TWO EXAMPLES OF WELL-POSEDNESS OF WEAK SOLUTIONS FOR QUASILINEAR EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. To establish a well-posedness theory for weak solutions of quasilinear evolutionary partial differential equations is a difficult task. There is little control over the time evolution of weak solutions constructed by compactness methods. The purpose of the present article is to explain the construction procedures of weak solutions for hyperbolic and viscous conservation laws. These procedures allow for the establishment of well-posedness theory. For system of hyperbolic conservation laws, the weak solutions are constructed using the Riemann solutions as building blocks. For the compressible Navier-Stokes equations, one uses the Green's function approach to construct the weak solutions by solving integral equations. Within these construction procedures, the traditional Hadamard well-posedness criteria are satisfied, and the regularity and time-asymptotic behaviors of the weak solution can be studied.

1. Introduction

The Hadamard well-posedness requirements for the initial value problem of an evolutionary partial differential equations are: Existence, Uniqueness, and Continuous Dependence of the solutions on the initial values.

For linear equations, well-posedness analysis usually is based on the a priori estimates, from which a proper function space is chosen. For the existence of solutions in the function space, one applies the functional analytic theorems such as the Riesz representation and Lax-Milgram theorem. The uniqueness and continuous dependence properties with respect to the function space result from the principle of linear superposition.

For nonlinear equations, the well-posedness theory for smooth solutions is less straightforward. For instance, consider system of quasilinear hyperbolic equations

$$\mathbb{A}_0(\boldsymbol{u})\boldsymbol{u}_t + \sum_{i=1}^{n} \mathbb{A}_i(\boldsymbol{u})\boldsymbol{u}_{x_i} = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{x},),\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}).$$

The theory for the well-posedness of smooth solutions is not done in one function space. Suppose that the the initial value is in the Sobolev space $\boldsymbol{u}_0(\boldsymbol{x} \in H^s(\boldsymbol{x}),\ s > n/2+1$. Then there exists classical solution locally in time in the same space $H^s(\boldsymbol{x}),\ s > n/2+1$; but the continuous dependence is on the lower norm space $H^1(\boldsymbol{x})$. This is the classical theory of Kato and Lax, c.f. Chapter 6 of [21]. After some finite time, solutions in general develop shock waves and one needs to consider the weak solutions.

There is the well-known example of weak solutions for the incompressible Navier-Stokes equations by Leray [18]:

$$\begin{aligned} &\boldsymbol{v}_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} \otimes \boldsymbol{v} + \nabla_{\boldsymbol{x}} p = \mu \Delta \boldsymbol{v}, \\ &\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0. \end{aligned}$$

Another example is the construction of weak solutions for the Boltzmann equation by DiPerna-Lions, [7]:

$$\partial_t \mathsf{f}(\boldsymbol{x}, t, \boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \mathsf{f}(\boldsymbol{x}, t, \boldsymbol{\xi}) = \frac{1}{k} Q(\mathsf{f}, \mathsf{f})(\boldsymbol{x}, t, \boldsymbol{\xi}).$$

There is no well-posedness theory for the weak solutions constructed in these two examples. In fact, substantial progresses have been made on the non-uniqueness of weak solutions for fluid dynamics equations, e.g. [25], [6], [4].

A natural way to formulate the well-posedness theory is to establish a solution algorithm that yields sufficient information on the structure of solutions so that it is possible to show that, within this construction, the solution is unique and depends continuously on its initial data, and allows for the analysis of the behavior of the solutions. This is the classical thinking before functional analytic approach became the normal way. Clearly, the classical way has its limitations. On the other hand, when it works, the well-posedness theory would include the solution behavior as a basic requirement. There are two well-posedness theories for weak solutions under this formulation that we will focus on. The first is for system of hyperbolic conservation laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,\ \boldsymbol{u} \in \mathbb{R}^n,\ n \geq 1. \tag{1.1}$$

The existence theory was established by Glimm [9] using the Riemann solutions as building blocks. Subsequent developments will be explained in the next section. Another one is for simplest viscous conservation laws in physics, the compressible, isentropic Navier-Stokes equations done recently by Liu-Yu [22] using the Green's function approach:

$$\begin{aligned} &v_t - u_x = 0,\\ &u_t + p(v)_x = \left(\kappa \frac{u_x}{v}\right)_x. \end{aligned} \tag{1.2}$$

This will be explained in Section 3. In both cases, the well-posedness analysis is first to establish the boundedness in BV norm and continuous dependence in $L_1(x)$ norm.

