A method for computing the bifurcation set of a complex polynomial mapping II

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Abstract

Bifurcation sets of complex polynomial mappings are considered in the context of symbolic computation. Based on a result of A. Parusiński, a new effective method is proposed for computing bifurcation sets of complex polynomial mappings. The keys of our approach are the concept of local cohomology and the Grothendieck local duality on residues. The resulting method can treat the case of a family of polynomial mappings that depend on deformation parameters.

1 Introduction

The bifurcation set plays a fundamental role in the study of topology of a polynomial mapping. After the study of S. A. Broughton [5, 6], many researchers have studied bifurcation sets from several different viewpoints. e.g. [1, 2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 24, 25, 26, 27]. Many interesting and deep results were obtained. However, the determination of bifurcation sets is still quite difficult problem [16, 24, 25]. We propose in this paper a new effective method for computing them. .

In Section 2, we recall the notion of a bifurcation set and a result of A. Parusiński [13]. In Section 3, we briefly recall the concept of local cohomology and the Grothendieck local duality on residues. In Section 4, we give a method for computing bifurcation sets. In Section 5, we show that the proposed method can compute parameter dependency of bifurcation sets of a family of polynomial mappings.

2 Bifurcation set and Milnor number at infinity

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a complex polynomial mapping Then, there exists a finite set $\Gamma_f \subset \mathbb{C}$ s.t.

$$f : \mathbb{C}^n - f^{-1}(\Gamma_f) \longrightarrow \mathbb{C} - \Gamma_f$$

is a locally trivial $\mathcal{C}^\infty$ fibration. The smallest set that satisfies the condition above is called bifurcation set, or atypical set [5, 6] of the mapping f.

Let $Sing(f)$ be the singular set of f and let C_f denote the image by f of $Sing(f)$. Let B_f denote the bifurcation set of f.

Example ([5]) $f(x, y) = x^2y - x$

Let $J_f \subset K[x, y]$ be the Jacobi ideal of f. Since $J_f = \langle 2xy - 1, x^2 \rangle = \langle 1 \rangle$,we have $C_f = \emptyset$.

Noe consider the fiber $f^{-1}(c)$ of $c \in \mathbb{C}$. Since $x^2y - x = x(xy - 1)$, we have $f^{-1}(c) \cong \mathbb{C}^*$ for $c \neq 0$, and $f^{-1}(0) \cong \mathbb{C} \cup \mathbb{C}^*$ for $c = 0$. Therefore $B_f = \{0\}$ which means in particular $C_f \neq B_f$.

Let $f(x)$ be a polynomial in n variables $x = (x_1, x_2, \ldots, x_n)$ of degree d. Let $\tilde{f}(x, \eta)$ denote the homogenization of f, i.e. $\tilde{f}(x, \eta) = \eta^d f(\frac{x_1}{\eta}, \frac{x_2}{\eta}, \cdots, \frac{x_n}{\eta})$, where η is a new variable. Now consider a homogeneous function $F_t(x, \eta)$ defined by

$$F_t(x, \eta) = \tilde{f}(x, \eta) - t\eta^d,$$

where t is regarded as a parameter.

Let A be the singular locus of $F_t(x, \eta)$ at infinity $\eta = 0$

$$A = \left\{ [x_1 : x_2 : \cdots : x_n : \eta] \in \mathbb{P}^n \;\middle|\; \frac{\partial F_t}{\partial x_1} = \frac{\partial F_t}{\partial x_2} = \cdots = \frac{\partial F_t}{\partial x_n} = \frac{\partial F_t}{\partial \eta} = \eta = 0 \right\}$$

Lemma Let $f = f_d(x) + f_{d-1}(x) + \cdots + f_0(x)$ be a polynomial of degree d, where $f_k(x)$ is a polynomial of degree k. Then , the following holds.

$$A = \left\{ [x_1 : x_2 : \cdots : x_n : 0] \in \mathbb{P}^n \;\middle|\; \frac{\partial f_d}{\partial x_1} = \frac{\partial f_d}{\partial x_2} = \cdots = \frac{\partial f_d}{\partial x_n} = f_{d-1} = 0 \right\}.$$

