An extension of the Floquet-Bloch theory to nilpotent groups and its applications

By

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Abstract

The Floquet-Bloch theory is a popular tool for the investigation of materials with periodic structures. For example, one can show that the spectrum of periodic Schrödinger operators have band structures. In the context of this note, this theory was applied to the following problems in the case of abelian extensions:

(1) A geometric analogue of the Chebotarev density theorem for prime closed geodesics in a compact Riemannian manifold with negative curvature

(2) A long time asymptotic expansion of the heat kernels of covering manifolds of compact Riemannian manifolds.

In this note, we shall develop our version of non-commutative Floquet-Bloch theory and give applications to these problems for nilpotent groups with emphasis on the second topic.

Moreover, as a by-product, we give another mathematical explanation of the semi-classical asymptotic expansion formula for the Harper operator due to Wilkinson, which is originally done by Helffer-Sjöstrand.

§ 1. Introduction

Our main concern in this note is an extension of the Floquet-Bloch theory to discrete nilpotent groups Γ.

We shall explain the above theory through its application to long time asymptotics of the heat kernels on Γ-covering spaces $X \to M = X/\Gamma$ of compact Riemannian manifold or a finite unoriented graph M. We denote by $k_X(t, p, q)$ the heat kernel (resp. the transition probability of simple random walks) on X when M is a compact Riemannian manifold (resp. a finite graph). For the brevity of the description of results, we assume that M is not a bipartite graph.

Our problem here is as follows:

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Problem 1.1. What is the asymptotic behavior of $k_X(t,p,q)$ as $t \to \infty$?

In the case when Γ is an abelian group, the following results are known.

Theorem 1.2. *If Γ is an abelian group of rank d, then we have*

$$k_X(t,p,q) \sim \frac{C}{t^{d/2}}\left(1+\frac{c_1}{t}+\frac{c_2}{t^2}+\cdots\right), \tag{1.1}$$

where $C, c_1, c_2, \dots$ are constants depending on the geometry of M.

This result is due to Kotani and Sunada [29]. The leading term of the above theorem is obtained by several authors [16], [25], [30] and [36], [28] independently and [21], [22] for non-symmetric random walks.

The following is one of our main results in application.

Theorem 1.3. *Let Γ be a finitely generated torsion-free nilpotent group.*

$$k_X(t,p,q) \sim \frac{C}{t^{d/2}}\left(1+\frac{c_1}{t}+\frac{c_2}{t^2}+\cdots\right),$$

where C can be computed in terms of $\zeta_H(d/2)$ the special value at $s = d/2$ of the spectral zeta function $\zeta_H(s)$ of a hypo-elliptic operator H and geometry of base space M. The constant d is the polynomial growth order of Γ and $c_1, c_2, \dots$ are also expressed in terms of Chen's iterated integrals.

It should be noted that in the case when M is a finite graph, the asymptotics for the leading term is already obtained by a combination of Alexopoulos [1] and Ishiwata [20] for general nilpotent groups Γ. However, their method seems not to give geometric nature of the leading coefficient C. There are several related results (cf. [8], [13], [14], [18], [34], [37]). Moreover, several extensions to non-symmetric random walks of [20] are done in [23], [24], [5], [6], [31].

In the case when Γ is the discrete Heisenberg group $\mathrm{Heis}_3(\mathbb{Z})$, H is the harmonic oscillator, $d = 4$ and $\zeta_H(d/2)$ can be computed explicitly in terms of the special value $\zeta(2) = \pi^2/6$ of the Riemann zeta function $\zeta(s)$.

Next, we point out a relation to the our methods in the case of the Heisenberg group and the analysis of the discrete magnetic Laplacian or the Harper operator on the square lattice $\mathbb{Z}^2$. Spectrum of the latter operators are expressed by the celebrated Hofstadter butterfly as the following figure:

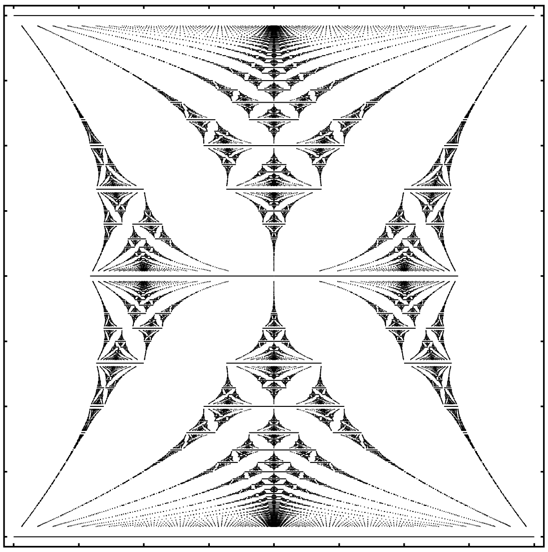

Figure 1. the Hofstadter's butterfly (created by Hisashi Naito). The vertical axis express the value of θ corresponding to the strength of a magnetic field or the magnetic flux density in the interval $[0, 2\pi]$ and the horizontal axis expresses the range of the spectrum of H_θ which are subsets of the closed interval $[-4, 4]$.

The Harper operator $H_\theta : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$ is defined as follows:

$$(H_\theta u)(m, n) = u(m+1, n) + u(m-1, n) + e^{\sqrt{-1}m\theta}u(m, n+1) + e^{-\sqrt{-1}m\theta}u(m, n-1)$$

This operator is a discrete analogue of the Laplacian (with some shift by identity operator Id) on the plane under constant magnetic field, which is the Hamiltonian of the Landau quantization.

For a relation to our extension of Floquet-Bloch theory is explained as follows: First, we point out that analysis of the Harper operator corresponds the case when the base space M is the graph $\mathcal{G} = (V, E)$ such that the set V of vertices has only one element p and the set E of edges has two loops u, v at the vertex p, namely if we realize $\mathcal{G}$ as a one dimensional complex, $\mathcal{G}$ is the bouquet of two S^1's, i.e. one point sum of two circles. If $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, then the Γ-covering space X is the Cayley graph of Γ with generators $\{u, v\}$, which is described by the following figures:

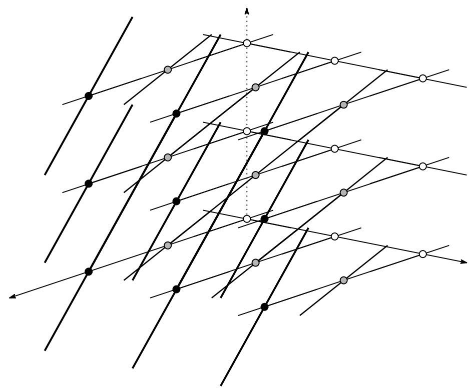

Figure 2. The Cayley graph X of $(\mathrm{Heis}_3(\mathbb{Z}), \{u, v\})$ created by Satoshi Ishiwata: This is a regular graph of degree four embedded in $\mathbb{R}^3$. Its edges are expressed by solid lines and the dotted line indicate z-axis, which does not mean edges.

Moreover, the square lattice $\mathbb{Z}^2$ appears as an intermediate space of coverings $\varpi_1 : X \to \mathbb{Z}^2$ and $\varpi_2 : \mathbb{Z}^2 \to M$ of $\pi = \varpi_2 \circ \varpi_1 : X \to M$. For $\theta \in \mathbb{Q}$, the Harper operator H_θ is a lift (with a shift by $4I$ with the identity operator I) of the twist discrete Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ on M with $x = (0, 0, \theta/2\pi) \in \hat{X}$ as an operator (without considering their domains) and H_θ (with a shift by $4I$ acting on $\ell^2(\mathbb{Z}^2)$) can be decomposed as a direct integral of $\Delta_{\rho_{\mathrm{fin},x}}$, $x \in \widehat{X}$ as in previous sections. These are described by,

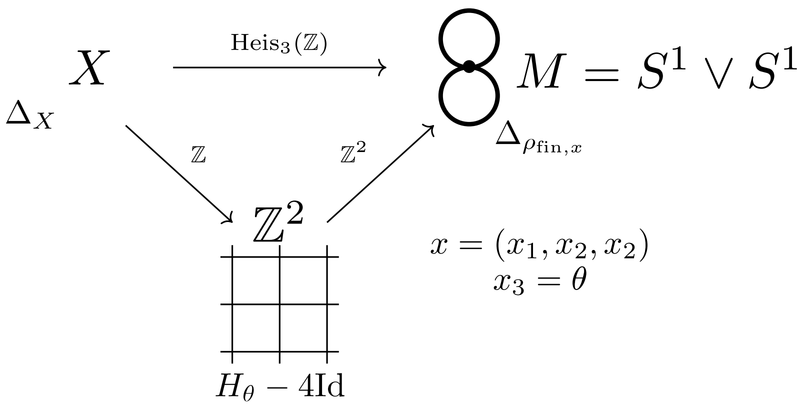

Figure 3. Relation between twisted Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ and Harper operator H_θ in terms of covering of graphs.