2. Hyperbolic conservation laws

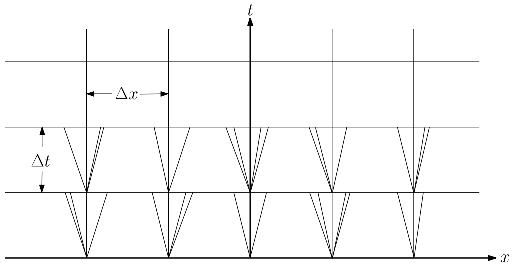
Consider the initial value problem for hyperbolic conservation laws (1.1):

$$\begin{aligned} &\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,\ \boldsymbol{u} \in \mathbb{R}^n,\ n \geq 1,\\ &\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),\ x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

The system is assumed to be strictly hyperbolic

$$\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}),\ \lambda_1(\boldsymbol{u}) < \lambda_2(\boldsymbol{u}) < \cdots < \lambda_n(\boldsymbol{u}).$$

The existence theory to the general initial value problem (2.1) is done in the seminal paper Glimm (1965) [9]. The Glimm scheme uses the Riemann solutions as the building blocks, see the figure below.

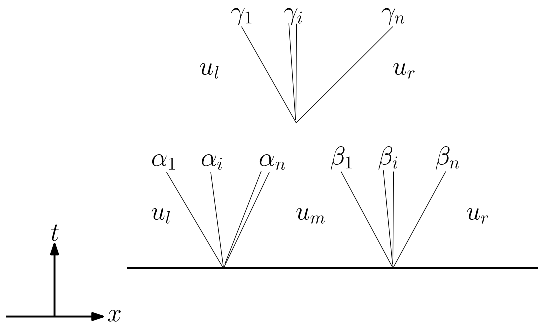


The Riemann problem is a special problem with two constant states as initial data:

$$\boldsymbol{u}_0(x) = \begin{cases} \boldsymbol{u}_l,\ x < 0,\\ \boldsymbol{u}_r,\ x > 0. \end{cases}$$

Both the conservation laws and the Riemann initial data are invariant under the dilation $(x,t) \to (cx, ct)$ for any constant $c > 0$. Thus the solution is self-similar $\boldsymbol{u}(x,t) = \boldsymbol{\phi}(\xi)$ for some function $\boldsymbol{\phi}$ of one independent variable $\xi = x/t$. Thus the Riemann problem is much easier to solve than the general initial value problem. On the other hand, solving the Riemann problem yields the basic information of the identification of the entropy condition and also the construction of elementary waves. The construction of elementary waves is explicit using differential and algebraic equations. An i-th elementary wave takes value around the characteristic direction $\boldsymbol{r}_i(\boldsymbol{u})$ and propagates around the characteristic speed $\lambda_i(\boldsymbol{u})$ see Lax (1956) [17], and Chapter 7 of [21] for general discussions. The Glimm scheme therefore makes use of the essential, concrete information of the equations.

The Glimm scheme does not induce numerical diffusion. It uses a random sequence to move the elementary waves. The approximate solutions consist of local interaction of two sets of Riemann wave pattern around each grid point $(i\Delta x, j\Delta t)$, $i = 0, \pm1, \pm2, \cdots$; $j = 1, 2, \cdots$, see figure below.



The key to the Glimm theory is the study of wave interaction of these local interactions:

$$\gamma_i = \alpha_i + \beta_i + O(1)D_d + O(1)D_s;$$
$$D_d \equiv \sum_{j>k} |\alpha_j \beta_k|, \text{interaction measure for distinct families;}$$
$$D_s \equiv |\alpha_j \beta_j|(|\alpha_j| + |\beta_j|), \text{interaction measure for same family.}$$

Geometrically, the above estimates are natural, as the interaction measures are zero if waves are not approaching to each other. These local estimates naturally lead to a global nonlinear functional $F(J)$ across a space-like curve J in (x,t) space:

$$F(J) = L(J) + KQ_d(J) + KQ_s(J),$$
$$L(J) \equiv \sum\{|\alpha_i|,\ \alpha_i \text{ crossing } J\};$$
$$Q_d(J) \equiv \sum\{|\alpha_j \beta_k|,\ \alpha_j \text{ and } \beta_k \text{ crossing } J \text{ and } \alpha_j \text{ lies to the left of } \beta_k,\ j > k\};$$
$$Q_s(J) \equiv \sum\{|\alpha_j \alpha_k|,\ \alpha_j \text{ and } \alpha_k\ i\text{-waves crossing } J \text{ and one of them a shock},\ i = 1, \cdots, n\}.$$

By the estimates, for the constant K sufficiently large, the functional is time decreasing. This implies that when the initial datum $\boldsymbol{u}_0(x)$ has small total variation TV then the approximate solutions have also small total variation, and strong compactness properties for BV space yields the existence theory.