Example $f(x, y) = x^2y^2 + x$

From $\tilde{f}(x, y, \eta) = x^2y^2 + x\eta^3$, $F_t(x, y, \eta) = x^2y^2 + x\eta^3 - t\eta^4$,

we have $A = \{[x : y : \eta] \in \mathbb{P}^2 \mid xy^2 = x^2y = \eta = 0\} = \{[0 : 1 : 0], [1 : 0 : 0]\}$

Example $f(x, y) = x^4 + y^4 - 4xy$

From $\tilde{f}(x, y, \eta) = x^4 + y^4 - 4xy\eta^2$, $F_t(x, y, \eta) = x^4 + y^4 + 4xy\eta^2 - t\eta^4$,

we have $A = \{[x : y : 0] \in \mathbb{P}^2 \mid x^3 = y^3 = 0\} = \emptyset$.

We assume hereafter that A is a finite set. For $A = \{a_1, a_2, \ldots, a_m\}$, with $a_j \in A$, let μ_{t,a_j} denote the Milnor number of F_t at the point a_j.

In 1995, A. Parusiński obtained the following result.

Theorem ([13]) Let $t_0 \notin C_f$. Then the following are equivalent.

(1) $t_0 \notin B_f$

(2) There is a neighborhood $U \subset \mathbb{C}$ of t_0 s.t.
$\mu_{t,a_j} = \mu_{t_0,a_j}, j = 1, 2, \ldots, m, \forall t \in U.$

The result implies in particular that if the Milnor number μ_{t,a_j} jumps at the point t_0, then, t_0 belongs to bifurcation set B_f.

3 Local cohomology and duality

In this section we recall some basics on local cohomology and duality. Let X be a neighborhood in $\mathbb{C}^n$ of the origin O. Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions and let $\mathcal{O}_{X,O}$ be the stalk at O of the sheaf $\mathcal{O}_X$. Let $\hat{\mathcal{O}}_{X,O}$ be the ring of formal power series at O

Let $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ be the sheaf of local cohomology supported at O defined as

$$\mathcal{H}^n_{\{O\}}(\mathcal{O}_X) = H^n(R\Gamma_{\{O\}}(\mathcal{O}_X)),$$

where $R\Gamma_O$ is a derived functor. Let $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ be algebraic local cohomology supported at O:

$$\mathcal{H}^n_{[O]}(\mathcal{O}_X) = \lim_{k \to \infty} Ext^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{m}^k, \mathcal{O}_X),$$

where $\mathfrak{m}$ is the maximal ideal of O.

The classical theory of Fréchet Schwartz and dual Fréchet Schwartz locally convex topological vector spaces implies that the pairings

$$\mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \times \mathcal{O}_{X,O} \longrightarrow \mathbb{C}, \quad \mathcal{H}^n_{[O]}(\mathcal{O}_X) \times \hat{\mathcal{O}}_{X,O} \longrightarrow \mathbb{C}$$

are perfect. Namely, $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ and $\mathcal{O}_{X,O}$ are mutually dual and $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ and $\hat{\mathcal{O}}_{X,O}$ are mutually dual.

Note that the pairings are defined by the Grothendieck local residues. Here we identified $\mathcal{O}_{X,O}$ with the set of germs of holomorphic n-forms $\Omega^n_{X.O}$ and $\hat{\mathcal{O}}_{X,O}$ with $\hat{\Omega}^n_{X,O}$.

Let $I \subset \mathcal{O}_X$ be an ideal s.t $V(I) = \{O\}$ and let $\hat{I} = I\hat{\mathcal{O}}_{X,O}$

We define H_I and $H_{\hat{I}}$ by

$$H_I = \{\sigma \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \mid I\sigma = 0\}, \quad H_{\hat{I}} = \{\sigma \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid \hat{I}\sigma = 0\}.$$

Since $V(I) = \{O\}$, H_I and $H_{\hat{I}}$ are finite dimensional vector spaces and they are isomorphic: $H_I \cong H_{\hat{I}}$.

A complex analytic version of the Grothendieck local duality yields the following [19, 20].

Theorem Let $I \subset \mathcal{O}_X$ be an ideal s.t. $V(I) = \{O\}$. Then, the following pairing are non-degenerate.

$$H_I \times \mathcal{O}_{X,O}/I \longrightarrow \mathbb{C}, \quad H_{\hat{I}} \times \hat{\mathcal{O}}_{X,O}/\hat{I} \longrightarrow \mathbb{C}.$$

The non-degeneracy immediately implies the following.

