ON LIPSCHITZ REGULARITY FOR LEVEL-SET FORCED MEAN CURVATURE FLOW UNDER THE NEUMANN BOUNDARY CONDITION

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Abstract. Here, we study a level-set forced mean curvature flow with the homogeneous Neumann boundary condition. We show that the solution is Lipschitz in time and locally Lipschitz in space. Then, under an additional condition on the forcing term, we prove that the solution is globally Lipschitz continuous, and we obtain the large time behavior of the solution in this setting. Also, we give an example to demonstrate that the additional condition on the forcing term is sharp, and without it, the solution might not be globally Lipschitz continuous.

The main purpose of this proceeding is to briefly and simply describe some results in [13], which have recently been obtained jointly with J. Jang, D. Kwon, H. V. Tran, on level-set forced mean curvature flow equations under the Neumann boundary condition.

1. Introduction

In this paper, we study the level-set equation for the forced mean curvature flow

$$\begin{cases} u_t = |Du| \operatorname{div}\left(\dfrac{Du}{|Du|}\right) + c(x)|Du| & \text{in } \Omega \times (0,\infty), \quad (1.1) \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0,\infty), \quad (1.2) \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}. \quad (1.3) \end{cases}$$

The domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is assumed to be bounded and $C^{2,\theta}$ for some $\theta \in (0,1)$. Here, $c = c(x)$ is a forcing term, which is in $C^1(\overline{\Omega})$, and $\vec{\mathbf{n}}$ is the outward unit normal vector to $\partial\Omega$. Throughout this paper, we assume that $u_0 \in C^{2,\theta}(\overline{\Omega})$, and $\frac{\partial u_0}{\partial \vec{\mathbf{n}}} = 0$ on $\partial\Omega$ for compatibility.

We first notice that the well-posedness and the comparison principle for (1.1)–(1.3) are well established in the theory of viscosity solutions (see [1, 2, 7, 8] for instance). Our main interest in this paper is to go beyond the well-posedness theory to understand the Lipschitz regularity and large time behavior of the solution. The Lipschitz regularity for

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the solution is rather subtle because of the competition between the forcing term and the mean curvature term together with the constraint on perpendicular intersections of the level sets of the solution with the boundary of Ω. It is worth emphasizing that the geometry of $\partial\Omega$ plays a crucial role in the analysis.

We now describe our main results. First of all, we show that u is Lipschitz in time and locally Lipschitz in space.

Theorem 1.1. *Let u be the unique viscosity solution u of* (1.1)–(1.3)*. Then, there exists a constant $M > 0$ and for each $T > 0$, there exists a constant $C_T > 0$ depending on T such that*

$$\begin{cases} |u(x,t) - u(x,s)| \leqslant M|t-s|, \\ |u(x,t) - u(y,t)| \leqslant C_T|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,T].$$

We next show that if we put some further conditions on the forcing term c, then we have the global Lipschitz estimate in x of the solution. Denote by

$$\begin{cases} C_0 := \max\{-\lambda \,:\, \lambda \text{ is a principal curvature of } \partial\Omega \text{ at } x_0 \text{ for } x_0 \in \partial\Omega\} \in \mathbb{R}, \\ K_0 := \min\{d \,:\, d \text{ is the diameter of an open ball inscribed in } \Omega\} > 0. \end{cases}$$

Theorem 1.2. *Assume that there exists $\delta > 0$ such that*

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > \max\left\{0,\ C_0|c(x)| + \frac{2nC_0}{K_0}\right\} \quad \text{for all } x \in \Omega. \tag{1.4}$$

Let u be the unique viscosity solution to (1.1)–(1.3)*. Then, there exist constants $M, L > 0$ depending only on the forcing term c and the initial data u_0 such that*

$$\begin{cases} |u(x,t) - u(x,s)| \leqslant M|t-s|, \\ |u(x,t) - u(y,t)| \leqslant L|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,\infty). \tag{1.5}$$

Let us now explain a bit the geometric meaning of K_0. For each $x \in \partial\Omega$, let

$$K_x = \max\{2r > 0 \,:\, B(x - r\vec{\mathbf{n}}(x), r) \subset \Omega\}.$$

Then, $K_0 = \min_{x\in\partial\Omega} K_x$. We notice next that if Ω is convex in Theorem 1.2, then we clearly have $C_0 \le 0$. In this case, (1.4) becomes $\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > 0$, a kind of coercive assumption, which often appears in the usage of the classical Bernstein method to obtain Lipschitz regularity (see [14] for instance).

