# Convex integration method and non-uniqueness of weak solutions for viscous fluids

Yachun Li[a], Peng Qu[b], Zirong Zeng[a], Deng Zhang[a]

[a]School of Mathematical Sciences, Shanghai Jiao Tong University.
[b]School of Mathematical Sciences, Fudan University.

**Abstract**

We review our recent results concerning the non-uniqueness of weak solutions to viscous fluid dynamics, based on the technique of convex integration. Concerning the hyperdissipative incompressible Navier-Stokes and MHD equations, we proved the sharp non-uniqueness of weak solutions in supercritical regimes with respect to the Ladyženskaja-Prodi-Serrin (LPS) criteria. For the hypo-viscous compressible Naver-Stokes equations, we proved that there exist infinitely many weak solutions with the same initial data, which provides the first non-uniqueness result of weak solutions to viscous compressible fluid.

## 1 Background

We are concerned with the three-dimensional incompressible Navier-Stokes equations (INS for short)

$$\begin{cases}\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla P=0,\\ \operatorname{div} u=0,\end{cases}\tag{1.1}$$

where $u=(u_1,u_2,u_3)^\top(t,x)\in\mathbb{R}^3$ and $P=P(t,x)\in\mathbb{R}$ represent the velocity field and pressure of the fluid, respectively, $\nu>0$ is the viscosity coefficient. In the groundbreaking paper [62], Leray demonstrated the existence of weak solutions to INS in the space $L^\infty_tL^2_x\cap L^2_t\dot H^1_x$ which obey the energy inequality

$$\|u(t)\|_{L^2}^2+2\nu\int_{t_0}^t\|\nabla u(s)\|_{L^2}^2\mathrm{d}s\le\|u(t_0)\|_{L^2}^2.\tag{1.2}$$

This solution is now referred to as Leray-Hopf solutions, also due to Hopf [51] in the case of bounded domains. Since then, the uniqueness of Leray-Hopf solutions has remained an challenging open problem in the theory of INS.

One criterion for the well-posedness/ill-posedness is the scaling exponent. It gives a useful classification of subcritical, critical and supercritical spaces. A general philosophy is that equations are well-posed in the subcritical spaces, while solutions may exhibit ill-posedness phenomena in the supercritical spaces. For the INS, the equation is invariant under the scaling

$$u(t,x)\mapsto\lambda u(\lambda^2t,\lambda x),\quad P(t,x)\mapsto\lambda^2P(\lambda^2t,\lambda x),\tag{1.3}$$

and the critical scaling in the mixed Lebsgue space $L^\gamma_tL^p_x$ is the so-called Ladyženskaja-Prodi-Serrin (LPS for short) condition

$$\frac{2}{\gamma}+\frac{d}{p}=1\tag{1.4}$$

with $d$ being the underlying spatial dimension.

It is well-known that weak solutions in the (sub)critical spaces $L^\gamma_tL^p_x$ with $2/\gamma+3/p\le1$ are unique and even are Leray-Hopf solutions. See [28, 40, 41] and references therein. See also [58, 59] for comprehensive discussions of scaling exponents for general nonlinear PDEs.

In contrast, on the flexible side (super-critical regime), based on the convex integration scheme, Buckmaster-Vicol [16] first proved the non-uniqueness of weak solutions to INS. The convex integration approach was introduced to 3D Euler equations by De Lellis and Székelyhidi in the pioneering papers [35, 36]. There have been significant progresses towards the non-uniqueness problem for various fluid models based on convex integrations. One milestone is the resolution of the flexible part of Onsager's conjecture for incompressible Euler equations, developed in [10, 12, 13, 37, 38] and finally settled by Isett [52] and Buckmaster-De Lellis-Székelyhidi-Vicol [14].

The main ingredient introduced by Buckmaster-Vicol [16] is the intermittent building blocks, which in particular permit to control the strong viscosity in INS. The intermittent convex integration also has been applied successfully to various other models. We refer, e.g., to [71] for hyperdissipative INS (1.1), [70] for 2D hypoviscous INS, [73] for stationary INS, and [32, 39] for the non-uniqueness of Leray solutions to hypodissipative INS. See also the surveys [17, 18] for other interesting applications. In particular, in the recent works [15, 77], the intermittent convex integration enables to achieve the regularity close to $1/2$, beyond the Onsager regularity $1/3$, for non-conservative solutions of 3D Euler equations, and provides a new proof of the flexible side of Onsager's conjecture. See also [49, 50] for the proof of the strong Onsager's conjecture.

In the recent remarkable paper [28], Cheskidov-Luo proved the sharp non-uniqueness of weak solutions in view of the LPS condition to INS in the spaces $L^\gamma_t L^\infty_x$ for any $1 \leq \gamma < 2$. Moreover, in the 2D case, they also proved the sharp non-uniqueness in $L^\infty_t L^p_x$, where $1 < p < 2$, which is another endpoint case of the LPS condition. Very recently, for the 3D hyperdissipative INS with viscosity beyond the Lions exponent $5/4$, we proved the sharp non-uniqueness at two endpoints of the LPS condition [63].