Although the structure of the spectrum of H_θ is far from complete understanding, it is already extensively studied (cf. [35]). For example, if the parameter θ is a rational number, then the spectrum of H_θ is a union of finite intervals, namely it has band structure and θ is an irrational number, it was conjectured that the spectrum is a Cantor set, which was known as M. Kac's "Ten Martini Problem" and was finally settled by Avila and Jitomirskaya [2]. Our interest here is the following semi-classical asymptotic expansion formula of the spectrum of H_θ as $\theta \to 0$ where we regard θ as semi classical parameters (3.9 in [33], similar formulas (6.9 in [38] and 6.3.1 in [17]),

$$E_n = -4 + (2n+1)\theta + O(\theta^2) \qquad n = 0, 1, 2, \dots \tag{1.2}$$

This formula first obtained by Wilkinson [38] by WKB arguments. Rammal and Bellissard [33] also derived this formula by a similar method. Formal derivation of the above expansion is not difficult as follows: It is known that H_θ is unitary equivalent to the following operator h_θ acting on $L^2(\mathbb{R})$,

$$h_\theta = -2\cos\left(\sqrt{\theta}\frac{d}{\sqrt{-1}ds}\right) - 2\cos(\sqrt{\theta}s). \tag{1.3}$$

Note that this expression h_θ is closely related to the Schrödinger representation of the Heisenberg Lie group $\mathrm{Heis}_3(\mathbb{R})$.

By the (formal) Taylor expansion formula, this operator can be expressed as

$$h_\theta = -4 + \left(-\frac{d^2}{ds^2} + s^2\right)\theta + O(\theta^2). \tag{1.4}$$

The coefficient of the linear part in θ is the harmonic oscillator $\widetilde{\mathcal{H}} := -\frac{d^2}{ds^2} + s^2$ whose eigenvalues are $n + \frac{1}{2}, n = 0, 1, 2, \dots$, which implies the above expansion (1.2). However, (1.4) is a form that a bounded operator h_θ is approximated by a unbounded operator $\widetilde{\mathcal{H}}$. We need to justify the above arguments mathematically. Note that there are a few other reasons of necessity for mathematical justifications.

This is first given by Helffer and Sjöstrand [17] using semi-classical analysis. Their method is described very roughly as follows: If the domain $L^2(\mathbb{R})$ of both operators h_θ and $\mathcal{H}$ are exhausted by a sequence of common invariant finite dimensional spaces V_k, $k = 1, 2, \dots$, then the expansion (1.4) can be considered as a limit of the Taylor expansion of the restriction $h_\theta|_{V_k}$ of h_θ. However there is no such sequence. In stead of V_k, Helffer and Sjöstrand used the space $\widetilde{V_k}$ spanned by eigenfunctions of $\widetilde{\mathcal{H}}$ associated with eigenvalues less than or equal to k. This space is not invariant by h_θ but errors are $O(h^\infty)$ with $h = \theta/2\pi$ and thus do not effect the asymptotic expansion (1.2). Their methods are based on the semi-classical localizations related to the harmonic oscillators.

We shall give another proof based on a comparison between unitary representations of $\mathrm{Heis}_3(\mathbb{Z})$ and the Heisenberg Lie group $\mathrm{Heis}_3(\mathbb{R})$, which will be explained briefly in Subsection 2.3. Note that this relation somewhat related to well known fact that the spectrum of the Harper operator H_θ for $\theta \in \mathbb{Q}$ has the band structure. However as far as we know, it seems that there are no arguments of using this fact to justify mathematically the asymptotic expansion formula (1.2). We believe our method has some advantage to the original proof in [17] since the arguments there need to consider the differential operator d/ds and the multipication operators conjointly to make the harmonic oscillator. However our method can handle them separately which seems to give more flexible applicabilities.

§ 2. Outline of the proofs: Floquet Bloch theory and applications to several asymptotics

We give an outline of our version of Floquet-Bloch theory in conjuction with their applications to asymptotics of heat kernels. To clarify the distinction between parts for the Floquet Bloch theory and other parts which are mainly pertutabation arguments for the twisted Laplacians, we denote (F's) for the former and (P's) for the latter in the starting points of the arguments.

