On stability of Hill's vortex and its applications

Kyudong Choi
Ulsan National Institute of Science & Technology

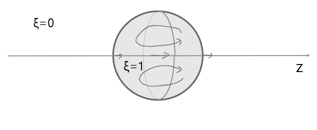
1 Introduction

This article summarizes the recent developments [6, 9], where the second one is a joint work with In-Jee Jeong.

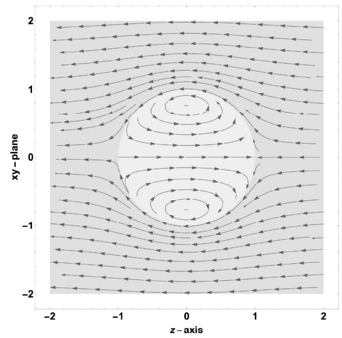
The Hill's vortex ξ_H, which was discovered by M. Hill in 1894 [17], is simply defined by

$$\xi_H(\mathbf{x}) = 1_B(\mathbf{x}),$$

where $B = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| < 1\}$ is the unit ball in $\mathbb{R}^3$ centered at the origin. In the incompressible Euler equations in $\mathbb{R}^3$, the sphere travels in a constant speed without changing its shape in axi-symmetric flow without swirl. In [6, 9], we consider its stability with possible consequences.



(a) A sketch in the fixed frame: The red arrows mean the direction of trajectories.



(b) The streamline in the moving frame

Figure 1: An illustration of Hill's vortex

We study stability of such a localized vortex moving without changing shape or size because we want to understand transport of mass, momentum and energy in large scale at a flow of high Reynolds number. At the same time, stability can produces interesting applications such as development of a long tail, vorticity growth, etc.

2 Preliminaries

2.1 3D incompressible Euler equations

We consider the incompressible Euler equations: u: velocity vector, P: pressure

$$\partial_t u + (u \cdot \nabla)u + \nabla P = 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad t > 0,$$
$$\operatorname{div} u = 0.$$

It is useful to study the quantity so called vorticity vector $\omega := curl(u)$, which is a vector field that describes the local spinning motion of a continuum near some point or the tendency of something to rotate. Then the Euler equations can be read in vorticity vector form:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \quad u = [\nabla \times \Delta_{\mathbb{R}^3}^{-1}]\omega.$$

2.2 Hill's vortex

M. Hill [17] found an explicitly written and compactly supported weak solution:

$$\omega(t, \mathbf{x}) = -x_2 \mathbf{1}_{B^{(2/15)t}}(\mathbf{x}) e_{x_1} + x_1 \mathbf{1}_{B^{(2/15)t}}(\mathbf{x}) e_{x_2},$$

where we define the translated unit ball B^τ, $\tau \in \mathbb{R}$ by

$$B^\tau := \{\mathbf{x} \in \mathbb{R}^3 \, : \, |\mathbf{x} - \tau e_{x_3}| < 1\}.$$

With the cylindrical coordinate $\mathbf{x} = (r, \theta, z)$, it is just

$$\omega(t, r, \theta, z) = r\mathbf{1}_{B^{(2/15)t}}(r, \theta, z) e_\theta(\theta) = \underbrace{r\mathbf{1}_{B^{(2/15)t}}(r, z)}_{\omega^\theta(r,z)} e_\theta(\theta).$$

Since the vorticity has only angular component and the component is independent of the angle variable θ, the flow is axisymmetric without swirl:

$$u(t, \mathbf{x}) = u^r(t, r, z) e_r(\theta) + u^z(t, r, z) e_z.$$

By denoting relative vorticity $\xi := \omega^\theta / r$, the Hill's vortex is simply written by

$$\xi(t, r, z) = \mathbf{1}_{B^{(2/15)t}}(r, z) \quad \text{with the initial data} \quad \xi(0, r, z) = \xi_H(r, z) = \mathbf{1}_B(r, z).$$

We remark that the axi-symmetric Euler equation without swirl in ξ:

$$\partial_t \xi + u \cdot \nabla \xi = 0. \tag{1}$$

In the sense of the above equation (1), the Hill's vortex is just a traveling wave solution with traveling speed $2/15$ along $z-$axis.