Corollary $\dim_{\mathbb{C}}(\mathcal{O}_{X,O}/I) = \dim_{\mathbb{C}}(H_{\hat{I}})$ holds.

Let $g(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a polynomial, s.t. $g(0) = 0$ and let $J_g = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \cdots, \frac{\partial g}{\partial x_n} \rangle$ denote the Jacobi ideal in the polynomial ring $\mathbb{C}[x]$.

Assume that, there is a neighborhood X of the origin $O \in \mathbb{C}^n$ s.t. $V(J_g) \cap X = \{O\}$.

Let

$$H_{J_g} = \{\psi \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \mid p\psi = 0, \forall p \in J_g\}$$

Then, we have the following. [19, 20, 22]

Proposition Let $\mu(g)$ be the Milnor number of g at the origin. Then, the following holds.

$$\mu(g) = \dim_{\mathbb{C}}(H_{J_g}).$$

Proof Let $\mathcal{O}_{X,O}J_g$ be the Jacobi ideal of g in the ring $\mathcal{O}_{X,O}$ of convergent power series at the origin. It is easy to see that

$$\{\psi \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \mid p\psi = 0, \forall p \in J_g\} = \{\psi \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \mid h\psi = 0, \forall h \in \mathcal{O}_{X,O}J_g\}.$$

Since the Milnor number $\mu(g)$ of g at the origin is defined as $\mu(g) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/(\mathcal{O}_{X,O}J_g))$, we have the result.

In [22, 23], an algorithm for computing a basis of the vector space H_I of local cohomology classes is described.

4 Algorithm

Let $t = (t_1, t_2, \cdots, t_\ell)$ and let $I_t \subset K[t][x_1, x_2, \ldots, x_n]$ be a family of ideals in the polynomial ring, where $x = (x_1, x_2, \ldots, x_n)$ are variables and $t = (t_1, t_2, \ldots, t_\ell)$ are regarded as parameters. Let $V(I_t)$ denote the zero loci in $\mathbb{C}^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{C}, i = 1, 2, \ldots, n\}$ depending on parameters $t_1, t_2, \ldots, t_\ell$. We assume that there is a neighborhood $X \subset \mathbb{C}^n$ s.t. $V(I_t) \cap X = \{O\}$, the origin.

Let H_{I_t} denote the set of algebraic local cohomology classes that are annihilated by the ideal I_t, where t are regarded as parameters. Structure of H_{I_t} depends on parameters.

In [11], the authors of the present paper gave an algorithm for computing a basis of the vector spaces H_{I_t}. More precisely, the algorithm compute parameter dependency of the space H_{I_t}.

By using this algorithm **ALCohomology** the Milnor numbers μ_{t,a_j} of $F_t(x,\eta)$ at the point $a_j \in \mathbb{P}^n$ are computable.

We give some examples for illustration.

Example(S. A. Broughton [2]) $f(x,y) = x^2y - x$

$F_t(x,y,\eta) = x^2y - x\eta^2 - t\eta^3,\ A = \{[0:1:0]\}$.

Let $x = uy, \eta = hy$. Then $F_t = y^3 g_t(u,h)$, where $g_t(u,h) = u^2 - uh^2 - th^3$.

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

(i) the case $t \neq 0$, $H_{J_{g_t}} = \mathrm{Span}\{\left[\begin{array}{c}1\\ uh\end{array}\right], \left[\begin{array}{c}1\\ uh^2\end{array}\right]\}$

(ii) the case $t = 0$, $H_{J_{g_0}} = \mathrm{Span}\{\left[\begin{array}{c}1\\ uh\end{array}\right], \left[\begin{array}{c}1\\ uh^2\end{array}\right], \left[\begin{array}{c}1\\ u^2h\end{array}\right] + 2\left[\begin{array}{c}1\\ uh^3\end{array}\right]\}$.

Accordingly, $\mu_t = 2$ for $t \neq 0$, and $\mu_0 = 3$ for $t = 0$. Therefore the bifurcation set B_f is equal to $\{0\}$.

Example $f(x,y) = x^2y^2 + x$

$F_t(x,y,\eta) = x^2y^2 + x\eta^3 - t\eta^4,\ A = \{[0:1:0],[1:0:0]\}$.