In the specific case where $c \equiv 0$ and Ω is convex and bounded, the global Lipschitz estimate of the solution was obtained in [6]. Moreover, a very interesting example was given in [6] to show that the solution is not globally Lipschitz continuous if Ω is not convex. Motivated by this example, we give two examples in [13] showing that u is not globally Lipschitz continuous if we do not impose (1.4). Furthermore, the examples demonstrate that condition (1.4) is sharp.

Let us note that the graph mean curvature flow with the Neumann boundary conditions has been studied much in the literature (see [10, 11, 15] and the references therein).

In Section 2 we give an example to show the non global Lipschitz phenomenon in Theorem 2.1 which demonstrates that condition (1.4), which is needed for the global Lipschitz regularity of u. .

As an application of Theorem 1.2 we establish the large time behavior of u under condition (1.4) by using the Lyapunov function method.

Theorem 1.3. *Assume* (1.4). *Let u be the unique viscosity solution to* (1.1)–(1.3). *Then,*

$$u(\cdot,t) \to v, \qquad \textit{as } t \to \infty,$$

uniformly on $\overline{\Omega}$ for some Lipschitz function v, which is a viscosity solution to

$$\begin{cases} -\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + c(x)\right)|Dv| = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We call v which is given by Theorem 1.3 the *large time profile* of the solution u. It is important to note that the stationary problem (1.6) may have various different solutions, and thus, the question on how the large time profile v depends on the initial data u_0 is rather delicate and challenging. This is still rather open expect for the radially symmetric case (see [13] for details).

We refer to [13] for the proofs of Theorems 1.1, 1.2 and 1.3.

Our problem (1.1)–(1.2) basically describes a level-set forced mean curvature flow with the homogeneous Neumann boundary condition. If a level set of the unknown u is a smooth enough surface, then it evolves with the normal velocity $V(x) = \kappa + c(x)$, where κ equals $(n-1)$ times the mean curvature of the surface at x, and it perpendicularly intersects $\partial\Omega$ (if ever). What is really interesting and delicate here is the competition between the forcing term $c(x)$ and the mean curvature term κ coupled with the constraint on perpendicular intersections of the level sets with the boundary. It is worth emphasizing that we do not assume Ω is convex, and the geometry of $\partial\Omega$ plays a crucial role in the behavior of the solution here. Indeed, analyzing the competition between the two constraints, the force and the boundary condition subjected to $\partial\Omega$, as time evolves in viscosity sense is the main topic of this paper.

We now briefly describe our approaches to get the aforementioned results. We use the maximum principle and rely on the classical Bernstein method to establish *a priori* gradient estimates for the solution. The main difficulty is when a maximizer is located on the boundary, which we cannot apply the maximum principle directly. We deal with this difficulty by considering a multiplier that puts the maximizer, with the homogeneous Neumann boundary condition, inside the domain so that the maximum principle is applicable. To the best of our knowledge, the idea of handling a maximizer in the proof of Theorem 1.2 for the level-set equation for forced mean curvature flows under the Neumann boundary condition is new in the literature. Once we get a global Lipschitz estimate for the solution, by using a standard Lyapunov function, we prove the convergence in Theorem 1.3.

2. The gradient growth as time tends to infinity in two dimensions

Let $n = 2$. Let the forcing term c be a positive constant in Ω, that is, $c(x) = c$ for all $x \in \overline{\Omega}$ for some $c > 0$. Consider the following nonconvex domain,

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < f(x_1)\}, \tag{2.1}$$

where $f(x) = \frac{m}{2}x^2 + k$ for fixed $m > 0$ and $k > 0$. Here, Ω is unbounded.