2.3 Velocity of Hill's vortex

Let's first compute the velocity field of the vortex. Since it is a traveling wave, it is enough to do at the initial time. The initial vortex is simply $\xi_H(\mathbf{x}) = 1_B(\mathbf{x})$. Then vorticity vector is

$$\omega_H(\mathbf{x}) = r1_B(\mathbf{x})e_\theta(\theta).$$

We can obtain the stream vector ϕ_H from solving the Poisson equation:

$$-\Delta\phi_H = \omega_H \quad \text{in} \quad \mathbb{R}^3.$$

To obtain the corresponding velocity u_H, we take curl:

$$u_H = \nabla \times \phi_H = (u_H^r e_r + u_H^z e_z),$$

which gives

$$\begin{aligned} u_H^r &= \begin{cases} \frac{3}{2}Wrz, & |\mathbf{x}| \le 1, \\ \frac{1}{2}W\frac{rz}{|\mathbf{x}|^5}, & |\mathbf{x}| > 1, \end{cases} \\ u_H^z &= \begin{cases} W\left(\frac{5}{2} - \frac{3}{2}|\mathbf{x}|^2 - \frac{3}{2}r^2\right), & |\mathbf{x}| \le 1, \\ \frac{W}{|\mathbf{x}|^3}(1 - \frac{3r^2}{2|\mathbf{x}|^2}), & |\mathbf{x}| > 1. \end{cases} \end{aligned} \tag{2}$$

We refer Figure 1.

2.4 Biot-Savart law

When considering general axi-symmetric flow without swirl, the corresponding velocity can be obtained easily in the following representation $u = \mathcal{K}[\xi]$ in terms of relative vorticity ξ:

For given $\xi(r, z)$, first consider

$$\omega = \omega^\theta(r, z)e_\theta(\theta) = r\xi(r, z)e_\theta(\theta).$$

As before, we take the (vector) potential $\phi = (-\Delta)^{-1}\omega$ so that it has only angular component. Then we denote (scalar relative) angular stream $\psi(r, z)$ by

$$\phi = r^{-1}\psi(r, z)e_\theta(\theta).$$

Then the angular stream ψ satisfies the (scalar) elliptic equation:

$$-\frac{1}{r^2}\mathcal{L}\psi = \xi, \qquad \mathcal{L} := r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

We simply denote $\mathcal{G} = \left(-\frac{1}{r^2}\mathcal{L}\right)^{-1}$ so that $\psi = \mathcal{G}[\xi]$. Then the velocity

$$u = \nabla \times \phi = (u^r e_r + u^z e_z)$$

is simply discovered by

$$u = -\frac{\partial_z\psi}{r}e_r + \frac{\partial_r\psi}{r}e_z =: \mathcal{K}[\xi].$$

This is called the axi-symmetric Biot Savart law.

2.5 As a vortex ring? A traveling wave!

Let's suppose that we are searching for *steady vortex rings*. In other words, we are interested in traveling wave solutions ξ to the active scalar equation (1):

$$\partial_t \xi + u \cdot \nabla \xi = 0, \quad u = \mathcal{K}[\xi]$$

of the form

$$\xi(\mathbf{x}, t) = \xi_{vr}(\mathbf{x} - Wte_z) \tag{3}$$

for some compactly supported $\xi_{vr}(\mathbf{x})$ and for some constant $W \in \mathbb{R}$ which is the propagation speed. We set

$$U = \mathcal{K}[\xi_{vr}] - We_z,$$

which gives the stationary equation:

$$\begin{cases} U \cdot \nabla \xi_{vr} = 0, & \mathbf{x} \in \mathbb{R}^3, \\ U(\mathbf{x}) \to -We_z & \text{as} \quad |\mathbf{x}| \to \infty. \end{cases}$$

By denoting the (scalar) stream $\psi_{vr} = \mathcal{G}[\xi_{vr}]$ and by denoting the adjusted stream

$$\Psi_{vr} = \psi_{vr} - \frac{1}{2}Wr^2 - \gamma, \quad \gamma \in \mathbb{R}(\text{flux constant}),$$

we have

$$\frac{\partial(\xi_{vr}, \Psi_{vr})}{\partial(r, z)} = 0.$$

In other words, we shall seek a function ξ_{vr} satisfying

$$\xi_{vr} = f(\Psi_{vr})$$

for some function

$$f : \mathbb{R} \to \mathbb{R},$$

which is so-called a *vorticity function*.