We also refer to another programme by Jia and Šverák [53, 54] towards the non-uniqueness of Leray-Hopf solutions, under a certain assumption for the linearized Navier-Stokes operator. Recently, Albritton-Brué-Colombo [1] constructed the non-uniqueness of Leray solutions to the forced incompressible INS. See also [2] for the case of bounded domain and [3] for the case of forced hypo-viscous INS in dimension two.

The compressible models is also one of the famed models of fluid dynamics and has been extensively studied in literature. For the Euler equations of inviscid compressible fluid, although the uniqueness and stability results have been well established in the 1D case with small BV initial data and mild assumptions (see, e.g., [8, 69]), it was a long standing open problem in the multi-dimensional case until it was observed by De Lellis-Székelyhidi [36], that non-unique bounded entropy solutions can be constructed for multi-dimensional compressible Euler equations. Afterwards, several achievements have been obtained and we refer to [9, 22, 29–31, 48, 60, 72, 74] and the references therein for the flexibility of compressible Euler equations.

In contrast to the above extensive studies of compressible Euler equations and INS, the non-uniqueness of weak solutions to compressible Navier-Stokes equations remains a challenging problem.

Besides the Navier-Stokes equations, non-uniqueness and related turbulence of MHD equations also attract significant interests in literature. Faraco-Lindberg [42] first constructed non-vanishing smooth strict subsolutions to 3D ideal MHD. Afterwards, Faraco-Lindberg-Székelyhidi [44, 45] constructed infinitly many bounded solutions with prescribed total energy and cross helicity. Based on the convex integration via staircase laminates, they also solve the conjecture that $L^3_{t,x}$ is the threshold for magnetic helicity conservation in [45]. Based on the intermittent convex integration scheme, Beekie-Buckmaster-Vicol [6] gave the first example of weak solutions with non-conserved magnetic helicity. It was also proved in [6] that Taylor's conjecture fails for the ideal MHD equations.

Concerning the non-uniqueness problem for the viscous and resistive MHD, different from the case of INS, the specific geometry of MHD restricts the choice of oscillation directions, and thus, in particular, limits the spatial intermittency of building blocks in the convex integration scheme. So, the control of viscosity and resistivity in (4.1) becomes significantly hard. In [66], we provided the first examples of sharp non-uniqueness for MHD equations near one endpoint of the critical-scaling LPS condition. This in particular extends the recent sharp non-uniqueness result in [28] in the context of Navier-Stokes equations. Another interesting phenomena observed in [66] is that the scaling-invariant LPS condition serves as the criterion to detect non-uniqueness: the uniqueness would fail even in the high viscous and resistive regime beyond the Lions exponent, if the (sub)criticality of state space is violated. We also refer to [34] for the case of Hall MHD equations, [65] for MHD equations with the viscosity and resistivity up to the Lions exponent and [76] for the non-uniqueness in $C_tL^2_x$ space. See also [24] and [25] for other ill-posedness phenomena.

**Organization of paper.** In §2 we present our recent results on the sharp non-uniqueness of weak solutions in view of the LPS criterion to hyperdissipative Navier-Stokes equations [63]. Then, §3 is devoted to the results of the non-uniqueness of weak solutions to hypoviscous compressible Navier-Stokes equations [64]. Finally, in §4 we review the results on the non-uniqueness of weak solutions to viscous and resistive MHD equations [65, 66].

## 2 Incompressible Navier-Stokes equations

In this section, we review our recent result concerning the sharp non-uniqueness of weak solutions to the following hyperdissipative Navier-Stokes equations [63] on the torus $\mathbb{T}^3 := [-\pi,\pi]^3$:

$$\begin{cases}\partial_t u + \nu(-\Delta)^\alpha u + (u\cdot\nabla)u + \nabla P = 0,\\ \operatorname{div} u = 0,\end{cases} \tag{2.1}$$

where $u = (u_1, u_2, u_3)^\top(t,x) \in \mathbb{R}^3$ and $P = P(t,x) \in \mathbb{R}$ represent the velocity field and pressure of the fluid, respectively, $\nu > 0$ is the viscosity coefficient, $\alpha \in [1,2)$ is the viscosity exponent, and the hyperdissipation operator $(-\Delta)^\alpha$ is defined on the flat torus via the Fourier transform

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

To begin with, let us formulate precisely the definition of weak solutions to equation (2.1).

**Definition 2.1.** *(Weak solutions) Given any weakly divergence-free initial datum $u_0 \in L^2(\mathbb{T}^3)$, we say that $u \in L^2([0,T]\times\mathbb{T}^3)$ is a weak solution for the hyperdissipative Navier-Stokes equations* (2.1) *if $u$ is divergence-free for a.e. $t \in [0,T]$, and*

$$\int_{\mathbb{T}^3} u_0\varphi(0,x)\mathrm{d}x = -\int_0^T\int_{\mathbb{T}^3} u(\partial_t\varphi - \nu(-\Delta)^\alpha\varphi + (u\cdot\nabla)\varphi)\mathrm{d}x\mathrm{d}t \tag{2.2}$$

*for any divergence-free test function $\varphi \in C_0^\infty([0,T)\times\mathbb{T}^3)$.*

The uniqueness of weak solutions to (2.1) in the critical mixed Lebesgue space $L^\gamma_tW^{s,p}_x$ is contained in Theorem 2.2 below.