(1) computation at $a_1 = [0:1:0]$

Let $x = uy, \eta = hy$. Then $F_t = y^4 g_t(u,h),\ g_t(u,h) = u^2 + uh^3 - th^4$

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

(i) the case $t \neq 0$, $H_{J_{g_t}} = \mathrm{Span}\{\left[\begin{array}{c}1\\ uh\end{array}\right], \left[\begin{array}{c}1\\ uh^2\end{array}\right], \left[\begin{array}{c}1\\ uh^3\end{array}\right]\}$

(ii) the case $t = 0$, $H_{J_{g_0}} = \mathrm{Span}\{\left[\begin{array}{c}1\\ uh\end{array}\right], \left[\begin{array}{c}1\\ uh^2\end{array}\right], \left[\begin{array}{c}1\\ uh^3\end{array}\right], \left[\begin{array}{c}1\\ u^2h\end{array}\right] + 2\left[\begin{array}{c}1\\ uh^4\end{array}\right], \left[\begin{array}{c}1\\ u^2h^2\end{array}\right] + 2\left[\begin{array}{c}1\\ uh^5\end{array}\right]\}$.

Accordingly, $\mu_{t,a_1} = 3$ for $t \neq 0$, and $\mu_{0,a_1} = 5$ for $t = 0$.

(2) computation at $a_2 = [1:0:0]$, We see that $\mu_{t,a_2} = 2$ for all t

Therefore, we have $B_f = \{0\}$.

Example (A. Dimca [8]) $f(x,y,z) = x^2y + y^2z + x$

$F_t(x,y,z,\eta) = x^2y + y^2z + x\eta^2 - t\eta^3,\ A = \{[0:0:1:0]\}$.

Let $x = uz, y = vz, \eta = hz$. Then. $F_t = z^3 g_t(u,v,h)$ with $g_t(u,v,h) = u^2v + v^2 + uh^2 - th^3$.

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial v}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

$$H_{J_{g_t}} = \mathrm{Span}\{\left[\begin{array}{c}1\\ uvh\end{array}\right], \left[\begin{array}{c}1\\ uvh^2\end{array}\right], \left[\begin{array}{c}1\\ u^2vh\end{array}\right], \left[\begin{array}{c}1\\ u^3vh\end{array}\right] - \tfrac{1}{2}\left[\begin{array}{c}1\\ uv^2h\end{array}\right],$$

$$\left[\begin{array}{c}1\\ u^4vh\end{array}\right] - \tfrac{1}{2}\left[\begin{array}{c}1\\ u^2v^2h\end{array}\right] + \tfrac{3}{2}\left[\begin{array}{c}1\\ u^2vh^2\end{array}\right] + \left[\begin{array}{c}1\\ uvh^3\end{array}\right]\}$$

$H_{J_{g_t}}$ depend on t. Whereas Milnor numbers does not depend on t. Therefore, we have $B_f = \emptyset$.

Remark: In 2004, A. Bodin obtained an algorithm for computing bifurcation sets. The method proposed in this section is different from the algorithm presented in [3].

5 Bifurcation sets and deformations

The proposed method can treat a family of polynomial map depending on deformation parameters. Namely, the algorithm **ALCohomology** can compute parameter dependency of bifurcation sets. We give some examples.

Example (A. Bodin [2]) $f = (x - s^2 + 1)(x^2y + 1)$, where s is a deformation parameter.

$\tilde{f}(x, y, \eta) = x^3y - (s^2 + 1)x^2y\eta + x\eta^3 - (s^2 + 1)\eta^4$

$F_t(x, y, \eta) = x^3y - (s^2 + 1)x^2y\eta + x\eta^3 - (s^2 + 1)\eta^4 - t\eta^4$

$A = \{[0 : 1 : 0]\}$.

Now let $x = uy, y = y, \eta = hy$. Then
$F_t = y^4 g_t(u, h)$ where $g_t(u, h) = u^3 - (s^2 + 1)u^2h + uh^3 - (s^2 + 1 + t)h^4$.

Compute bases of parametric local cohomology $H_{J_{g_t}}$ by using **ALCohomology**. We regard s and t as parameters. The output consists of four cases.

(i) $s^2 + 1 = 0, t = 0$,

(ii) $s^2 + 1 = 0, t \neq 0$

(iii) $s^2 + 1 + t = 0, s^2 + 1 \neq 0, t \neq 0$,

(iv) $(s^2 + 1)t(s^2 + 1 + t) \neq 0$

For each cases, local cohomology are given as follows.