§ 2.1. Finite extensions

(P0): Our concern is a long time asymptotics of the heat kernel $k_X(t,p,q)$ on X for a Γ-covering $\pi : X \to M$. We start from easier case that Γ is a finite group. In this case, X is compact and thus, $k_X(t,p,q)$ can be expressed as follows:

$$k_X(t,p,q) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(p)\varphi_i(q), \tag{2.1}$$

where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

are eigenvalues of the Laplacian Δ_X on X and $\{\varphi_i\}_{i=0}^{\infty}$ is a complete orthonormal system of eigenfunctions φ_i associated to eigenvalues λ_i. Note that the 0-th eigenfunction φ_0 is a constant function $1/\sqrt{\mathrm{vol}(X)}$ with the volume $\mathrm{vol}(X)$ of X. If $i \geq 1$, then $\lambda_i > 0$ and thus $e^{-\lambda_i t}$ decays to 0 if $t \to \infty$. Then, we have

$$\lim_{t\to\infty} k_X(t,p,q) = \lim_{t\to\infty} e^{-\lambda_0 t}\varphi_0(p)\varphi_0(q) = \varphi_0(p)\varphi_0(q) = \frac{1}{\mathrm{vol}(X)}.$$

§ 2.2. Infinite abelian extensions

Here we recall the strategy in the case when Γ is an infinite abelian group, let us recall the proof of the leading term of the asymptotic expansion (1.1). Namely, we shal show that if Γ is an infinite abelian group of rank d, then there exist a constant $C > 0$ such that

$$k_X(t,p,q) \sim \frac{C}{t^{d/2}}, \tag{2.2}$$

For simplicity, we assume $\Gamma = \mathbb{Z}^d = H_1(M, \mathbb{Z})$. Essential points in the arguments are already appeared in this case and it is not so difficult to extend to general finitely generated abelian groups (cf. [29] for detailed arguments). Here X is noncompact. Although there exists a similar formula to (2.1) in this case using the spectral decomposition of Δ_X, it seems useless for our problem since it contains parts of continuous spectrum.

Step 1 (F1): In stead of the above spectral decomposition formula, we regard the domain $L^2(X)$ of the Laplacian Δ_X on X as a space "$L^2(M) \otimes L^2(\Gamma)$", which is rigorously formulated as a space of L^2-sections $L^2(E_R)$ of the flat vector bundle E_R associated to the right regular representation R of Γ, a.k.a. local system associated to R. Namely, E_R is described as follows:

$$E_R \simeq X \times L^2(\Gamma)/\sim, \quad (p, \varphi) \sim (\gamma p, R(\gamma^{-1})\varphi) \quad \text{for} \quad \gamma \in \Gamma.$$

In the case when Γ is abelian, it can be analyzed by the usual Floquet-Bloch theory. To adjust later generalizations, we formulate the Floquet-Bloch theory geometrically as follows: Since $\Gamma = \mathbb{Z}^d$, the right regular representation R can be written as a direct integral of one dimensional irreducible unitary representations, i.e. characters, over the unitary dual $\widehat{\Gamma}$ of Γ which is the space of equivalence class of characters $\chi : \Gamma \to U(1)$ with one dimensional unitary group $U(1)$. Note that $\widehat{\Gamma}$ is isomorphic to a d-dimensional torus $U(1)^d$, which can be identified with the Brillouin zone in usual terminology of the Floquet-Bloch theory in condensed matter physics. Associated with this formula, we have the following direct integral decomposition

$$L^2(E_R) \simeq \int_{\widehat{\Gamma}}^{\oplus} L^2(E_\chi) d\chi,$$

where $L^2(E_\chi)$ is the space of sections of the flat line bundle associated with χ, which is similarly defined as $L^2(E_R)$. The space $L^2(E_\chi)$ can be identified with the space

$$H_\chi = \{f : X \to \mathbb{C} \mid f(\gamma p) = \chi(\gamma) f(p) \text{ for all } \gamma \in \Gamma\}.$$

Using this identification, the heat kernel $k_X(t,p,q)$ can be decomposed as

$$k_X(t,p,q) = \int_{\widehat{\Gamma}} k_\chi(t,p,q) d\chi \tag{2.3}$$

where the integrand $k_\chi(t,p,q)$ is the Fourier transform of $k_X(t,p,q)$ with respect to $\Gamma = \mathbb{Z}^d$ as

$$k_\chi(t,p,q) = \sum_{\gamma \in \Gamma} \chi(\gamma) k_X(t,p,\gamma q). \tag{2.4}$$

and (2.3) is the Fourier inversion formula.