**Theorem 2.2** (Uniqueness of weak solutions in $L^\gamma_tW^{s,p}_x$, [63])**.** *Let $u \in L^\gamma_tW^{s,p}_x$ and $(s,\gamma,p)$ satisfy*

$$\frac{2\alpha}{\gamma} + \frac{3}{p} = 2\alpha - 1 + s, \tag{2.3}$$

*with $\alpha \geq 1$, $s \geq 0$, $2 \leq \gamma \leq \infty$, $1 \leq p \leq \infty$ and $0 \leq \frac{1}{p} - \frac{s}{3} \leq \frac{1}{2}$. If $u$ is a weak solution to* (2.1) *in the sense of Definition 2.1, then $u$ is a unique Leray-Hopf solution.*

**Main results.** Our mains results include the sharp non-uniqueness at two LPS endpoints, the strong non-uniqueness for the high dissipativity beyond the Lions exponent, the partial regularity of weak solutions, and the vanishing viscosity limit.

**(i) Sharp non-uniqueness at two LPS endpoints.** We focus on the non-uniqueness of weak solutions in the following two supercritical regimes, whose borderlines contain two endpoints of the generalized LPS condition (2.3).

More precisely, in the case $\alpha \in [5/4, 2)$ we consider the supercritical regime $\mathcal{A}_1$ given by

$$\mathcal{A}_1 := \left\{(s,\gamma,p) \in [0,3) \times [1,\infty] \times [1,\infty] : 0 \le s < \frac{4\alpha-5}{\gamma} + \frac{3}{p} + 1 - 2\alpha\right\}, \tag{2.4}$$

and in the case $\alpha \in [1,2)$ we consider supercritical regime $\mathcal{A}_2$ given by

$$\mathcal{A}_2 := \left\{(s,\gamma,p) \in [0,3) \times [1,\infty] \times [1,\infty] : 0 \le s < \frac{2\alpha}{\gamma} + \frac{2\alpha-2}{p} + 1 - 2\alpha\right\}. \tag{2.5}$$

The supercritical regimes $\mathcal{A}_1$ and $\mathcal{A}_2$ in the case $s = 0$ can be seen in Figure 1 below.

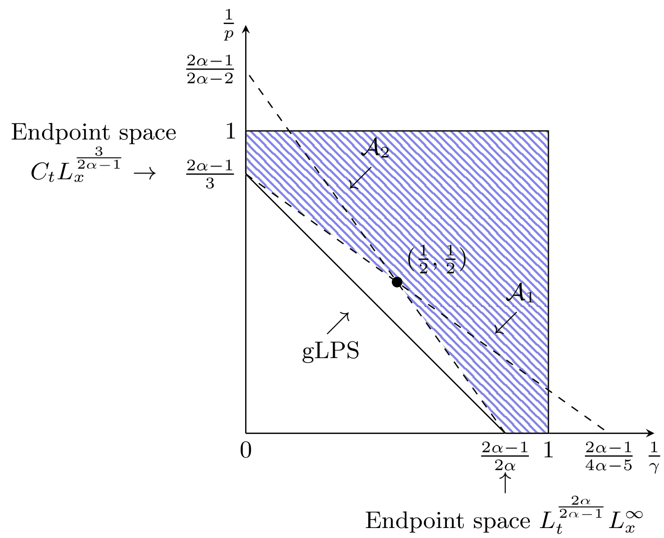


Figure 1: The supercritical regimes $\mathcal{A}_1$ and $\mathcal{A}_2$ ($\alpha \in [\frac{5}{4}, 2), s = 0$)

Concerning the case of $\alpha = 1$, Cheskidov-Luo [28] proved that for any $\gamma < 2$, there exist non-unique solutions in $L^\gamma_t L^\infty_x$ to INS in all dimensions $d \ge 2$, which is sharp at the endpoint case of $(s,\gamma,p) = (0, 2, \infty)$. The sharp non-uniqueness for the other endpoint case $(s,\gamma,p) = (0,\infty,2)$ was also achieved in [26] for the 2D INS.

When $\alpha \ge 1$, the sharp non-unqiueness of hyperdissipative Navier-Stokes equations (2.1) is presented in Theorem 2.3 below.

**Theorem 2.3.** *(Sharp non-uniqueness, [63]) Let* $\alpha \in [5/4, 2)$. *Then, for any weak solution* $\widetilde{u}$ *to* (2.1)*, there exists a different weak solution* $u \in L^\gamma_t W^{s,p}_x$ *to* (2.1) *with the same initial data, where* $(s,\gamma,p) \in \mathcal{A}_1 \cup \mathcal{A}_2$.

In particular, when $\alpha = 5/4$, Theorem 2.3 provides the non-uniqueness of weak solutions in $C_t L^p_x$ to (2.1) for any $p < 2$, which is sharp in view of Theorem 2.2.

