Convergence of approximating solutions of the Navier-Stokes equations

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1 Introduction

Let us consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n (n \geq 2)$;

$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0 \text{ in } \mathbb{R}^n \times (0,T), \\ \text{div } u = 0 \text{ in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = a \text{ in } \mathbb{R}^n \end{cases}$$

where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $\pi = \pi(x,t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and the time $t \in (0,T)$, respectively, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ is the given initial data of velocity. In the famous paper of Kato [1], he proved that for every $a \in L^n_\sigma(\mathbb{R}^n) \equiv PL^n(\mathbb{R}^n)$, there exist $0 < T < \infty$ and a unique solution $u \in BC([0,T); L^n_\sigma(\mathbb{R}^n))$ of integral equation

$$\text{(IE) } u(t) = u_0(t) - \int_0^t P\nabla \cdot e^{-(t-s)A}(u \otimes u)(s)ds, \ 0 < t < T,$$
$$\text{with } u_0(t) = e^{-tA}a$$

satisfying properties

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}u(t) \in BC([0,T); L^p(\mathbb{R}^n)) \text{ for all } n \leq p \leq \infty, \tag{1.1}$$

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})+\frac{1}{2}}\nabla u(t) \in BC([0,T); L^p(\mathbb{R}^n)) \text{ for all } n \leq p < \infty \tag{1.2}$$

where P is the Helmholtz projection onto the solenoidal vector fields and $A = -P\Delta$ denotes the Stokes operator. We call such a solution u the *mild solution* of (N-S) on

$(0,T)$. It is known that the mild solution u necessarily satisfies $u \in C((0,T); W^{2,n}(\mathbb{R}^n)) \cap C^1((0,T); L^n_\sigma(\mathbb{R}^n))$ and fulfills the abstract evolution equation

$$\text{(E)} \begin{cases} \dfrac{du}{dt} + Au + P(u\cdot\nabla)u = 0 \text{ in } L^n_\sigma(\mathbb{R}^n),\ 0<t<T, \\ u(0) = a \end{cases}$$

due to the theory of the holomorphic semigroup. We call such a solution u the *strong solution* of (N-S) on $(0,T)$. Kozono-Okada-Shimizu [3] constructed the strong solution of (N-S) for more general initial data in homogeneous Besov space as well as its analyticity. Let us define the approximating solutions $\{u_j\}_{j=0}^\infty$ of (IE).

$$\begin{cases} u_0(t) = e^{-tA}a, \\ u_{j+1}(t) = u_0(t) - \displaystyle\int_0^t P\nabla\cdot e^{-(t-s)A}(u_j\otimes u_j)(s)ds,\ j=0,1,2,\cdots. \end{cases}$$

In this work, we define a *very mild solution* u as a solution of (IE) only satisfying (1.1) and show that it necessarily becomes the strong solution of (N-S) on $(0,T)$. We also show that the convergence of the approximating solutions $\{u_j\}_{j=0}^\infty$ corresponding to the norm in (1.1)

$$\sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u_j(t)-u(t)\|_p \to 0 \text{ as } j\to\infty \tag{1.3}$$

for some $n \le p < \infty$, necessarily yields that $\{u_j\}_{j=0}^\infty$ converge to u in the topology of $C((0,T); W^{2,n}(\mathbb{R}^n))$ and $C^1((0,T); L^n_\sigma(\mathbb{R}^n))$. It should be noted that (1.3) is the most fundamental scaling invariant norm of u in L^p with the time weight. Throughout this paper, we denote by $\|\ \|_p$ the usual L^p-norm on $\mathbb{R}^n$.