As an example, for the Hill's spherical vortex, we simply take the vorticity function

$$(\text{Heaviside function})\ f_H(s) := \begin{cases} 1, & s > 0, \\ 0, & s \leq 0. \end{cases}.$$

Its scalar stream $\psi_H := \mathcal{G}[\xi_H] = \left(-\frac{1}{r^2}\mathcal{L}\right)^{-1}[\xi_H]$ is

$$\psi_H(\mathbf{x}) = \begin{cases} \frac{1}{2}Wr^2\left(\frac{5}{2} - \frac{3}{2}|\mathbf{x}|^2\right), & |\mathbf{x}| \leq 1, \\ \frac{1}{2}Wr^2\frac{1}{|\mathbf{x}|^3}, & |\mathbf{x}| > 1. \end{cases}$$

In this setting, the vortex

$$\xi(t, x) = \xi_H(x - Wte_{x_3}), \quad W = (2/15)$$

is a steady vortex ring to the equation (3) because the adjusted stream (for $\gamma = 0$)

$$\Psi_H = \psi_H - \frac{1}{2}Wr^2$$

satisfies

$$\xi_H = f_H(\Psi_H) = 1_{\{\Psi_H > 0\}}.$$

The Euler scaling gives more solutions (for two parameters)

$$\xi_{H(\lambda,a)} = \lambda \xi_H(x/a) \quad \lambda, a > 0.$$

We remark that there are huge number of vortex rings including the ones from [14], [22], [23], etc.

2.6 Derivation

Amick-Fraenkel [2] in 1986 showed the uniqueness of the Hill's vortex by *moving plane method*. Indeed, suppose that we have ψ satisfying

$$-\frac{1}{r^2}\mathcal{L}\psi(r,z) = f_H(\psi(r,z) - (1/2)Wr^2), \quad r > 0, \quad z \in \mathbb{R}.$$

Then we set $\eta : \mathbb{R}^5 \to \mathbb{R}$ by

$$\eta(\mathbf{y}) = \frac{\psi(r,z)}{r^2}, \quad \mathbf{y} = (y_1, y_2, y_3, y_4, y_5) = (y', y_5) \in \mathbb{R}^5,$$

where $r^2 = |y'|^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$ and $z = y_5$. We observe

$$-\Delta_{\mathbb{R}^5}\eta = -\Delta_{\mathbb{R}^5}\frac{\psi}{r^2} = -\frac{1}{r^2}\mathcal{L}\psi.$$

It implies

$$-\Delta_{\mathbb{R}^5}\eta = f_H(\psi - (1/2)Wr^2) = f_H(r^2 \cdot [\eta - (1/2)W]) = f_H(\eta - (1/2)W) \geq 0, \text{ in } \quad \mathbb{R}^5.$$

The moving plane method [26, 16] says that the above Poisson equation in η gives *monotone radial symmetry* to η in $\mathbb{R}^5$ (up to translation in y_5-direction) Then, it only remains to solve for *monotone* unknown $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and some unknown constant $a > 0$ to the following ODE:

$$\begin{aligned} -\frac{1}{t^4}(t^4\eta')' &= 1, \quad 0 < t < a, \\ -\frac{1}{t^4}(t^4\eta')' &= 0, \quad t > a, \\ \eta(a) &= \frac{1}{2}W, \quad \eta(\infty) = 0. \end{aligned}$$

Fortunately, we can easily solve the problem by setting $\eta = \psi_H/r^2$ for $W = (2/15)$ with $a = 1$.

2.7 Literature related stability

We briefly introduce some previous results related stability questions:

Benjamin [4] in 1976 expected that Hill's vortex $\xi_H = 1_B$ maximizes the kinetic energy among other axi-symmetric patch-type functions $\zeta = 1_A$ having the same impulse

$$\int_A r^2 d\mathbf{x} = \int_B r^2 d\mathbf{x}.$$

Friedman–Turkington [15] in 1981 further developed Benjamin's setting for vortex rings. 7 years later, Wan [29] showed that the Hill's vortex is a nondegenerate local maximum of the kinetic energy under certain constraints.

On the other hand, Moffatt-Moore [20] in 1978, Pozrikidis [24] in 1986, and Protas-Elcrat [25] in 2016 focused on linearized (or approximated) responses to patch-type perturbations to *its boundary* and/or related numerical computations. Among them, we would like to emphasize Pozrikidis's simulation showing that an initially prolate perturbation develops into a long tail, which was the main motivation of the joint work [9] (see Figure 2).