Furthermore, from the viewpoint of the LPS criteria (2.3), Theorem 2.3 also provides the sharp non-uniqueness for the hyperdissipative INS (2.1) at two endpoints, i.e., $(3/p + 1 - 2\alpha, \infty, p)$ for $\alpha \in [5/4, 2)$, and $(2\alpha/\gamma + 1 - 2\alpha, \gamma, \infty)$ for $\alpha \in (1, 2)$.

We would also expect the non-uniqueness in the supercritical regimes where $\alpha \in [2, 5/2)$ or near the endpoint $(3/p+1-2\alpha, \infty, p)$ when $\alpha \in [1, 5/4)$, which probably require a new convex integration scheme different from the existing $L^2_{t,x}$-critical convex integration. The reason is that, in the present $L^2_{t,x}$-critical scheme, the temporal intermittency has been explored to upgrade temporal integrability by allowing $\gamma > 2$, but this in turn requires to reduce the spatial integrability with $p < 2$. However, in the endpoint case $(0,\infty,3/(2\alpha-1))$ where $\alpha<5/4$, both the temporal and spatial integrability exponents are larger than two, which is thus beyond the present $L^2_{t,x}$ scheme.

**(ii) Strong non-uniqueness for the high dissipativity beyond the Lions exponent.** Besides the sharp non-uniqueness, the strong non-uniqueness of weak solutions to (2.1) also holds in the hyperdissipative case where $\alpha\in[5/4,2)$, which is formulated in Theorem 2.4 below.

**Theorem 2.4.** *(Strong non-uniqueness, [63]) Let $\alpha\in[5/4,2)$, for every divergence-free $L^2_x$ initial data, there exist infinitely many weak solutions in $L^\gamma_tW^{s,p}_x$ to (2.1) which are smooth almost everywhere in time.*

It is folklore that one has global solvability in the high dissipative case when $\alpha\geq 5/4$. In fact, based on the works [11, 68, 71], the critical threshold for the $C_tL^2_x$ well-posedness for (2.1) is precisely at $\alpha=5/4$. This indicates that weak solutions in $C_tL^2_x$ to (2.1) are unique if $\alpha$ is greater than or equal to $5/4$, but they become non-unique if $\alpha$ is less than $5/4$.

Theorem 2.4 shows that the non-unique weak solutions still exhibit in the high-dissipative regime where $\alpha\geq 5/4$. More specifically, it reveals that the uniqueness breaks down in the space $L^\gamma_tW^{s,p}_x$, if the exponents $(s,\gamma,p)$ lies in the supercritical regime $\mathcal{A}_1\cup\mathcal{A}_2$, defined in (2.4) and (2.5), respectively.

**(iii) Partial regularity of weak solutions.** In the pioneering paper [62], Leray proves that the Leray-Hopf solutions to INS are smooth outside a closed singular set of times, which has zero Hausdorff $\mathcal{H}^{1/2}$ measure. This provides an alternative approach to address the global existence problem. Following the works of Scheffer [79, 80], Caffarelli-Kohn-Nirenberg [19] proved a space-time regularity version and showed the existence of global Leray-Hopf solutions which have singular sets in $\mathbb{R}^+\times\mathbb{R}^3$ of zero Hausdorff $\mathcal{H}^1$ measure. The simplified proofs were obtained by Lin [67] and Vasseur [83]. In [47], Giga established partial regularity of solutions in mixed $L^q_tL^p_x$ spaces. For hyperdissipative INS with $\alpha\in(1,5/4]$, Katz-Pavlović [57] proved that the Hausdorff dimension of singular set at the time of first blow-up is at most $5-4\alpha$. More recently, Colombo-De Lellis-Massaccesi [33] proved a stronger version of the Katz-Pavlović result and establish the existence of Leray-Hopf solutions with singular space-time sets of zero Hausdorff $\mathcal{H}^{5-4\alpha}$ measure, thus extending the Caffarelli-Kohn-Nirenberg theorem to hyper-dissipative INS.

Concerning the partial regularity of weak solutions to hyper-dissipative INS, in [63, Theorem 1.2] we prove that, in the highly dissipative regime $\alpha\in[5/4,2)$, for any $\eta_*>0$, there exist weak solutions to (2.1) in any small $L^\gamma_tW^{s,p}_x$-neighborhood of Leray-Hopf solutions, where $(s,\gamma,p)\in\mathcal{A}_1\cup\mathcal{A}_2$. The weak solutions coincide with Leray-Hopf solutions near $t=0$, and have singular temporal sets with zero Hausdorff $\mathcal{H}^{\eta_*}$ measure.

**(iv) Vanishing viscosity limit.** Theorem 2.5 below contains the vanishing viscosity limit result, which relates INS and Euler equations.