2 Main Results

Before stating our main theorems, let us define a function space X_p by

$$X_p = \{u \in C((0,T); L^p_\sigma(\mathbb{R}^n));\ t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}u \in BC([0,T); L^p(\mathbb{R}^n))\}$$

for $n \le p < \infty$. X_p is a Banach space with the norm

$$\|u\|_{X_p} = \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_p. \tag{2.1}$$

Our main results now read:

Theorem 2.1. (Koizumi-Taniguchi [2]) *Let $a \in L^n_\sigma(\mathbb{R}^n)$. Suppose that u is a very mild solution of (N-S) on $(0,T)$. Then, it necessarily holds that*

(i) $t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}}\nabla u \in BC([0,T); L^q(\mathbb{R}^n))$ *for all* $n \le q < \infty$;
(ii) $u \in C^1((0,T); L^n(\mathbb{R}^n)) \cap C((0,T); W^{2,n}(\mathbb{R}^n))$ *with*

$$\sup_{0<t<T} t\|Au(t)\|_n + \sup_{0<t<T} t\|\partial_t u(t)\|_n < \infty;$$

(iii) u *satisfies the differential equation (E) with* $\|u(t) - a\|_n \to 0$ as $t \to +0$.

Theorem 2.2. (Koizumi-Taniguchi [2]) *Let* $a \in L^n_\sigma(\mathbb{R}^n)$. *Suppose that* u *is a very mild solution of (N-S) on* $(0,T)$. *Let* $\{u_j\}_{j=0}^\infty$ *be approximating solutions of (IE). Then, it holds that* $u_j \in X_p$ *for all* $n \le p < \infty$ *and all* $j = 0, 1, \cdots$ *. If*

$$\|u_j - u\|_{X_p} \to 0 \tag{2.2}$$

for some $n \le p < \infty$, *then we have the following properties* (i) *and* (ii).
(i)

$$\sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}}\|\nabla u_j(t) - \nabla u(t)\|_q \to 0 \text{ as } j \to \infty \tag{2.3}$$

for all $n \le q < \infty$;
(ii)

$$\sup_{0<t<T} t\|Au_j(t) - Au(t)\|_n \to 0 \text{ as } j \to \infty, \tag{2.4}$$

$$\sup_{0<t<T} t\|\partial_t u_j(t) - \partial_t u(t)\|_n \to 0 \text{ as } j \to \infty. \tag{2.5}$$

Remark. (i) It should be noted that convergences (2.4) and (2.5) are obtained only in terms of (2.2). Furthermore, we will clarify that even (2.6) is a consequence of (2.2). We emphasize that (2.2) is closely related to a scaling invariant norm. Indeed, for the norm $\|\cdot\|_{X_p}$ defined in (2.1) with $T = \infty$, it holds that

$$\|u\|_{X_p} = \|u_\lambda\|_{X_p} \text{ for all } \lambda > 0,$$

where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$. Hence, our theorem exhibits that once approximating solutions $\{u_j\}_{j=0}^\infty$ converge to the very mild solution u like (2.2) in such a scaling invariant norm in X_p as (2.1), the convergence in higher ordered Sobolev spaces like (2.4) and (2.5) necessarily holds.
(ii) As a further result of Theorem 2, Koizumi [3] proved that (1.3) necessarily implies that

$$\sup_{0<t<T} t^{m+\frac{|\alpha|}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}\|\partial_t^m D^\alpha u_j(t) - \partial_t^m D^\alpha u(t)\|_q \to 0 \text{ as } j \to \infty \tag{2.6}$$

for all $n \le q < \infty$, all $m \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$ with $D^\alpha = \partial_x^\alpha$. Moreover, if we define the approximating solutions of the pressure $\{p_j(t)\}_{j=1}^\infty$ of (N-S) by $p_j(t) = (-\Delta)^{-1}\text{div div}(u_{j-1} \otimes u_{j-1})(t)$, as a consequence of (2.6), $\{p_j(t)\}_{j=1}^\infty$ satisfy

$$\sup_{0<t<T} t^{m+\frac{|\alpha|}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}}\|\partial_t^m D^\alpha p_j(t) - \partial_t^m D^\alpha p(t)\|_q \to 0 \text{ as } j \to \infty \tag{2.7}$$

for all $n \le q < \infty$, all $m \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$, where p is the pressure determined by (E) i.e., $p(t) = (-\Delta)^{-1}\mathrm{div}\ \mathrm{div}(u \otimes u)(t)$.