Moreover, $k_\chi(t,p,q)$ can be expressed by a twisted average as follows:

$$k_\chi(t,p,q) = \sum_{i=0}^{\infty} e^{-\lambda_i(\chi)t}\varphi_{i,\chi}(p)\varphi_{i,\chi}(q), \tag{2.5}$$

where $\lambda_i(\chi)$ and $\varphi_{i,\chi}$ are the i-th eigenvalue and eigenfunction of the twisted Laplacian $\Delta_\chi := \Delta_X \left|_{H_\chi}\right.$. Note that $\lambda_i(\chi)$ are discrete as eigenvalues λ_i of the usual Laplacian Δ_M on M and $\lambda_i = \lambda_i(\mathbf{1})$ for the trivial character $\mathbf{1}$.

Step 2 (P1): The second step is an analysis of eigenvalues $\lambda_i(\chi)$. It is not difficult to show that there is a positive constant c such that $\lambda_i(\chi) \geq c$ for $i \geq 1$. We also have

$$\lambda_0(\chi) \geq 0, \qquad \lambda_0(\chi) = 0 \Longleftrightarrow \chi = \mathbf{1} \tag{2.6}$$

and $\lambda_0(\chi)$ depends on χ smoothly. From this fact, we easily find the first derivative with respect to χ at $\mathbf{1}$ is zero. The essential point is to show that the hessian of $\lambda_0(\chi)$ at $\chi = \mathbf{1}$ is positive definite.

To prove this fact, we use perturbation arguments as follows: Although the usual setting of the perturbation theory is that operators defined on a fixed domain are varied, our situation here are converse that defining domains H_χ are varied but the operator Δ_χ are fixed since it is the restriction of the Laplacian Δ_X on X to the space H_χ.

To reduce our situation to usual setting of the perturbation theory, we construct a canonical section s_χ of the line bundle E_χ (or better to consider as a section of $U(1)$-principal bundle) and identify $L^2(M)$ with $L^2(E_\chi)$ by the following correspondence:

$$L^2(M) \ni f \longleftrightarrow fs_\chi \in L^2(E_\chi) \simeq H_\chi.$$

Associated with this correspondence, we have a unitary equivalence between the twisted operator $L_\chi := s_\chi^{-1} \circ \Delta_\chi \circ s_\chi$ acting on $L^2(M)$ and the twisted Laplacian Δ_χ acting on $L^2(E_\chi) \simeq H_\chi$.

To construct s_χ, by the de Rham-Hodge Theorem, we can take a harmonic 1-form ω satisfying

$$\chi([\gamma]) := \chi_\omega([\gamma]) := \exp\left(2\pi\sqrt{-1}\int_\gamma \omega\right)$$

for a closed curve γ and its homology class $[\gamma] \in H_1(M,\mathbb{Z})$.

Since χ is a homomorphism from $\Gamma \simeq H_1(M,\mathbb{Z}) \simeq \pi_1/[\pi_1,\pi_1]$ to an abelian group $U(1)$, we can lift χ to a character $\widetilde{\chi} : \pi_1(M) \to U(1)$ canonically. Then, taking a lift $\widetilde{\omega}$ of ω to the universal covering $\widetilde{M}$ of M, we define a function $\widetilde{s}_{\widetilde{\omega}}$ on $\widetilde{M}$ by

$$\widetilde{s}_{\widetilde{\omega}}(p) = \exp\left(2\pi\sqrt{-1}\int_{p_0}^{p} \widetilde{\omega}\right) \tag{2.7}$$

for some reference point $p_0 \in \widetilde{M}$. Note that this is well defined since the line integral $\int_{p_0}^{p} \widetilde{\omega}$ does not depend on the choice of a curve from p_0 to p by the homotopy invariance property due to the fact that $\widetilde{\omega}$ is a closed form. Since $\widetilde{s}_{\widetilde{\omega}}(\gamma p) = \chi_\omega([\gamma])\widetilde{s}_{\widetilde{\omega}}(p)$, this function can be identified with a section $s_\omega = s_{\chi_\omega}$ of $L^2(E_\chi)$.

Then $L_\chi = s_{\chi_\omega}^{-1} \circ \Delta_{\chi_\omega} \circ s_{\chi_\omega}$ can be computed as

$$L_\chi f = \Delta_M f - 4\pi\langle \omega, df\rangle + 4\pi^2|\omega|^2 f$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the inner product and the norm of cotangent space induced from the Riemannian metric of M respectively.