As a remark, we can expect only nonlinear stability *up to a translation* since, by scaling, we have a two-parameter family $\xi_{H(\lambda,a)}$ for each $\lambda > 0$(vortex strength) and $a > 0$(size of vortex). Indeed, we just imagine two Hill's vortices with distinct speeds. Due to the different speeds, they will be disjoint after finite time.

3 Stability theorem with applications

Here is the orbital stability statement of Hill's vortex for patch-perturbations.

Theorem 3.1 (Theorem 1.1 of [6])**.** *The Hill's vortex is orbitally stable in axi-symmetric patch-type perturbations in the sense that for $\varepsilon > 0$, there exists $\delta > 0$ such that for any axi-symmetric measurable subset $A_0 \subset \mathbb{R}^3$ satisfying $A_0 \subset \{0 \le r < R\}$ for some $R < \infty$ and*

$$\int_{A_0 \triangle B} (1 + r^2)\, d\mathbf{x} \le \delta,$$

the corresponding solution $\xi(t) = 1_{A_t}$ for the initial data $\xi_0 = 1_{A_0}$ satisfies

$$\int_{A_t \triangle B^{\tau(t)}} (1 + r^2)\, d\mathbf{x} \le \varepsilon \quad \text{for all} \quad t \ge 0, \tag{4}$$

for some shift function $\tau(\cdot_t) : [0, \infty) \to \mathbb{R}$ satisfying $\tau(0) = 0$, where

$$B^\tau := \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x} - \tau e_z| < 1\}$$

is the unit ball centered at $\tau e_{x_3} \in \mathbb{R}^3$.

The value of τ is not uniquely defined but is multiply defined. In [6], we can allow any non-negative initial data as perturbed initial data. The strategy (and structure) for the proof is somewhat parallel to the previous joint work [1] with K. Abe on stability of Lamb's dipole for the 2D Euler equations. The key difference lies how to characterize the patch maximizers as traveling waves.

In the above orbital stability theorem, we have shown the existence of a shift function $\tau(\cdot_t)$ satisfying

$$\sup_{t\geq 0} \|\xi(t) - 1_{B^{\tau(t)}}\|_X \leq \varepsilon, \qquad \|f\|_X = \|f\|_{L^1\cap L^2} + \|r^2 f\|_{L^1}.$$

Here, the shift $\tau(t)$ is defined only implicitly from a contradiction argument. So natural questions are arising:
Q1) Do we have an estimate of $\tau(t)$? We recall the speed $W = \frac{2}{15}$ for the vortex. Thus we may expect

$$\tau(t) \sim Wt \qquad \text{or} \qquad \dot{\tau} \sim W.$$

Q2) The numerical experiment from Pozrikidis [24] in 1986 showed that an initially prolate perturbation develops into a long tail (see Figure 2). Can we prove such a filamentation phenomenon?

We partly answer the questions above.

Theorem 3.2. *(I) [Theorem 1.2 of [6]] For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Given axi-symmetric ξ_0 satisfying $\xi_0, r\xi_0 \in L^\infty(\mathbb{R}^3)$, $\xi_0 \geq 0$, and*

$$\|\xi_0 - 1_B\|_X \leq \delta,$$

the corresponding solution $\xi(t)$ satisfies

$$\|\xi(t) - 1_{B^{\tau(t)}}\|_X \leq \varepsilon, \quad t \geq 0 \tag{5}$$

for some shift function $\tau(\cdot_t) : [0,\infty) \to \mathbb{R}$ satisfying $\tau(0) = 0$.
(II) [Theorem B of [9]] There exists an absolute constant $\varepsilon_0 > 0$ such that, for each $M > 0$, if $\|\xi_0\|_{L^\infty} \leq M$ and if $\varepsilon \in (0, \varepsilon_0)$, then the shift τ satisfies

$$|\tau(t) - tW| \leq C_M\sqrt{\varepsilon}(t+1), \quad t \geq 0, \tag{6}$$

where $C_M > 0$ is a constant depending only on M.

The statement (II) above says that the moving speed of the perturbed vortex is close to that of the original Hill's vortex if the perturbation is sufficiently small at the initial time.

3.1 Ideas for orbital stability

To give a concrete idea first, consider the 2D Euler equations and its simplest steady solution: vortex patch on disk ($\bar{\omega} = \mathbf{1}_D$). Then the basic strategy for showing Lyapunov $\varepsilon - \delta$ stability of $\bar{\omega}$ consists of

1. to set E : certain energy (e.g. $\int |x|^2 \omega \, dx$) and X : some space (e.g. $\{f \in L^1 \,|\, f = \mathbf{1}_A\}$).