**Theorem 2.5.** *(Strong vanishing viscosity limit) Let $\alpha\in(1,2)$ and $u\in H^{\widetilde{\beta}}([-2T,2T]\times\mathbb{T}^3)$ ($\widetilde{\beta}>0$) be any mean-free weak solution to the Euler equation,*

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+\nabla P=0,\\ \operatorname{div}u=0.\end{cases}\tag{2.6}$$

*Then, there exist $\beta'\in(0,\widetilde{\beta})$ and a sequence of weak solutions $u^{(\nu_n)}\in H^{\beta'}_{t,x}$ to (2.1), where $\nu_n$ is the viscosity coefficient, such that as $\nu_n\to 0$,*

$$u^{(\nu_n)}\to u\quad\textit{strongly in } H^{\beta'}_{t,x}.\tag{2.7}$$

Theorem 2.5 shows that, in the $H^{\widetilde{\beta}}_{t,x}$ topology, the set of accumulation points of weak solutions to hyper-dissipative INS (2.1) contain all the weak solutions in $H^{\widetilde{\beta}}_{t,x}$ to the Euler equations, where $\widetilde{\beta}$ can be any small positive constant. Thus, as in the INS context [16], being a strong limit of weak solutions to the hyper-disspative INS cannot serve as a selection criteria for weak solutions to Euler equations.

## 3 Compressible Navier-Stokes equations

This section presents the results in [64] about the non-uniqueness of weak solutions to hypo-viscous compressible Navier-Stokes equations. We consider the $d$-dimensional isentropic hypo-viscous compressible Navier-Stokes equations (CNS for short) on the torus $\mathbb{T}^d := [-\pi, \pi]^d$,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mu(-\Delta)^\alpha u - (\mu + \nu)\nabla \mathrm{div} u + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = 0. \end{cases} \tag{3.1}$$

where $d \geq 2$, $\rho : [0, T] \times \mathbb{T}^d \to (0, \infty)$ and $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ are the mass density and velocity of fluid, respectively. The pressure $P(\rho)$ is $C^2$ with respect to $\rho$. The coefficients $\mu$ and $\nu$ are constants, $\mu$ is the shear viscosity coefficient, $\nu + \frac{2}{d}\mu$ is the bulk viscosity coefficient satisfying the physical assumptions

$$\mu > 0, \quad \nu + \frac{2}{d}\mu \geq 0. \tag{3.2}$$

The hypo-viscosity $(-\Delta)^\alpha$, $\alpha \in (0, 1)$, is defined by the Fourier transform

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^d.$$

We note that, (3.1) is the classical compressible Navier-Stokes equations when $\alpha = 1$, and when the viscosities vanish ($\mu = \nu = 0$), (3.1) reduces to the compressible Euler equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = 0. \end{cases} \tag{3.3}$$

It would also be convenient to formulate (3.1) in terms of the density and momentum $m := \rho u$ as follows

$$\begin{cases} \partial_t \rho + \mathrm{div} m = 0, \\ \partial_t m + \mu(-\Delta)^\alpha(\rho^{-1}m) - (\mu + \nu)\nabla \mathrm{div}(\rho^{-1}m) + \mathrm{div}(\rho^{-1}m \otimes m) + \nabla P(\rho) = 0. \end{cases} \tag{3.4}$$

The weak solutions to (3.4) is understood in the distributional sense.

**Definition 3.1.** *(Weak solutions) Let $0 < T < \infty$. Given any initial data $\rho_0 \in L^\infty(\mathbb{T}^d)$, $\rho_0 > 0$, and $u_0 \in L^2(\mathbb{T}^d)$, we say that $(\rho, m) \in L^\infty([0, T] \times \mathbb{T}^d) \times L^2((0, T) \times \mathbb{T}^d)$ is a weak solution to (3.4) if*

- *$\rho \geq 0$ a.e. and*

$$\int_{\mathbb{T}^d} \rho_0(x)\varphi(0, x)\mathrm{d}x = -\int_0^T \int_{\mathbb{T}^d} \rho \partial_t \varphi + (m \cdot \nabla)\varphi \mathrm{d}x\mathrm{d}t$$

  *for any test function $\varphi \in C_0^\infty([0, T) \times \mathbb{T}^d)$.*

- *$m = 0$ whenever $\rho = 0$, and*

$$\int_{\mathbb{T}^d} m_0 \varphi(0, x)\mathrm{d}x = -\int_0^T \int_{\mathbb{T}^d} m \cdot \partial_t \varphi - \frac{1}{\rho}m \cdot (\mu(-\Delta)^\alpha \varphi - (\mu + \nu)\nabla \mathrm{div}\varphi) + (\frac{1}{\rho}m \otimes m) : \nabla \varphi + P\mathrm{div}\varphi \mathrm{d}x\mathrm{d}t$$

  *for any test function $\varphi \in C_0^\infty([0, T) \times \mathbb{T}^d)$, where $m_0 := \rho_0 u_0$ .*

**Main results** Our main results include the non-uniqueness of weak solutions, the sharp viscosity threshold for $L^2_tC_x$ well-posedness, and the vanishing viscosity limit.

**(i) Non-uniqueness of weak solutions.** The non-uniqueness of weak solutions to the hypo-viscous CNS (3.4) is formulated in Theorem 3.2 below.

**Theorem 3.2** (Non-uniqueness for hypo-viscous CNS)**.** *There exist $\rho_0 \in L^\infty(\mathbb{T}^d)$, $\rho_0 > 0$, and $m_0 \in L^2(\mathbb{T}^d)$, such that for any exponents $(p, s)$ satisfying*

$$\alpha + s - \frac{2\alpha}{p} < 0, \quad (p, s) \in [1, 2] \times [0, 1), \tag{3.5}$$

*there exist infinitely many weak solutions $(\rho, m) \in C_tC^1_x \times L^p_tC^s_x$ to the hypo-viscous CNS* (3.4) *with the same initial data $(\rho_0, m_0)$.*

To the best of our knowledge, Theorem 3.2 provides the first result for the non-uniqueness of weak solutions to the compressible fluid with viscosity. It also shows that, the hypo-vsicosity cannot rule out the non-uniqueness mechanism, which, actually, coincides with the Ladyženskaja-Prodi-Serrin criteria (see also result (ii) below).

