Big(\|A^{1/2}u_0\|_H + \varepsilon^{1/2}\|u_1\|_H\Big)^2. \end{aligned}$$

Consequently, we obtain the first estimate in Theorem 1.1:

$$\begin{aligned} \|u(t) - e^{-tA}u_0\|_H &\le \|u(t) - v_\varepsilon(t)\|_H + \varepsilon\|e^{-tA}J_\varepsilon u_1\|_H \\ &= \varepsilon\|U'(t)\|_H + \varepsilon\|e^{-tA}J_\varepsilon u_1\|_H \\ &\le \frac{5}{2}\Big(\varepsilon^{1/2}\|A^{1/2}u_0\|_H + \varepsilon\|u_1\|_H\Big). \end{aligned}$$

For the second estimate for $(u_0,u_1)\in D(A^{3/2})\times D(A^{1/2})$, we divide the solution u_ε as $u_\varepsilon = u_{1\varepsilon}+u_{2\varepsilon}$ with

$$\begin{cases}\varepsilon u_{1\varepsilon}''(t)+Au_{1\varepsilon}(t)+u_{1\varepsilon}'(t)=0, & t\in(0,\infty),\\ (u_{1\varepsilon},u_{1\varepsilon}')(0)=(u_0,-Au_0),\end{cases}\tag{4.3}$$

$$\begin{cases}\varepsilon u_{2\varepsilon}''(t)+Au_{2\varepsilon}(t)+u_{2\varepsilon}'(t)=0, & t\in(0,\infty),\\ (u_{2\varepsilon},u_{2\varepsilon}')(0)=(0,g),\end{cases}\tag{4.4}$$

where we put $g=Au_0+u_1$ for short. Since typical terms related to the initial layer do not appear in the solution $u_{1\varepsilon}$ of (4.3), we shall only explain the strategy for the analysis of the other solution $u_{2\varepsilon}$. By using Lemma 1 with $(\varepsilon J_\varepsilon g,-\varepsilon J_\varepsilon g,-J_\varepsilon g)$ to (4.4), we have $u_{2\varepsilon}=\varepsilon e^{-tA}J_\varepsilon g+\varepsilon\widetilde{U}_{2\varepsilon}'$ with

$$\begin{cases}\varepsilon \widetilde{U}_{2\varepsilon}''(t)+A\widetilde{U}_{2\varepsilon}(t)+\widetilde{U}_{2\varepsilon}'(t)=\varepsilon Ae^{-tA}J_\varepsilon g, & t\in(0,\infty),\\ (\widetilde{U}_{2\varepsilon},\widetilde{U}_{2\varepsilon}')(0)=(-\varepsilon J_\varepsilon g,-J_\varepsilon g).\end{cases}\tag{4.5}$$

Then setting

$$V_{2\varepsilon}(t)=-2e^{-tA}J_\varepsilon g+tAe^{-tA}J_\varepsilon g$$

(in view of the same manner in Lemma 1), we find that $U_{2\varepsilon}=\widetilde{U}_{2\varepsilon}-\varepsilon V_{2\varepsilon}$ satisfies

$$\begin{cases}\varepsilon U_{2\varepsilon}''(t)+AU_{2\varepsilon}(t)+U_{2\varepsilon}'(t)=-\varepsilon V_{2\varepsilon}''(t), & t\in(0,\infty),\\ (U_{2\varepsilon},U_{2\varepsilon}')(0)=(\varepsilon J_\varepsilon g,-g).\end{cases}\tag{4.6}$$

This suggests the relation $U_{2\varepsilon}+u_{2\varepsilon}=\varepsilon W_{2\varepsilon}$, where $w_{2\varepsilon}$ is the solution of

$$\begin{cases}\varepsilon W_{2\varepsilon}''(t)+AW_{2\varepsilon}(t)+W_{2\varepsilon}'(t)=-V_{2\varepsilon}''(t), & t\in(0,\infty),\\ (W_{2\varepsilon},W_{2\varepsilon}')(0)=(J_\varepsilon g,0).\end{cases}\tag{4.7}$$

Consequently, connecting the above decomposition, we arrive that

$$\begin{aligned}u_{2\varepsilon}(t)&=\varepsilon e^{-tA}J_\varepsilon g+\varepsilon\widetilde{U}_{2\varepsilon}'(t)\\ &=\varepsilon e^{-tA}J_\varepsilon g+\varepsilon V_{2\varepsilon}'(t)+\varepsilon U_{2\varepsilon}'(t)\\ &=\varepsilon e^{-tA}J_\varepsilon g+\varepsilon V_{2\varepsilon}'(t)-\varepsilon u_{2\varepsilon}'(t)+\varepsilon^2 W_{2\varepsilon}'(t)\end{aligned}$$

which can be regarded as the first order differential equation for $u_{2\varepsilon}$. By estimating $W_{2\varepsilon}'$ via the energy method and solving this equation, we can reach

Theorem 4.1 ([**6**]). *Let $u_{2\varepsilon}$ be the solution of* (4.4). *Then there exists a positive constant C such that for every $t\geq 0$,*

$$\|u_{2\varepsilon}(t)-\varepsilon(e^{-tA}g-e^{-t/\varepsilon}g)\|_H\leq C\varepsilon^{3/2}\|A^{1/2}g\|_H.$$

Although the estimate in Theorem 4.1 differs from the second estimate in Theorem 1.1, it can be seen (and can be verified) that the derivative of the profile (of order ε) satisfies

$$\frac{d}{dt}\Big[\varepsilon(e^{-tA}g-e^{-t/\varepsilon}g)\Big]=-\varepsilon Ae^{-tA}g+e^{-t/\varepsilon}g=e^{-t/\varepsilon}g+O(\varepsilon)$$

which is nothing but the initial layer term found by Kisynski.

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