2. (easy direction) to show

$$|E(\omega) - E(\bar{\omega})| \le C\|\omega - \bar{\omega}\|_X$$

for general $\omega \in X$.

3. (hard direction) to show $\bar{\omega}$ is a *non-degenerate* energy minimum. For instance, we can show

$$E(\omega) - E(\bar{\omega}) \ge c\|\omega - \bar{\omega}\|_X^2$$

for $\omega \in X$ with some constraint(e.g. $\int \omega\, dx = \int \bar{\omega}\, dx$) around $\bar{\omega}$.

4. (by recalling $E(\omega(t)) = E(\omega_0)$ for solutions) to get stability under the constraint by

$$\begin{aligned}\|\omega(t) - \bar{\omega}\|_X^2 &\lesssim E(\omega(t)) - E(\bar{\omega}) \\ &= E(\omega_0) - E(\bar{\omega}) \lesssim \|\omega_0 - \bar{\omega}\|_X, \quad \forall t \ge 0.\end{aligned}$$

5. To remove the constraint.

We refer to the wonderful textbook [19] of Machioro–Pulvirenti whose idea is going back to Kelvin [28] in 1880, Arnold [3] in 1966, Wan–Pulvirenti [30] in 1985, etc.

When considering traveling waves of the Euler equations, the situation becomes complicated. Following the strategy of Burton-Lopes-Lopes [5] and [1], we can show Lyapunov orbital $\varepsilon - \delta$ stability of certain traveling wave solution $\bar{\omega}$:

1. to set a proper variational problem. i.e. decide 1. an admissible class with some constraints and 2. a conserved energy functional.

2. (existence) to show $\exists$ a minimizer for the functional in the class.

3. (uniqueness) to show the target $\bar{\omega}$ is the unique minimizer up to translation.

4. (compactness) to show any sequence converges up to translation to $\bar{\omega}$ as long as the involved norms converge to those of $\bar{\omega}$.

5. to get $\varepsilon - \delta$ stability up to translation via a contradiction argument.

For the Hill's vortex case, each step is non-trivial. However, fortunately, there have been remarkable developments by many mathematicians. We list some:

1. variational framework based on vorticity from Benjamin [4] in 1976, Friedman–Turkington [15] 1981, . . . ,

2. concentrated compactness lemma from Lions [18] in 1984, . . . ,

3. uniqueness of Hill's vortex in a certain situation by Amick–Fraenkel [2] in 1984,

4. existence of weak solution with required conservations from Nobili–Seis in [21] in 2020, . . . ,

5. the notion of metrical boundary (e.g. Croft [11] in 1982, Szenes [27] in 2011, . . .).

Here is the setting for our proof:

1. Kinetic energy $E[\xi] = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} \xi \mathcal{G}[\xi] dx, \quad \mathcal{G} = (-\frac{1}{r^2}\mathcal{L})^{-1}$.
2. We follow the variational setting from Friedman-Turkington [15].
   For $\mu > 0$, set the space of admissible functions
   $$\mathcal{P}_\mu = \left\{ \xi \in L^\infty(\mathbb{R}^3) \;\middle|\; \xi = 1_A \text{ for some axi-sym. } A \subset \mathbb{R}^3\,,\ \frac{1}{2}\|r^2\xi\|_1 = \mu,\ \|\xi\|_1 \le 1 \right\}.$$
3. Variational problem of maximizing the energy E on $\mathcal{P}_\mu$.
   $$\mathcal{I}_\mu = \sup_{\xi \in \mathcal{P}_\mu} E[\xi].$$
4. (Existence) The set S_μ of maximizers is non-empty.
5. (Property) Every maximizer is a steady vortex ring (i.e. a traveling wave):
   $\exists! W > 0, \gamma \ge 0$
   $$\xi = 1_{\{\Psi > 0\}} \quad \text{a.e.} \quad \text{for} \quad \Psi = \mathcal{G}[\xi] - \frac{1}{2}Wr^2 - \gamma.$$
6. (Uniqueness) $S_\mu = \{\xi_{H(a)}(\cdot + ce_z) \,|\, c \in \mathbb{R}\}$ for small μ where $\xi_{H(a)} = 1_{B_a} = 1_{\{|x|<a\}}$.
7. (compactness) Using concentrated compactness lemma gives strong convergence up to a translation.
8. (contradiction argument) Any solutions which initially close to the Hill's vortex stay closed to *some translated* Hill's vortex.