In this unbounded setting, let $R_0 > 0$ be a sufficiently large constant. Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded $C^{2,\theta}$ domain such that

$$\Omega \cap B(0, R_0) \subset \widetilde{\Omega} \subset \Omega.$$

We say that u is a solution (resp., subsolution, supersolution) of (1.1)–(1.3) on $\overline{\Omega} \times [0,\infty)$ if there exists $\alpha \in \mathbb{R}$ such that

$$u - \alpha = u_0 - \alpha = 0 \quad \text{on } (\overline{\Omega} \setminus B(0, R_0)) \times [0,\infty), \tag{2.2}$$

and u is a solution (resp., subsolution, supersolution) of (1.1)–(1.3) with $\widetilde{\Omega}$ in place of Ω.

Let u be the solution to (1.1)–(1.3). If a level set of u is a smooth curve, then it is evolved by the forced curvature flow equation $V = \kappa + c$, where V is the normal velocity and κ is the curvature in the direction of the normal. Then, the classical Neumann boundary condition becomes the right angle condition for the level-set curves with respect to $\partial\Omega$, that is, if a smooth level curve and $\partial\Omega$ intersect, then their normal vectors are perpendicular at the points of intersections.

We show that if c is too small and fails to satisfy (1.4), then there exist discontinuous viscosity solutions to (1.6). In particular, we find that one such discontinuous solution of (1.6) is stable in the sense that the solution of (1.1)–(1.3) with a suitable choice of initial data converges to this discontinuous stationary solution as time goes to infinity. This implies that the global Lipschitz estimate for the solution of (1.1)–(1.3) does not hold. The following is the main result of this section.

Theorem 2.1. *Let Ω be the set given by (2.1), and $c(x) = c$ for all $x \in \overline{\Omega}$ for $c \in (0, r_{\min}^{-1})$, where $r_{\min}$ is defined by (2.7). Let $u \in C(\overline{\Omega} \times [0,\infty))$ be the solution of (1.1)–(1.3) with the given initial data $u_0 \in C^{2,\theta}(\overline{\Omega})$ satisfying that $\frac{\partial u_0}{\partial \vec{\mathbf{n}}} = 0$ on $\partial\Omega$ and there exist constants l_1, l_2, α and β such that $l_1 \in (0, a_1)$, $l_2 \in (0, a_2 - a_1)$, $\alpha < \beta$,*

$$u_0(x) = \begin{cases} \beta & \text{for } x = (x_1, x_2) \in U(a_1 - l_1), \\ \alpha & \text{for } x = (x_1, x_2) \in \overline{\Omega} \setminus \overline{U(a_1 + l_2)}, \end{cases} \tag{2.3}$$

and $\alpha \leqslant u_0 \leqslant \beta$, where $U(a)$ is defined by (2.6) for $a > 0$, and $0 < a_1 < a_2$ is given in Theorem 2.4. Then,

$$\lim_{t\to\infty} u(x,t) = \begin{cases} \beta & \text{if } x \in U(a_1), \\ \alpha & \text{if } x \in \overline{\Omega} \setminus \overline{U(a_1)}. \end{cases}$$

2.1. **Set-theoretic stationary solutions.** For $a > 0$, consider a family of curves with constant curvature in Ω,

$$X(a,\theta) = (X_1(a,\theta), X_2(a,\theta)) = p(a) + r(a)(\cos\theta, \sin\theta), \quad |\theta| < \arctan(ma), \tag{2.4}$$

where we choose $p(a)$, $r(a)$ so that the curve

$$\Gamma := \{(X_1(a,\theta), X_2(a,\theta)) \,:\, |\theta| < \arctan(ma)\} \cup \{(-X_1(a,\theta), X_2(a,\theta)) \,:\, |\theta| < \arctan(ma)\}$$

has a constant curvature, and is perpendicular to the boundary $\partial\Omega$. Indeed, set

$$p(a) := \left(\frac{a}{2} - \frac{k}{ma}, 0\right).$$

Then, we see that the tangent line for $\{(x_1, x_2) \mid x_2 = f(x_1)\}$ at $x_1 = a$ goes through $p(a)$. Moreover, setting

$$r(a) := \left|\left(a, \frac{ma^2}{2} + k\right) - p(a)\right| = \left(\frac{a}{2} + \frac{k}{ma}\right)\sqrt{m^2a^2 + 1},$$

by elementary geometry, we can check that

$$\Gamma \perp \partial\Omega.$$

The parameter a will be specified so that

$$c = \frac{1}{r(a)}$$

in Lemma 2.3.