From this expression, we can compute the hessian $\mathrm{Hess}_0\lambda_0$ of $\lambda_0(\chi)$ at $\chi = \chi_\omega = \mathbf{1}$ (i.e. $\omega = 0$) which is written as

$$\mathrm{Hess}_0\lambda_0(\omega,\omega) = \frac{8\pi^2}{\mathrm{vol}(M)}\int_M |\omega|^2 dv_g$$

where $\mathrm{vol}(M)$ is the Riemannian volume and dv_g is the Riemannian measure of M.

By the Morse lemma, we can take a local coordinates $(U, (x^1, x^2, \ldots, x^d))$ around the trivial character $\mathbf{1}$ on $\widehat{\Gamma} \simeq (U(1))^d$ such that

$$\lambda_0(\chi) = \lambda_0(x^1, x^2, \ldots, x^d) = \sum_{k=1}^{d}(x^i)^2.$$

Then we get the conclusion by the following computation:

$$\begin{aligned}
k_X(t,p,q) &= \int_{\widehat{\Gamma}} k_\chi(t,p,q)d\chi \\
&= \int_{\widehat{\Gamma}} \sum_{n=0}^{\infty} e^{-\lambda_n(\chi)t}\varphi_{n,\chi}(p)\varphi_{n,\chi}(q)d\chi \\
&= \int_U e^{-\lambda_0(\chi)t}\varphi_{0,\chi}(p)\varphi_{0,\chi}(q)d\chi + \text{error term} \\
&= C\int_U e^{-\sum_{i=1}^{d}(x^i)^2 t}dx^1dx^2\ldots dx^d + \text{error term} \\
&\sim C\prod_{i=1}^{d}\left\{\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-(y^i)^2}dy^i\right\} \\
&= \frac{C}{t^{d/2}},
\end{aligned}$$

where U is a neighborhood of $\mathbf{1}$ in $\widehat{\Gamma}$. Note that this arguments is sometimes called the Laplace method.

§ 2.3. Heisenberg extensions

For generalization of the above arguments to nilpotent groups, we point out arising difficulties and give a strategy to overcome them in the case when $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$. The reasons why we explain this case separately are the following:

(1) Representation theory of $\mathrm{Heis}_3(\mathbb{R})$ and some part of theory of $\mathrm{Heis}_3(\mathbb{Z})$ are explicit and can be explained as concrete form without knowledge of orbit methods.

(2) Main innovation in this note is already appeared in this case as the simplest form.

Step 1 (F2): In the first step of the previous arguments, we have used the irreducible decomposition formula of the right regular representation R of abelian groups. However, if Γ is non abelian nilpotent, its representation theory is not of type I. In this case, although there exists abstract decomposition formulas of the regular representation, it is not unique and the unitary dual $\widehat{\Gamma}$ is a wild space. As a conclusion, it seems that there is no computable formula in practice by the (full) unitary dual.

In the case that $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, if we restrict to finite dimensional unitary representations, we can use the Plancherel formula (Theorem due to Pytlik [32]. In this formula, the Plancherel measure μ is finitely additive and supported on finite dimensional irreducible unitary representations. This formula is indeed useful. However dimensions of representations appearing in the formula are varied, which makes some difficulties to apply perturbation arguments similar to Step 2 (P1) in the previous subsecion. To overcome this point, we relate these finite dimensional representations to infinite dimensional irreducible unitary representations which are called the Schrödinger representations of $\mathrm{Heis}_3(\mathbb{R})$. These representations ρ_h are parametrized by $h \in \mathbb{R} \setminus \{0\}$ and their representation spaces are common $L^2(\mathbb{R})$ for all $h \in \mathbb{R} \setminus \{0\}$.

Let us explain briefly the above relation. Finite dimensional irreducible unitary representations $\rho_{\mathrm{fin},x}$ of Γ are parametrized by $x = (x_1, x_2, x_3) \in \widehat{X} := [0,1] \times [0,1] \times (\mathbb{Q} \cap [0,1])$ and the above Plancherel measure μ is the product measure of the Lebesgues measure m on the interval $[0,1]$ for the first and second factors and the finitely additive measure $\widetilde{m}$ on the third factor $\mathbb{Q} \cap [0,1]$ determined by

$$\widetilde{m}(\mathbb{Q} \cap [a,b]) = b - a.$$

The representations $\rho_{\mathrm{fin},x}$ are essentially determined from x_3 which is, in some sense, a discrete version of the Schrödinger representation. If we express x_3 as the irreducible fraction p/q, then the dimension of $\rho_{\mathrm{fin},x}$ is q and the role of x_1, x_2 for $\rho_{\mathrm{fin},x}, x = (x_1, x_2, x_3)$ are abelian perturbations of $\rho_{\mathrm{fin},(0,0,x_3)}$. The point of our arguments is the following fact:

Fact 2.1. *The fluctuation by x_1, x_2 is like $O(1/q)$ which is independent to the numerator p of $x_3 = p/q$.*

Moreover, the subset of $\widehat{X}$ consisting $x = (x_1, x_2, x_3)$ whose third component $x_3 = p/q$ has the denominator smaller than a fixed constant is a null set with respect to the above Plancherel measure μ.

To relate the Schrödinger representation ρ_h, we note that if we restrict ρ_h to the discrete subgroup Γ when the parameter $h = x_3$ is a rational number p/q, the restriction $\rho_h|_\Gamma$ to Γ of ρ_h can be decomposed as a direct integral

$$\rho_h|_\Gamma = \int_0^1 \int_0^{1 \oplus} \rho_{\mathrm{fin},(x_1,\{qx_3x_2\},x_3)} dm(x_1) dm(x_2), \tag{2.8}$$

where $\{a\}$ denotes the fractional part $a - [a]$. We regard this decomposition formula as the approximation formula of the left hand side $\rho_h|_\Gamma$ by $\rho_{\mathrm{fin},x}$. It means that the left hand side is the "average" of the integrand in the right hand side and the fluctuation of the integrand are $O(1/q)$. Moreover, for any h near 0, it can be approximated by rational numbers with arbitrarily large denominators, which implies an infinite dimensional representation ρ_h can be approximated by finite dimensional representations $\rho_{\mathrm{fin},x}$ by arbitrarily orders. This means that for a mathematical justification of problems of convergence, e.g. the Fourier transform $\rho_h(f)$ of L^1-function f does not in the trace class in general, we use the above Plancherel formula which include only finite dimensional representations but for some formal computation of the eigenvalues, we can use the Schrödinger representation since it varies smoothly in h. Namely we can exchange freely finite dimensional representations and infinite dimensional representations. This is what we call the extension of the Floquet-Bloch theory to the Heisenberg group.

Step 2 (P2): As in the second step of arguments for abelian groups in the previous subsection, we need to investigate the behavior of the eigenvalues of Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ twisted by $\rho_{\mathrm{fin},x}$ near the trivial representation $\mathbf{1}$. By Step 1 (F2), it can be reduced to a formal computation of the twisted Laplacian Δ_{ρ_h} associated to ρ_h replacing $\Delta_{\rho_{\mathrm{fin},x}}$. These formal computations are, as a spirit, same as abelian case. However, technically speaking, it is necessary to modify several points in the arguments to proceed actual computations. One of them is to replace the line integral $\int_{p_0}^p \widetilde{\omega}$ in (2.7) by the Lie integral of the Lie algebra valued 1-form (cf. [7]), which is equivalent to the Chen's iterated integrals [10]. Moreover our computation is somewhat complicated since we need to carry out in infinite rank vector bundles. For example, in this procedure, the hessian $\mathrm{Hess}_0\lambda_0$ should be replaced by the quadratic forms associated with a modified harmonic oscillator $\mathcal{H}$ acting on the fiber $L^2(\mathbb{R})$ of $E_{\rho_h}(L^2(\mathbb{R}))$. However, the arguments are essentially a blending of the so called Schrödinger method in physics literatures (cf. [33]) and a computation carried in [29] in abelian case.

§ 2.4. Nilpotent extensions

Our strategies to deal with these cases are roughly summarized as follows:

Step 1 (F3): Since it seems to no known gereralization of Pytlik theorem for $\mathrm{Heis}_3(\mathbb{Z})$ to general nilpotent groups, we recover them as follows: We first decompose the right regular representation R of discrete nilpotent group Γ more coarsely, into factor representations ρ_l . Then, we extend ρ_l to a monomial representation ρ_l^G and decompose them into irreducible representations ρ_l^Ω. Then restrict them to Γ. If $\rho_l^\Omega|_\Gamma$ satisfies some rationality conditions, we further decompose to finite dimensional representations by two steps.