3.2 Ideas for dynamic stability

When estimating the shift position $\tau(t)$, the first difficulty appears because the shift function may not be continuous. Indeed, there are infinitely many functions satisfying the stability estimate (4) (or (5)). The difficulty is removed by noting that discontinuity (if exists) is limited only to small jumps (Lemma 3.1 in [9]):

$$|\tau(t) - \tau(t')| \lesssim \varepsilon^{1/2}, \quad t, t' \ge 0$$

whenever $|t - t'| \lesssim \varepsilon^{1/2}$. Then we take two bootstrap hypotheses:
One is for the difference of the shift function in time

$$|\tau(t) - tW| \ll_\varepsilon 1$$

while the other is for the particle trajectory $\phi(t, B)$:

$$\left|\phi(t, B) \setminus B^{\tau(t)}\right| \ll_\varepsilon 1.$$

We prove that they are well-controlled for short time $[0, t_0]$ by assuming smallness on the bound $t_0 > 0$. Then, a bootstrap argument gives local control in global-time:

Proposition 3.3 (Proposition 3.2 in [9]). *Any function* τ *satisfies*

$$|\tau(t') - \tau(t) - W(t' - t)| \lesssim \varepsilon^{1/2} \tag{7}$$

for any $t \geq 0$ *and for any* $t' \in (t, t + \alpha_0)$.

By summing the estimate (7), we obtain

$$|\tau(t) - Wt| \lesssim \sqrt{\epsilon}(t + 1).$$

4 Applications

As applications of the stability, we present examples of filamentation and vorticity growth.

4.1 Filamentation

We recall that the core of perturbed solutions moves along the axis *linearly* by (6). As an application, we are interested in linear in time filamentation phenomena. By *linear in time filamentation*, we simply mean

$$\mathrm{diam}(\mathrm{supp}(\xi(t, \cdot))) \gtrsim t.$$

Such a filamentation phenomenon was already observed numerically by Pozrikidis [24] for prolate spheroid in 1986 at least for short time. In Corollary 1.1 of [9], we confirmed such a development rigorously (see Figure 2).

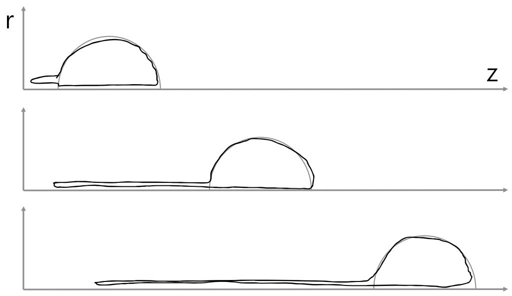


Figure 2: An illustration of development of a long tail obtained [9]

To produce filamentation, we use

1. L^p-smallness in vorticity level of perturbations gives L^∞-smallness in velocity $u = K[\xi]$ level. In the axisymmetric setting, the corresponding estimate can be found in Feng-Sverak [13] in 2015:

$$\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \|r^2\xi\|_{L^1(\mathbb{R}^3)}^{1/4} \|\xi\|_{L^1(\mathbb{R}^3)}^{1/4} \|\xi\|_{L^\infty(\mathbb{R}^3)}^{1/2}.$$

2. Since the operator K for $u = K[\xi]$ is linear, e.g. we see

$$u - u_H = K[\xi] - K[\xi_H] = K[\xi - \xi_H],$$

so

$$\|u - u_H\|_{L^\infty(\mathbb{R}^3)} \lesssim \|r^2(\xi - \xi_H)\|_{L^1(\mathbb{R}^3)}^{1/4} \|\xi - \xi_H\|_{L^1(\mathbb{R}^3)}^{1/4} \|\xi - \xi_H\|_{L^\infty(\mathbb{R}^3)}^{1/2}.$$

3. The above argument says

$$u(t) = \underbrace{u(t) - u_H^{\tau(t)}}_{L^\infty\text{-small by stability for all time}} + u_H^{\tau(t)}.$$

4. Lastly, we recall $\tau(t) \sim Wt$ by (6) while we perfectly know the velocity field u_H (2) induced by ξ_H.

4.2 Growth in vorticity maximum

In this time, we study growth of vorticity maximum $\|\omega(t)\|_{L^\infty(\mathbb{R}^3)}$. We first recall *Beale–Kato–Majda criterion* saying

a solution can blow up in finite time

only if the vorticity maximum $\|\omega(t)\|_{L^\infty(\mathbb{R}^3)}$ does at the time.