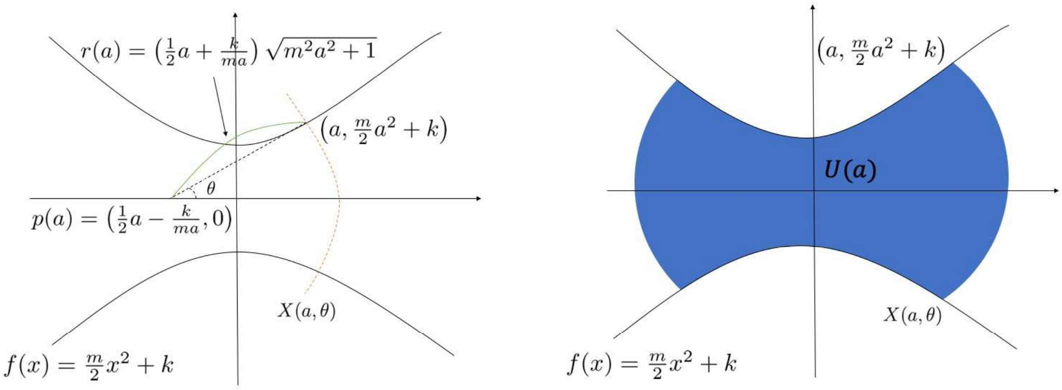

FIG. 1. Illustrations of (2.4) and (2.6)

The following definition is taken from [3, Definition 5.1.1].

Definition 2.2. *Let G be a set in $\mathbb{R}^n \times J$, where J is an open interval in $(0, T)$. We say that G is a set-theoretic subsolution (resp., supersolution) of*

$$V = \kappa + c \quad \text{on } \Gamma_t \quad \text{with} \quad \Gamma_t \perp \partial\Omega \tag{2.5}$$

if χ_G^* is a viscosity subsolution (resp., $(\chi_G)_*$ is a viscosity supersolution) of (1.1)–(1.2) *in $\mathbb{R}^n \times J$, where $\chi_G(x,t) = 1$ if $(x,t) \in G$, and $\chi_G(x,t) = 0$ if $(x,t) \notin G$, and χ_G^* and*

$(\chi_G)_*$ *denote the upper semicontinuous envelope and the lower semicontinuous envelope of* χ_G*, respectively. If* G *is both a set-theoretic subsolution and supersolution of* (2.5)*,* G *is called a set-theoretic solution of* (2.5)*.*

Set

$$U(a) := \{(x_1, x_2) \in \Omega : |x_1| < X_1(a, \theta), |x_2| < X_2(a, \theta), |\theta| < \arctan(ma)\}, \tag{2.6}$$

and

$$r_{\min} := \inf\{r(a) : a > 0\}. \tag{2.7}$$

Then, $r_{\min}$ is positive since r is a continuous positive function in $(0, \infty)$ and

$$\lim_{a\to 0} r(a) = \lim_{a\to\infty} r(a) = \infty. \tag{2.8}$$

Moreover, by direct computation, we have

$$r'(a) = \frac{1}{\sqrt{m^2a^2+1}}\left(m^2a^2 + \frac{1}{2} - \frac{k}{ma^2}\right).$$

Therefore, r has only one critical point $a_* = \frac{1}{2m}\sqrt{-1+\sqrt{1+16mk}}$ in $(0, \infty)$ and $r_{\min} = r(a_*)$. In addition,

$$r'(a) < 0 \text{ if } a < a_*, \text{ and } r'(a) > 0 \text{ if } a > a_*. \tag{2.9}$$

Lemma 2.3. *If* $c = \frac{1}{r(a)}$ *for some* $a > 0$*, then* $U(a)$ *is a set-theoretic stationary solution of* (1.1)–(1.2)*.*

Proof. As a consequence of the nice characterization of set-theoretic solutions in [3, Theorem 5.1.2], $U(a)$ is a set-theoretic stationary solution of (2.5) if and only if $0 = \kappa + c$ on $\partial U(a) \cap \Omega$ and the right angle condition holds. The equality follows from the fact that $\partial U(a) \cap \Omega$ contains two arcs of two circles of the same radius $r(a)$ and curvature $\kappa = -r(a)^{-1} = -c$.