(i) $s^2 + 1 = 0, t = 0$, $\dim_K(H_{J_{g_t}}) = 7$

$$H_{J_{g_t}} = \mathrm{Span}\{[\begin{array}{c}1\\ uh\end{array}], [\begin{array}{c}1\\ uh^2\end{array}], [\begin{array}{c}1\\ u^2h\end{array}], [\begin{array}{c}1\\ uh^3\end{array}], [\begin{array}{c}1\\ u^2h^2\end{array}], [\begin{array}{c}1\\ uh^4\end{array}] - \tfrac{1}{3}[\begin{array}{c}1\\ u^3h\end{array}], [\begin{array}{c}1\\ uh^5\end{array}] - \tfrac{1}{3}[\begin{array}{c}1\\ u^3h^2\end{array}]\}.$$

(ii) $s^2 + 1 = 0,\ t \neq 0$, $\dim_K(H_{J_{g_t}}) = 6$

$$H_{J_{g_t}} = \mathrm{Span}\{[\begin{array}{c}1\\ uh\end{array}], [\begin{array}{c}1\\ uh^2\end{array}], [\begin{array}{c}1\\ u^2h\end{array}], [\begin{array}{c}1\\ uh^3\end{array}], [\begin{array}{c}1\\ u^2h^2\end{array}], [\begin{array}{c}1\\ uh^4\end{array}] - \tfrac{1}{3}[\begin{array}{c}1\\ u^3h\end{array}] + \tfrac{4}{3}t[\begin{array}{c}1\\ u^2h^3\end{array}]\}$$

(iii) $s^2 + 1 + t = 0,\ s^2 + 1 \neq 0,\ t \neq 0$, $\dim_K(H_{J_{g_t}}) = 6$

$$H_{J_{g_t}} = \mathrm{Span}\{[\begin{array}{c}1\\ uh\end{array}], [\begin{array}{c}1\\ uh^2\end{array}], [\begin{array}{c}1\\ u^2h\end{array}], [\begin{array}{c}1\\ uh^3\end{array}], t[\begin{array}{c}1\\ uh^4\end{array}], -\tfrac{1}{2}[\begin{array}{c}1\\ u^2h^2\end{array}], t^3[\begin{array}{c}1\\ uh^5\end{array}] - \tfrac{1}{2}t^2[\begin{array}{c}1\\ u^2h^3\end{array}] - \tfrac{9}{4}[\begin{array}{c}1\\ u^2h^2\end{array}], +\tfrac{3}{2}t[\begin{array}{c}1\\ u^3h\end{array}]\}$$

(iv) $(s^2 + 1 + t)(s^2 + 1)t \neq 0$, $\dim_K(H_{J_{g_t}}) = 5$

$H_{J_{g_t}}$: omitted

Therefore bifurcation sets:are given as follows.

If $s^2 + 1 = 0,\ \ B_f = \{0\}$

If $s^2 + 1 \neq 0,\ \ B_f = \{-s^2 - 1\}$

The following example is due to M. Tibar.

Example (M. Tibar [26]) $f(x, y, z) = x^2y + x + z^2 + sz^3$, where s is a deformation parameter.

The singular set at infinity consists of one point: $A = \{[0 : 1 : 0 : 0]\}$.

Compute bases of parametric local cohomology $H_{J_{g_t}}$ by using **ALCohomology**.

The output consists of 5 cases. Here we give the conditions and Milnor numbers.

(i) $s = 0, t = 0$, $\dim_K(H_{J_{g_t}}) = 5$

(ii) $s = 0, t \neq 0$, $\dim_K(H_{J_{g_t}}) = 4$

(iii) $t = 0, s \neq 0$, $\dim_K(H_{J_{g_t}}) = 5$

(iv) $27s^2t - 4 = 0$, $\dim_K(H_{J_{g_t}}) = 5$

(v) $(27s^2t - 4)st \neq 0$, $\dim_K(H_{J_{g_t}}) = 4$

It is easy to see that bifurcation sets are given as follows.

If $s = 0$, $B_f = \{0\}$

If $s \neq 0$, $B_f = \{0, \frac{4}{27s^2}\}$

Notice that the bifurcation sets depend on deformation parameter s.

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