We remark that there is a recent breakthrough by Elgindi [12] constructing a finite-time blow-up of vorticity maximum $\|\omega(t)\|_{L^\infty(\mathbb{R}^3)}$ for some initial $\omega_0 \in C^\gamma$ data $0 < \gamma \ll 1$.

What if ω_0 is more regular than C^γ? It is interesting to search for *smoother* initial data which can lead to *large growth* of the vorticity maximum. To obtain growth, we note the equation (1) for relative vorticity $\xi := \omega^\theta / r$:

$$\partial_t \xi + u \cdot \nabla \xi = 0.$$

We recall that ξ is just transported

$$\xi(t, \Phi(t, \mathbf{x}_0)) = \xi_0(\mathbf{x}_0)$$

by the flow $\mathbf{x} = \Phi(t, \mathbf{x}_0)$ induced by the solution:

$$\frac{d}{dt}\Phi(t, \mathbf{x}_0) = u(t, \Phi(t, \mathbf{x}_0)).$$

Thus when considering patch-type ξ_0, we observe that the vorticity maximum is controlled by maximum of $r-$height of trajectories:

$$\|\omega(t)\|_{L^\infty} = \sup_{\mathbf{x}_0 \in \operatorname{supp} \xi_0} \Phi^r(t, \mathbf{x}_0).$$

Thus, one way to get growth in vorticity maximum is to find a particle in support of ω_0 escaping into large r-region.

As another corollary of the stability theorem for Hill's vortex, we get growth of vorticity maximum $\|\omega(t)\|_{L^\infty}$ for finite (but arbitrarily large) time with some non-negative vorticity:

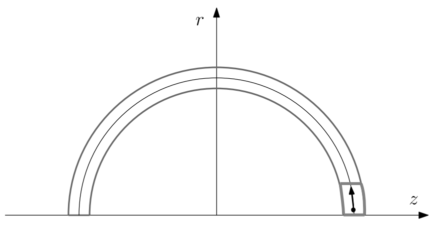
Corollary 4.1 (Corollary 1.3 of [9]). *For $L > 1$, there exists a compactly supported initial data $\omega_0 \in C^\infty(\mathbb{R}^3)$ satisfying*

$$\omega_0^\theta \geq 0, \quad \|\omega_0\|_{L^\infty(\mathbb{R}^3)} \leq 1, \quad \textit{and} \quad \sup_{t\in[0,c\log(L)]}\|\omega(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \geq L.$$

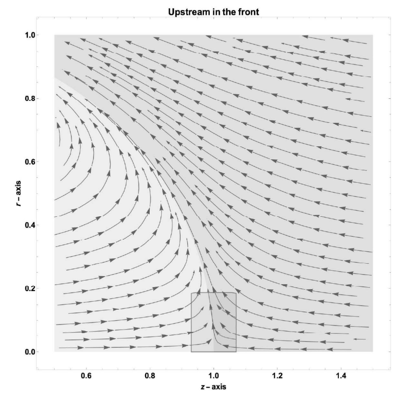
The key idea is to observe that the velocity of Hill's vortex satisfies the upstream:

$$u_H^r(r,z) \sim r$$

near the front $z = 1$ (see Figure 3).



(a) The small front band of upwind



(b) The streamline

Figure 3: An illustration of the upstream in the front

We remark that such a norm-inflation statement in three-dimensional Euler equations is not new. For instance, we have already seen the finite-time blow-up of Elgindi [12] from Holder data $\omega_0 \in C^\gamma$. What differentiates our result from others is that our vorticity ω_0 is compactly supported, C^∞-smooth, and non-negative. Very recently in [7], we found an *infinite-growth* example $\omega_0 \in C^{1,\gamma}(\mathbb{R}^3)$ having both signs with

$$\|\omega(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \to \infty \quad \text{as} \quad t \to \infty.$$

5 Some open questions

We finish this review paper mentioning interesting open problems remained.

1. It is interesting to investigate where $\frac{d}{dt}\tau(t)$ goes eventually. We might guess that it should be close to some $W' > 0$ which is the speed of a (scaled) Hill's vortex whose impulse is equal to the perturbed data.

2. Can we obtain stability of general vortex rings near the Hill's vortex? How about vortex rings far from the Hill's vortex such as vortex sheets?