3 Outline of the Proof

In what follows we use the symbols $\varepsilon_j(t) = u_j(t) - u(t), \mathcal{E}_j(t) = (u_j \otimes u_j)(t) - (u \otimes u)(t)(j = 0, 1, \dots)$. The following lemma is essential for the proof of our main theorems.

Lemma. *Let $a \in L^n_\sigma(\mathbb{R}^n)$. Suppose that $\{u_j\}_{j=0}^\infty$ and u are approximating solutions of (IE) and a very mild solution of(N-S) on $(0, T)$, respectively. Then we have the following estimates (i),(ii) and (iii).*

(i)

$$\|u\|_{Y_q} \le C(\|u_0\|_{X_q} + \|u\|^2_{X_{2q}} + \|u\|^2_{X_{2r}} + \|u\|_{X_{2q}}(\|u_0\|_{X_{2q}} + \|u\|^2_{X_{2q}})) \tag{3.1}$$

with $n \le q < \infty$ and $n/2 < r < n$.

(ii)

$$\|\varepsilon_j\|_{Y_q} \le C\|\varepsilon_{j-1}\|_{X{q2}}\big(\|u_0\|_{X_{q_1}} + \|u\|^2_{X_{q_1}} + \|u_{j-2}\|^2_{X_q} + (\|u\|_{X_{q_1}} + \|u_{j-1}\|_{X_{q_1}})(\|u\|_{X_{q_2}} + \|u_{j-2}\|_{X_{q_2}})\big) \tag{3.2}$$

with $n \le q, q_1, q_2 < \infty$ and $1/q = 1/q_1 + 1/q_2$.

(iii)

$$\sup_{0<t<T} t\|A\varepsilon_j(t)\|_n + \sup_{0<t<T} t\|\partial_t\varepsilon_j(t)\|_n \tag{3.3}$$

$$\le C\Big(\|\varepsilon_{j-1}\|_{Y_q}\big(\|u_0\|_{X_p} + \|u\|_{X_p} + \|u\|^2_{X_p}\big) + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}} + \|\varepsilon_{j-2}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}}$$
$$+ \|\varepsilon_{j-1}\|_{X_p}\big(\|u_0\|_{X_q} + \|u_{j-1}\|_{Y_q} + \|u_{j-2}\|_{X_{r_1}}\|u_{j-2}\|_{Y_{r_2}}\big) + \|\varepsilon_{j-2}\|_{X_p}\|u_{j-1}\|_{Y_q}\big(\|u\|_{X_p} + \|u_{j-2}\|_{X_p}\big)$$
$$+ \|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-2}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-2}\|_{Y_{r_2}}\Big)$$

with $n < p, q < \infty, 1/n = 1/p + 1/q, n \le r_1, r_2 < \infty$ and $1/n < 1/r = 1/r_1 + 1/r_2 < 1/q + 1/n$.

The proof of the lemma is based on Hölder continuity in time of $u(t), \varepsilon_j(t)$ and $\mathcal{E}_j(t)$ as L^p-valued functions. We only prove (3.3) using the following proposition which shows Hölder continuity in time of $\nabla\mathcal{E}_j(t)$ as an L^n-valued function.

Proposition. *Let $1/n = 1/p + 1/q$ with $n < p, q < \infty$. Assume that $1/n < 1/r = 1/r_1 + 1/r_2 < 1/q + 1/n$ with $n \le r_1, r_2 < \infty$. Suppose that $\{u_j\}_{j=0}^\infty$ and u are approximating solutions of (IE) and a very mild solution of (N-S) on $(0, T)$, respectively. Assume that $u_j \in X_{\tilde{p}}$ for all $j = 0, 1, \cdots$ and all $n \le \tilde{p} < \infty$. Then it holds that*