On the other hand, these arcs intersect with $\partial\Omega$ at four points $(a, \pm f(a))$, $(-a, \pm f(a))$. By symmetry, it suffices to prove the right angle condition at $(a, f(a))$. Notice that

$$(a, f(a)) = (X_1(a, \arctan(ma)), X_2(a, \arctan(ma))) = p(a) + \frac{r(a)}{\sqrt{m^2a^2+1}} \cdot (1, ma).$$

Therefore, the line joining $(a, f(a))$ and $p(a)$, the center of the arc, is tangent to $\partial\Omega$ at $(a, f(a))$. Thus, $\partial U(a) \cap \Omega$ satisfies the right angle condition at $(a, f(a))$. □

Theorem 2.4. *If* $c \in (0, \frac{1}{r_{\min}})$*, then there exist two positive constants* $a_1 < a_2$ *such that* $U(a_i)$ *is a set-theoretic stationary solution of* (2.5) *for* $i = 1, 2$*.*

Proof. Thanks to (2.7)–(2.9), there exist two positive constants a_1, a_2 with $a_1 < a_* < a_2$ such that

$$r(a_1) = r(a_2) = \frac{1}{c}. \tag{2.10}$$

By Lemma 2.3, $U(a_i)$ is a set-theoretic stationary solution of (2.5) for $i = 1, 2$. □

2.2. **Stability.** Let a_i be the constants given by Theorem 2.4 for $i = 1, 2$. In this section, we prove that $U(a_1)$ given by (2.6) is a set-theoretic solution which is stable in the sense of Theorem 2.1.

Lemma 2.5. *Let $l_1 \in (0, a_1)$, $l_2 \in (0, a_2 - a_1)$ and $\delta > 0$. Set $\underline{a}(t) := a_1 - l_1 e^{-\delta t}$ and $\overline{a}(t) := a_1 + l_2 e^{-\delta t}$. There exists $\delta_0 = \delta_0(m, k, l_1, l_2)$ such that $U(\underline{a}(t))$ and $U(\overline{a}(t))$ are a set-theoretic subsolution and supersolution to (2.5) for all $\delta \in (0, \delta_0)$, respectively.*

Proof. We only prove that $U(\underline{a}(t))$ is a set-theoretic subsolution, since we can similarly prove that $U(\overline{a}(t))$ is a set-theoretic supersolution. Let $\tilde{X}(t) := X(\underline{a}(t), \theta)$. From the characterization of set-theoretic solutions in [3, Theorem 5.1.2], it suffices to show that for $t \geqslant 0$,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} \leqslant -\frac{1}{r(\underline{a}(t))} + c \quad \text{for all } t > 0, \tag{2.11}$$

where $\vec{\mathbf{n}}$ is the outward normal vector $\vec{\mathbf{n}}$ of $U(\underline{a}(t))$, that is, $\vec{\mathbf{n}} = (\cos\theta, \sin\theta)$.

Note that

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \frac{\partial \underline{a}}{\partial t}\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta l_1 e^{-\delta t}\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta(a_1 - \underline{a}(t))\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}}.$$

Also, for any constant $L > 0$, there exists $C = C(m, k, L) > 0$ such that

$$\begin{aligned}\frac{\partial X}{\partial a}(a, \theta) \cdot \vec{\mathbf{n}} &= p'(a) \cdot \vec{\mathbf{n}} + r'(a) \leq |p'(a)| + r'(a) \\ &= \frac{1}{2} + \frac{m^2 a^2 + \frac{1}{2}}{\sqrt{m^2 a^2 + 1}} + \frac{mk}{m^2 a^2 + 1 + \sqrt{m^2 a^2 + 1}} \leq C\end{aligned}$$

for all $a \in (0, L)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \delta(a_1 - \underline{a}(t))\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} \leqslant C\delta(a_1 - \underline{a}(t)).$$