The proof of Theorem 3.2 in [64] also applies to the case of compressible Euler equations (3.3).

**Theorem 3.3** (Non-uniqueness for compressible Euler equations, [64])**.** *There exist $\rho_0 \in L^\infty(\mathbb{T}^d)$ and $m_0 \in L^2(\mathbb{T}^d)$, such that for any exponents $(p, s)$ satisfying* (3.5) *with any $\alpha \in (0, 1)$, there exist infinitely many weak solutions $(\rho, m) \in C_tC^1_x \times L^p_tC^s_x$ to the compressible Euler equations* (3.3) *with the initial data $(\rho_0, m_0)$.*

Theorem 3.3 provides new examples of anomalous dissipation in the class $L^1_tC^{1-}_x$ for the compressible Euler equations, by taking $p = 1$ and $\alpha$ close to 1. We note that a similar result for the incompressible Euler equations was provided in [28] with a different convex integration scheme.

**(ii) Sharp viscosity threshold for $L^2_tC_x$ well-posedness.** As in the proof of Theorem 3.2, we also have the non-uniqueness result for hypo-viscous INS.

**Theorem 3.4** (Non-uniqueness for hypo-viscous INS)**.** *Consider the incompressible Navier-Stokes equations*

$$\begin{cases} \partial_t u + \mu(-\Delta)^\alpha u + \operatorname{div}(u \otimes u) + \nabla P = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{3.6}$$

*where $\mu > 0$, $\alpha \in (0, 1)$. Then, there exists $u_0 \in L^2(\mathbb{T}^d)$ such that for any exponents $(p, s)$ satisfying* (3.5)*, there exist infinitely many weak solutions $u \in L^p_tC^s_x$ with the initial datum $u_0$.*

When $(p, s) = (2, 0)$, Theorem 3.4 provides the non-uniqueness of weak solutions in the space $L^2_tC_x$ for the hypo-viscous INS, which is sharp due to the well-posedness in $L^2_tL^\infty_x$, in view of the Ladyženskaja-Prodi-Serrin criteria.

Thus, Theorem 3.4 reveals that $\alpha = 1$ is the sharp threshold viscosity for the $L^2_tC_x$ well-posedness for the INS. It would serve as a counterpart of the recent results in [11, 71] regarding the sharp threshold of the Lions exponent $\alpha = 5/4$ for the $C_tL^2_x$ well-posedness of the 3D INS. It is worth noting that, both the spaces $L^2_tC_x$ and $C_tL^2_x$ are the endpoint spaces of the Ladyženskaja-Prodi-Serrin criteria.

This also complements the very recent work [3], where the non-uniqueness of Leray-Hopf solutions is proved for the 2-D hypo-viscous INS with a force.

**(iii) Vanishing viscosity limit.** The following strong vanishing viscosity limit relates the hypo-viscous CNS with compressible Euler equations.

**Theorem 3.5** (Strong vanishing viscosity limit for hypo-viscous CNS)**.** *Let $\alpha \in (0, 1)$ and $\rho, m \in C^{\widetilde{\beta}}_{t,x}$, $\widetilde{\beta} > 0$, be any weak solution to the compressible Euler equations* (3.3)*, such that $c_1 \le \rho \le c_2$ for some* *constants* $c_1, c_2 > 0$. *Then, there exist* $\beta' \in (0, \widetilde{\beta})$ *and a sequence of weak solutions* $(\rho^{(n)}, m^{(n)}) \in C_{t,x} \times H_t^{\beta'} C_x$ *to the hypo-viscous CNS* (3.1) *with viscous coefficients* $\mu = \kappa_n \mu_*$ *and* $\nu = \kappa_n \nu_*$*, such that as* $\kappa_n \to 0$,

$$\rho^{(n)} \to \rho \quad \text{strongly in } C_{t,x}, \quad \text{and} \quad m^{(n)} \to m \quad \text{strongly in } H_t^{\beta'} C_x. \tag{3.7}$$

Regarding the compressible case, the 1D case with artificial viscosity was solved by Bianchini-Bressan [7]. But it is still open for the multi-dimensional case with general initial data. By virtue of Theorem 3.5, we see that in the vanishing viscosity limit, the set of accumulation points of weak solutions to the hypo-viscous CNS (3.4) contains all the Hölder continuous weak solutions to the compressible Euler equations (3.3).

In particular, the entropy solutions in the Hölder spaces recently constructed in [48] may be obtained as the strong vanishing viscosity limits of weak solutions to the hypo-viscous CNS (3.1) when $d = 3$.