$$\|\nabla\mathcal{E}_j(t + h) - \nabla\mathcal{E}_j(t)\|_n$$

$$\leq C\Big((\|\varepsilon_j\|_{Y_q}\|u_0\|_{X_p} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}}$$
$$+ \|\varepsilon_{j-1}\|_{X_p}\|u_j\|_{Y_q}(\|u\|_{X_p} + \|u_{j-1}\|_{X_p}) + \|\varepsilon_{j-1}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-1}\|_{Y_{r_2}}$$
$$+ \|\varepsilon_j\|_{X_p}(\|u_0\|_{X_q} + \|u_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}})\big)h^{\alpha}t^{-\alpha-1}$$
$$+ \big(\|\varepsilon_j\|_{Y_q}\|u\|^2_{X_p} + \|\varepsilon_{j-1}\|_{X_p}\|u_j\|_{Y_q}(\|u\|_{X_p} + \|u_{j-1}\|_{X_p})\big)h^{\beta_{n,p}}t^{-1-\beta_{n,p}}$$
$$+ \big(\|\varepsilon_j\|_{X_p}\|u_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-1}\|_{Y_{r_2}}$$
$$+ \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}}\big)h^{\beta_{n,q,r}}t^{-1-\beta_{n,q,r}}\Big),$$
$$h > 0,\ 0 < t < T,\ \beta_{n,p} \equiv \frac{1}{2} - \frac{n}{2p},\ \beta_{n,q,r} \equiv \frac{1}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})$$

for all $0 < \alpha < \beta_{n,q,r}$*, where* $C = C(n,p,q,r,\alpha)$ *is a constant independent of* h *and* t*.*

Proof of (3.3). By (IE) and the definition of u_j we have

$$A\varepsilon_j(t) = P(e^{-\frac{t}{2}A} - 1)((u_{j-1}\cdot\nabla)u_{j-1})(t) - ((u\cdot\nabla)u)(t)) \tag{3.4}$$
$$- \int_0^{\frac{t}{2}} PAe^{-(t-s)A}((u_{j-1}\cdot\nabla)u_{j-1})(s) - ((u\cdot\nabla)u)(s))ds$$
$$+ \int_{\frac{t}{2}}^{t} PAe^{-(t-s)A}(\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s))ds$$
$$\equiv J_1(t) + J_2(t) + J_3(t),\ 0 < t < T.$$

It holds by the Hölder inequality and the bounds of $\{e^{-tA}\}_{t>0}$ and P in L^p that

$$\|J_1(t)\|_n \leq C\|(u_{j-1}\cdot\nabla)u_{j-1})(t) - ((u\cdot\nabla)u)(t)\|_n \tag{3.5}$$
$$\leq C(\|\varepsilon_{j-1}\|_p\|\nabla u_{j-1}(t)\|_q + \|u(t)\|_p\|\nabla\varepsilon_{j-1}(t)\|_q)$$
$$\leq Ct^{-1}(\|\varepsilon_{j-1}\|_{X_p}\|u_{j-1}\|_{Y_q} + \|\varepsilon_{j-1}\|_{Y_q}\|u\|_{X_p})$$

for all $0 < t < T$ with $C = C(n)$. It holds by the Hölder inequality, L^p-L^q estimate of Stokes semigroup and the bounds of P in L^p that

$$\|J_2(t)\|_n \leq \int_0^{\frac{t}{2}} \|Ae^{-(t-s)A}((u_{j-1}\cdot\nabla)u_{j-1})(s) - ((u\cdot\nabla)u)(s))\|_n ds \tag{3.6}$$
$$\leq C\int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})}\|((u_{j-1}\cdot\nabla)u_{j-1})(s) - ((u\cdot\nabla)u)(s)\|_r ds$$
$$\leq C\int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})}(\|\varepsilon_{j-1}(s)\|_{r_1}\|\nabla u_{j-1}(s)\|_{r_2} + \|u(s)\|_{r_1}\|\nabla\varepsilon_{j-1}(s)\|_{r_2})ds$$
$$\leq C(\|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}})\int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})}s^{-1-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}ds$$
$$\leq C(\|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}})t^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})}\int_0^{\frac{t}{2}} s^{-1-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}ds$$

$$= Ct^{-1}(\|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}})$$

for all $0 < t < T$ with $C = C(n, r)$. By the analiticity of $\{e^{-tA}\}_{t>0}$ we have