The following are a little more explanation:

- Substep 0-1: First we use a direct integral decomposition formula of the right regular representation R of Γ to their "factor" representations ρ_l due to [26] or [4]. The reason why we wrote "factor" is that generic representations of integrand are indeed factor representations but there are not factor representations which consist dense subset in the definig domain of the Plancherel measure.
  In addition, we note that all of the above representation ρ_l are obtained as induced representation from charaters l of the center of Γ and their representation spaces are isomorphic subspaces of $L^2(\Gamma)$.
- Subtep 0-2: Then, since the group multiplication law of Γ can be expressed as forms of polynomials of certain coordinate functions, similarly as construction of the Malcev completion G from Γ, we can extend ρ_l to representations ρ_l^G of the nilpotent Lie group G whose representation spaces are isomorphic subspaces of $L^2(G)$.
  We remark that if $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, then ρ_l^G is the infinite direct sum of a copy of fixed Schödinger representation ρ_h with $l(Z+Y) = hZ$ for Z in the center of $G = \mathrm{Heis}_3(\mathbb{Z})$ and the polarized algebra $\mathfrak{m} = \{Z, Y\}$ which is a special case of the formula in [11], [15].
- Subtep 0-3: Then, they are monomial representations. By Fujiwara's Plancherel theorem (Theorémè 2 in [15]), ρ_l^G can be decomposed as a direct integral of irreducible unitary representations ρ_l^Ω which are pametrized by certain parameter Ω.
- Subtep 0-4: By later arguments for asymptotic expansions, we may assume G is stratified in later sections. Namely, the Lie algebra $\mathfrak{g}$ of G is generated by $\mathfrak{g}^{(1)}$ in the following decomposition:
  $$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(m)},$$
  where $\mathfrak{g}^{(i)} \simeq \mathfrak{g}_i/\mathfrak{g}_{i+1}$, $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$, $\quad i > 1$. and $\mathfrak{g}^{(i)}$ satisfies the relation $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}$.
- Substep 0-5: Next we choose parameters Ω suitably to be dense in the domain of the above direct integral and restrict ρ_l^Ω again to Γ. Then we apply the branching formula due to Bekka and Driutti (Theorem 1.3 in [3]) which gives a direct integral

decomposion of $\rho_l^\Omega|_\Gamma$.

- Substep 0-6: In the above decomposition, some of representations which appear in the integrand of the direct integral are irreducible but others are not.In the latter case, extreme one, which satisfies "rationality conditions", can be decomposed into finite dimensional irreducible representations. This formula is similar to (2.8) in spirit. Moreover we can choose them so that these cases are also dense in the domain of the above direct integral.
- Substep 0-7: Combinig with decomposions in Substeps 0-1, 0-3, 0-5, 0-6, we obtain a substitute of Pytlik theorem for general nilpotent groups.
- Substep 0-8: Finite dimensional irreducible unitary representations ρ_{fin} of finitely generated torsion-free discrete nilpotent groups Γ are classified in [19]. Each representations have similar properties of fluctuation of Fact 2.1 generalizing to Γ from $\mathrm{Heis}_3(\mathbb{Z})$.
- Substep 0-9: Therefore, as in the Heisenberg case, formal computation can be reduced to spectral analysis associated to irreducible representation $\rho_{l,\Omega}$ of G. The hypo-elliptic operator H in Theorem 1.3 is written as he following form:

$$H := -\left(d\rho_l^\Omega(X_1)^2 + \cdots + d\rho_l^\Omega(X_k)^2\right),$$

  where $X_1, \ldots, X_k$ is a $\mathbb{Q}$-basis of $\mathfrak{g}^{(1)}$. If $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, this operator H is essentially the harmonic oscillator $-\frac{d^2}{du^2} + u^2$ which is already discussed in the previous subsection. It should be noted that in more general frame work, the corresponding operator has the form $-(E_1^2 + \cdots E_k^2) + B$, where E_i is coming from $\mathfrak{g}^{(1)}$ and B from $\mathfrak{g}^{(2)}$. In oue case, the property $B = 0$ is a consequence of the symmmetry of operators.
- Substep 0-10: Finally we notice that analysis of the above irreducible unitary representations of discrete nilpotent groups reduce to those of nilpotent Lie groups. This phenomena also holds even in corresponding Plancherel measures. Thus we can untilize the Plancherel theorem and the Fourier inversion formula of the latter as a formal model of the former case.

Step 2 (P3): As for perturbation analysis, the essentially same arguments as $\mathrm{Heis}_3(\mathbb{Z})$ can be applied but less explicit, since explicit value of eigenvalues of the hypo-elliptic operator H are not known in general nilpotent groups.

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