# 4 Incompressible MHD equations

In this section, we review our recent results in [65, 66] concerning the non-uniqueness of weak solutions to the following three-dimensional viscous and resistive magnetohydrodynamic (MHD for short) system on the torus $\mathbb{T}^3 := [-\pi, \pi]^3$:

$$\begin{cases} \partial_t u - \nu_1 \Delta u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla P = 0, \\ \partial_t B - \nu_2 \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{4.1}$$

where $u = (u_1, u_2, u_3)^\top (t, x) \in \mathbb{R}^3$, $B = (B_1, B_2, B_3)^\top (t, x) \in \mathbb{R}^3$ and $P = P(t, x) \in \mathbb{R}$ correspond to the velocity field, magnetic field and pressure of the fluid, respectively, and $\nu_1, \nu_2 \geq 0$ are the viscous and resistive coefficients, respectively. In particular, in the case without magnetic fields, system (4.1) reduces to the classical INS.

In order to formulate the main results more generally, we consider the MHD equations with hyper viscosity and resistivity:

$$\begin{cases} \partial_t u + \nu_1 (-\Delta)^\alpha u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla P = 0, \\ \partial_t B + \nu_2 (-\Delta)^\alpha B + (u \cdot \nabla) B - (B \cdot \nabla) u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{4.2}$$

where $\alpha \in [1, 3/2)$. Note that, the viscosity and resistivity exponents can be larger than the Lions exponent $5/4$.

System (4.2) is invariant under the scaling

$$u(t, x) \mapsto \lambda^{2\alpha - 1} u(\lambda^{2\alpha} t, \lambda x), \quad B(t, x) \mapsto \lambda^{2\alpha - 1} B(\lambda^{2\alpha} t, \lambda x), \tag{4.3}$$

and $P(t, x) \mapsto \lambda^{4\alpha - 2} P(\lambda^{2\alpha} t, \lambda x)$. The critical exponent $(s, \gamma, p)$ of the mixed Lebesgue spaces $L_t^\gamma W_x^{s,p}$ satisfies the generalized Ladyženskaja-Prodi-Serrin condition

$$\frac{2\alpha}{\gamma} + \frac{3}{p} = 2\alpha - 1 + s. \tag{4.4}$$

We show that the non-uniqueness of weak solutions also exhibit in the spaces $L_t^\gamma W_x^{s,p}$, where the exponents $(s, \gamma, p)$ lie in the following two supercritical regimes, with respect to the scaling (4.3):

$$\mathcal{S}_1 := \left\{ (s, \gamma, p) \in [0, 3) \times [1, \infty] \times [1, \infty] : 0 \leq s < \frac{2\alpha}{\gamma} + \frac{2\alpha - 2}{p} + 1 - 2\alpha \right\}, \tag{4.5}$$

and

$$\mathcal{S}_2 := \Big\{(s,\gamma,p) \in [0,3)\times[1,\infty]\times[1,\infty] : 0 \le s < \frac{4\alpha-4}{\gamma} + \frac{2}{p} + 1 - 2\alpha\Big\}. \tag{4.6}$$

The weak solutions to (4.2) are taken similarly in the distributional sense.

**Definition 4.1** (Weak solutions). *Given any divergence free initial data $(u_0, B_0) \in L^2(\mathbb{T}^3)$, we say that $(u,B) \in L^2([0,T]\times\mathbb{T}^3)$ is a weak solution to (4.2) if*

- *For all $t \in [0,T]$, $(u(t,\cdot), B(t,\cdot))$ are divergence free in the sense of distributions and have zero spatial mean.*
- *Equations (4.2) hold in the sense of distributions, i.e., for any divergence-free test functions $\varphi \in C_0^\infty([0,T]\times\mathbb{T}^3)$,*

$$\int_{\mathbb{T}^3} u_0\varphi(0,x)\mathrm{d}x + \int_0^T \int_{\mathbb{T}^3} \partial_t\varphi\cdot u - \nu_1(-\Delta)^\alpha\varphi\cdot u + \nabla\varphi : (u\otimes u - B\otimes B)\mathrm{d}x\mathrm{d}t = 0,$$

$$\int_{\mathbb{T}^3} B_0\varphi(0,x)\mathrm{d}x + \int_0^T \int_{\mathbb{T}^3} \partial_t\varphi\cdot B - \nu_2(-\Delta)^\alpha\varphi\cdot B + \nabla\varphi : (B\otimes u - u\otimes B)\mathrm{d}x\mathrm{d}t = 0.$$

Concerning the weak solutions to (4.1) in the sense of Definition 4.1, we have the following sharp non-uniqueness result.