$$\|J_3(t)\|_n \le \int_{\frac{t}{2}}^{t} \|Ae^{-(t-s)A}(\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s))\|_n ds$$
$$\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1}\|\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s)\|_n ds$$

for all $0 < t < T$ with $C = C(n)$. Set $\alpha = \beta_{n,q,r}/2$. Changing variable $s \to \tau = t - s$ of integration and using the Proposition, from the above estimate we obtain

$$\|J_3(t)\|_n \tag{3.7}$$
$$\le C\int_0^{\frac{t}{2}} \tau^{-1}\|\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(t-\tau)\|_n d\tau$$
$$\le C\Big(\big(\|\varepsilon_j\|_{Y_q}\|u_0\|_{X_p} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}} + \|\varepsilon_{j-1}\|_{X_p}\|u_j\|_{Y_q}(\|u\|_{X_p} + \|u_{j-1}\|_{X_p})$$
$$+ \|\varepsilon_{j-1}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_j\|_{X_p}(\|u_0\|_{X_q} + \|u_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}})\big)t^{-1-\alpha}\int_0^{\frac{t}{2}}\tau^{-1+\alpha}d\tau$$
$$+ \big(\|\varepsilon_j\|_{Y_q}\|u\|_{X_p}^2 + \|\varepsilon_{j-1}\|_{X_p}\|u_j\|_{Y_q}(\|u\|_{X_p} + \|u_{j-1}\|_{X_p})\big)t^{-1-\beta_{n,p}}\int_0^{\frac{t}{2}}\tau^{-1+\beta_{n,p}}d\tau$$
$$+ \big(\|\varepsilon_j\|_{X_p}\|u_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-1}\|_{Y_{r_2}}$$
$$+ \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}}\big)t^{-1-\beta_{n,q,r}}\int_0^{\frac{t}{2}}\tau^{-1+\beta_{n,q,r}}d\tau\Big)$$
$$\le Ct^{-1}\Big(\|\varepsilon_{j-1}\|_{Y_q}\big(\|u_0\|_{X_p} + \|u\|_{X_p}^2\big) + \|\varepsilon_{j-2}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}}$$
$$+ \|\varepsilon_{j-1}\|_{X_p}\big(\|u_0\|_{X_q} + \|u_{j-2}\|_{X_{r_1}}\|u_{j-2}\|_{Y_{r_2}}\big)$$
$$+ \|\varepsilon_{j-2}\|_{X_p}\|u_{j-1}\|_{Y_q}\ \big(\|u\|_{X_p} + \|u_{j-2}\|_{X_p}\big) + \|\varepsilon_{j-2}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-2}\|_{Y_{r_2}}\big)\Big)$$

for all $0 < t < T$ with $C = C(n, p, q, r)$. Now the desired estimate for $A\varepsilon_j(t)$ follows from (3.4)-(3.7). The estimate for $\partial_t\varepsilon_j(t)$ is easily deduced from the one for $A\varepsilon_j(t)$. In fact, due to the semigroup argument we have

$$\partial_t\varepsilon_j(t) = A\varepsilon_j(t)\ + (P(u_{j-1}\cdot\nabla)u_{j-1})(t) - P((u\cdot\nabla)u)(t)).$$

The second term of the above equality is estimated as

$$\|P(u_{j-1}\cdot\nabla)u_{j-1})(t) - P((u\cdot\nabla)u)(t)\|_n \tag{3.8}$$
$$\le C(\|\varepsilon_{j-1}(t)\|_p\|\nabla u_{j-1}(t)\|_q + \|u(t)\|_p\|\nabla\varepsilon_{j-1}(t)\|_q)$$
$$\le Ct^{-1}(\|\varepsilon_{j-1}\|_{X_p}\|u_{j-1}\|_{Y_q} + \varepsilon_{j-1}\|_{Y_q}\|u\|_{X_p}\|).$$

Combining (3.5)-(3.7) and (3.8), we have (3.3). □

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