The observation (2.9) implies that $r(\underline{a}(t)) > r(a_1) = c^{-1}$ for all $t \geqslant 0$, and thus we get

$$\left(\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}}\right)\left(-\frac{1}{r(\underline{a}(t))} + c\right)^{-1} \leqslant \delta C \frac{a_1 - \underline{a}(t)}{\frac{1}{r(a_1)} - \frac{1}{r(\underline{a}(t))}}.$$

Thus, (2.11) holds for $\delta \in (0, \delta_0)$, where

$$\delta_0 := \left(C \sup_{a \in [a_1 - l_1, a_1 + l_2]} h(a)\right)^{-1}.$$

Here the function $h : [a_1 - l_1, a_1 + l_2] \to \mathbb{R}$ is given by

$$h(a) := \begin{cases} \dfrac{a_1 - a}{\frac{1}{r(a_1)} - \frac{1}{r(a)}} & \text{for } a \in [a_1 - l_1, a_1 + l_2] \setminus \{a_1\}, \\[2ex] \dfrac{-r^2(a_1)}{r'(a_1)} & \text{for } a = a_1. \end{cases}$$

Since $a_1+l_2 < a_2$, by (2.9) we have $r(a) \neq r(a_1)$ in $[a_1-l_1, a_1+l_2]\setminus\{a_1\}$ and $r'(a_1) < 0$. Therefore, h is well-defined and continuous in $[a_1-l_1, a_1+l_2]$. Thus, h is bounded in $[a_1-l_1, a_1+l_2]$, and hence, $\delta_0 > 0$ is well-defined, which implies that (2.11) holds for all $\delta \in (0, \delta_0)$. □

Proof of Theorem 2.1. We let $\alpha = 0$ and $\beta = 1$ for simplicity. Set

$$\underline{u}(x,t) := \chi_{\overline{U(\underline{a}(t))}}(x) \quad \text{and} \quad \overline{u}(x,t) := \chi_{U(\overline{a}(t))}(x)$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$, where $\underline{a}$ and $\overline{a}$ are the functions defined in Lemma 2.5. By Lemma 2.5, we see that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of (1.1)–(1.2), respectively. Due to (2.3), we get

$$\underline{u}(\cdot,0) = \chi_{\overline{U(\underline{a}(0))}} \leqslant u_0 \leqslant \chi_{U(\overline{a}(0))} = \overline{u}(\cdot,0) \text{ on } \overline{\Omega}.$$

In addition, since

$$U(a) \subset V(a) := [-(|p(a)| + r(a)), |p(a)| + r(a)] \times [-f(a), f(a)]$$

by construction for $p(a)$ and $r(a)$ given in (2.4) and $f(a) = \frac{m}{2}a^2 + k$, we obtain

$$\operatorname{supp}(\underline{u}) \subset \bigcup_{a\in[a_1-l_1,a_1]} V(a) \times [0,\infty) \text{ and } \operatorname{supp}(\overline{u}) \subset \bigcup_{a\in[a_1,a_1+l_2]} V(a) \times [0,\infty).$$

As $|p(\cdot)| + r(\cdot)$ and f are continuous on $[a_1-l_1, a_1+l_2]$, there exists a constant $R_0 > 0$ satisfying (2.2).

By the comparison principle for (1.1)–(1.3), we get

$$\underline{u}(\cdot,t) \leqslant u(\cdot,t) \leqslant \overline{u}(\cdot,t) \quad \text{on } \overline{\Omega} \quad \text{for all } t > 0.$$

On the other hand, since both $a_1 - l_1e^{-\delta t}$ and $a_1 + l_2e^{-\delta t}$ converge to a_1 as t goes to infinity,

$$\lim_{t\to\infty} \underline{u}(x,t) = \lim_{t\to\infty} \overline{u}(x,t) = 1 \quad \text{for } x \in U(a_1),$$

and

$$\lim_{t\to\infty} \underline{u}(x,t) = \lim_{t\to\infty} \overline{u}(x,t) = 0 \quad \text{for } x \in \overline{\Omega} \setminus \overline{U(a_1)},$$

which finish the proof. □

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