**Theorem 4.2** (Sharp non-uniqueness for MHD). *Consider the viscous and resistive MHD (4.1). Then, for any $1 \le \gamma < 2$, there exist infinitely many weak solutions in $L^\gamma_t L^\infty_x$ with the same initial data.*

**Remark 4.3.** *One significant difference between the INS and MHD is the impact of the strong coupling between velocity and magnetic fields in MHD, which limits the available choices for oscillating directions. As a result, constructing spatial building blocks with strong spatial intermittency that align with the specific geometric structure of MHD becomes quite challenging.*

*It should be mentioned that, the 1D spatial intermittency and intermittent flows were first constructed in [6]. In [65, 66], the 2D intermittent flows were constructed. However, both intermittent flows are not strong enough to control the viscosity $-\Delta$. The novelty in [66] is to explore the additional temporal intermittency.*

*Another distinction between INS and MHD can be seen in the construction of the amplitudes of velocity and magnetic perturbations. This distinction arises from the anti-symmetry of magnetic nonlinearity, which requires a second geometric lemma in a small neighbourhood of the null matrix. This is different from the geometric lemma used for velocity perturbations, which holds in the neighborhood of the identity matrix. Additionally, when constructing velocity perturbations, a new matrix $\mathring{G}^B$ needs to be introduced, which does not appear in the context of the Navier-Stokes equations. These differences lead to specific algebraic identities for the nonlinear effects associated with magnetic and velocity perturbations.*

Besides the non-uniqueness results, it also holds the following strong vanishing viscosity and resistivity limits for the MHD equations (4.1). Its relationship to the Taylor's conjecture will be discussed in Remark 4.5 below.

**Theorem 4.4** (Strong vanishing viscosity and resistivity limit). *Let $\alpha \in [1, 3/2)$ and $\widetilde{\beta} > 0$. Let $(u,B) \in H^{\widetilde{\beta}}_{t,x} \times H^{\widetilde{\beta}}_{t,x}$ be any mean-free weak solution to the ideal MHD. Then, there exist $\beta' \in (0,\widetilde{\beta})$ and a sequence of weak solutions $(u^{(\nu_n)}, B^{(\nu_n)}) \in H^{\beta'}_{t,x} \times H^{\beta'}_{t,x}$ to the hyper viscous and resistive MHD (4.2), where $\nu_n = (\nu_{1,n}, \nu_{2,n})$ and $\nu_{1,n}$, $\nu_{2,n}$ are the viscosity and resistivity coefficients, respectively, such that as $\nu_n \to 0$,*

$$(u^{(\nu_n)}, B^{(\nu_n)}) \to (u,B) \quad \textit{strongly in } H^{\beta'}_{t,x} \times H^{\beta'}_{t,x}. \tag{4.7}$$

**Remark 4.5.** *It would be interesting to see the relationship between the Taylor's conjecture and the vanishing viscosity and resistivity limit in Theorem 4.4.*

*The smooth solution to the ideal MHD has several global invariants:*

- *The total energy:* $\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)|^2 + |B(t,x)|^2 \mathrm{d}x$*;*
- *The cross helicity:* $\mathcal{H}_{\omega,B}(t) = \int_{\mathbb{T}^3} u(t,x)\cdot B(t,x)\mathrm{d}x$*;*
- *The magnetic helicity:* $\mathcal{H}_{B,B}(t) := \int_{\mathbb{T}^3} A(t,x)\cdot B(t,x)\mathrm{d}x$*.*

*Here $A$ is a mean-free periodic vector field satisfying* $\mathrm{curl}A = B$*.*

*It is widely acknowledged in the field of plasma physics that magnetic helicity remains conserved as the conductivity approaches infinity. This phenomenon is known as Taylor's conjecture. It has been proved by Faraco-Lindberg [43] that Taylor's conjecture holds true when dealing with the weak limits of Leray-Hopf solutions MHD in simply connected, magnetically closed domain. See also Faraco-Lindberg-MacTaggart-Valli [46] in the case of multiply connected domains.*

*On the other hand, Beekie-Buckmaster-Vicol [6] demonstrated that Taylor's conjecture does not holds by constructing distributional solutions in $C_tL^2_x$ breaking the magnetic helicity conservation. The delicate point here is that the conservation of magnetic helicity demands a level of regularity that is much milder than that of total energy and cross helicity. Specifically, the weak solution in $B^{\alpha}_{3,\infty}$ with $\alpha > 1/3$ or $B^{1/3}_{3,c(\mathbb{N})}$ conserve the energy and cross helicity, while the magnetic helicity is conserved in the less regular space $B^{\alpha}_{3,\infty}$ with $\alpha > 0$ or in the endpoint space $L^3_{t,x}$. See [4, 20, 42, 55].*

*Theorem 4.4 shows that, in contrast to weak ideal limits, even for the hyper viscous and resistive MHD beyond the Lions exponent, there exists certain sequence of non-Leray-Hopf weak solutions such that the Taylor's conjecture fails in the limit along this sequence.*

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Yachun Li
School of Mathematical Sciences, CMA-Shanghai, MOE-LSC, and SHL-MAC
Shanghai Jiao Tong University
Shanghai 200240
PR China
E-mail address: `ycli@sjtu.edu.cn`

Peng Qu
School of Mathematical Sciences & Shanghai Key Laboratory for Contemporary Applied Mathematics
Fudan University
Shanghai 200433
PR China
E-mail address: `pqu@fudan.edu.cn`

Zirong Zeng
School of Mathematical Sciences,
Shanghai Jiao Tong University
Shanghai 200240
PR China
E-mail address: `beckzzr@sjtu.edu.cn`

Deng Zhang
School of Mathematical Sciences, CMA-Shanghai
Shanghai Jiao Tong University
Shanghai 200240
PR China
E-mail address: `dzhang@sjtu.edu.cn`