3. Is the sign-condition for perturbed data necessary? It seems that Lyapunov stability from energy maximiser cannot avoid the sign condition for perturbed data (e.g. see other settings [8], [10]). If it is necessary, one may try to find an example having both signs which develop instability.

References

[1] K. Abe and K. Choi. Stability of Lamb dipoles. *Arch. Ration. Mech. Anal.*, 244(3):877–917, 2022.

[2] C. J. Amick and L. E. Fraenkel. The uniqueness of Hill's spherical vortex. *Arch. Rational Mech. Anal.*, 92:91–119, 1986.

[3] V. I. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.

[4] T. B. Benjamin. The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics. pages 8–29. Lecture Notes in Math., 503, 1976.

[5] G. R. Burton, H. J. Nussenzveig Lopes, and M. C. Lopes Filho. Nonlinear stability for steady vortex pairs. *Comm. Math. Phys.*, 324:445–463, 2013.

[6] K. Choi. Stability of Hill's spherical vortex. *Comm. Pure Appl. Math., https://doi.org/10.1002/cpa.22134.*

[7] K. Choi and I.-J. Jeong. On vortex stretching for anti-parallel axisymmetric flows. *Amer. J. Math., to appear.*

[8] K. Choi and I.-J. Jeong. Stability and instability of Kelvin waves. *Calc. Var. Partial Differential Equations*, 61(6):Paper No. 221, 38, 2022.

[9] K. Choi and I.-J. Jeong. Filamentation near Hill's vortex. *Comm. Partial Differential Equations*, 48(1):54–85, 2023.

[10] K. Choi and D. Lim. Stability of radially symmetric, monotone vorticities of 2D Euler equations. *Calc. Var. Partial Differential Equations*, 61(4):Paper No. 120, 27, 2022.

[11] H. T. Croft. Three lattice-point problems of Steinhaus. *Quart. J. Math. Oxford Ser. (2)*, 33(129):71–83, 1982.

[12] T. Elgindi. Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. *Ann. of Math. (2)*, 194(3):647–727, 2021.

[13] H. Feng and V. Šverák. On the Cauchy problem for axi-symmetric vortex rings. *Arch. Ration. Mech. Anal.*, 215:89–123, 2015.

[14] L. E. Fraenkel. On steady vortex rings of small cross-section in an ideal fluid. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 316:29–62, 1970.

[15] A. Friedman and B. Turkington. Vortex rings: existence and asymptotic estimates. *Trans. Amer. Math. Soc.*, 268:1–37, 1981.

[16] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68:209–243, 1979.

[17] M. J. M. Hill. On a spherical vortex. *Philos. Trans. Roy. Soc. London Ser. A*, 185:213–245, 1894.

[18] P. L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1:109–145, 1984.

[19] C. Marchioro and M. Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[20] H. K. Moffatt and D. W. Moore. The response of Hill's spherical vortex to a small axisymmetric disturbance. *J. Fluid Mech.*, 87:749–760, 1978.

[21] C. Nobili and C. Seis. Renormalization and energy conservation for axisymmetric fluid flows. *Math. Ann. https://doi.org/10.1007/s00208-020-02050-0*, 2020.

[22] J. Norbury. A steady vortex ring close to Hill's spherical vortex. *Proc. Cambridge Philos. Soc.*, 72:253–284, 1972.

[23] J. Norbury. A family of steady vortex rings. *J. Fluid Mech.*, 57:417–431, 1973.

[24] C. Pozrikidis. The nonlinear instability of hill's vortex. *Journal of Fluid Mechanics*, 168:337367, 1986.

[25] B. Protas and A. Elcrat. Linear stability of Hill's vortex to axisymmetric perturbations. *J. Fluid Mech.*, 799:579–602, 2016.

[26] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[27] A. Szenes. Exceptional points for Lebesgue's density theorem on the real line. *Adv. Math.*, 226(1):764–778, 2011.

[28] W. Thomson (Lord Kelvin). Maximum and minimum energy in vortex motion, Nature 22, no. 574, 618–620 (1880). In *Mathematical and Physical Papers 4*, pages 172–183. Cambridge: Cambridge University Press, 1910.

[29] Y. H. Wan. Variational principles for Hill's spherical vortex and nearly spherical vortices. *Trans. Amer. Math. Soc.*, 308:299–312, 1988.

[30] Y. H. Wan and M. Pulvirenti. Nonlinear stability of circular vortex patches. *Comm. Math. Phys.*, 99:435–450, 1985.