Theorem 2.1. *For the initial datum $\boldsymbol{u}(x,0)$ of small total variation TV, there exists a global weak solution to the system of hyperbolic conservation laws $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$. Moreover, the total variation of $\boldsymbol{u}(\cdot,t),\ t \geq 0$, is $O(1)TV$ and*

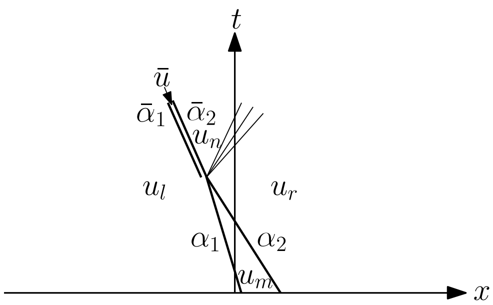
$$\int_{-\infty}^{\infty} |\boldsymbol{u}(x,t_2) - \boldsymbol{u}(x,t_1)|dx = O(1)TV|t_2 - t_1|,\ t_1, t_2 \geq 0.$$

Another consequence of the Glimm estimates is that the total amount of interaction measures is finite:

$$\sum_{(i\Delta x, j\Delta t)} \left(D_d^{i,j} + D_s^{i,j}\right) = O(1)TV^2.$$

The paper by Glimm-Lax (1970) [10] uses this global estimates to gain control of the time evolution of the solutions and proves decay of periodic solutions for 2×2 systems. For this, the notion of generalized characteristics and conservation of waves were introduced.

Another step toward better understanding of time evolution of solutions is the devise of wave splitting, see figure below, so that the Glimm solutions can be traced in time. This is the done in Liu (1977), see Section 3 of Chapter 9 of [21].In the figure the shock wave $(\boldsymbol{u}_l, \boldsymbol{u}_n)$ is splitted into $\bar{\alpha}_1$ and $\bar{\alpha})2$ so that α_i can be traced with $\bar{\alpha}_i,\ i = 1, 2$, with an error of the order of the interaction measure. Another way of wave tracing is to approximate the solutions by piecewise smooth functions, starting with the work of Dafermos (1970) [5].



With the understanding of the time evolution of the Glimm weak solutions, two methodologies are introduced for the proof of the continuous dependence on the initial data. One is the homotopy method of Bressan, [2], and the other is the functional method of Liu-Yang, [20].

Theorem 2.2. *Consider functions two solutions $\boldsymbol{u}_1(x,0),\ \boldsymbol{u}_2(x,0)$ with small total variation and two corresponding solutions $\boldsymbol{u}_1(x,t),\ \boldsymbol{u}_2(x,t)$ constructed by the Glimm scheme. Then there exists a constant independent of t such that*

$$\int_{-\infty}^{\infty} |\boldsymbol{u}_1(x,t) - \boldsymbol{u}_2(x,t)|dx \leq C \int_{-\infty}^{\infty} |\boldsymbol{u}_1(x,0) - \boldsymbol{u}_2(x,0)|dx,\ t \geq 0.$$

The bound C above for the $L_1(x)$ continuous dependence on the initial data depends on the total variation of the solutions. Thus it is bounded the BV space, and continuous dependence in the $L_1(x)$ space. For general weak solutions, such as those constructed using the theory of compensated compactness, there is no well-posedness theorem.

The Glimm construction and subsequent analyses yield deep understanding of the evolution of the weak solutions. Consequently it is possible to study the regularity and time-asymptotic behavior of the weak solutions. For scalar conservation laws $u \in \mathbb{R}$, the nonlinearity of the flux $f(u)$ implies that waves compress and expand. The boundedness of total wave interaction measure allows for the study of these nonlinear properties for the systems $\boldsymbol{u} \in \mathbb{R}^n,\ n > 1$. For details of the following regularity and time-asymptotic behavior of solutions, see Chapter 9 of [21].

Theorem 2.3. *There exist countable points of interaction, countable Lipschitz curves of shock waves, and the solution is continuous outside of the points of interaction and shock curves.*

Theorem 2.4. *The solution tends to the Riemann solution* $(\boldsymbol{u}_l, \boldsymbol{u}_r),\ \boldsymbol{u}_l \equiv \boldsymbol{u}_0(-\infty),\ \boldsymbol{u}_r \equiv \boldsymbol{u}_0(\infty)$.

The uniqueness of weak solutions has been expressed within the Glimm algorithm. There are interesting results on the uniqueness of solutions within some function spaces, e.g. [3].

The Glimm functional can be generalized to viscous conservation laws of the form $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \kappa \boldsymbol{u}_{xx}$ to obtain, in the zero dissipation limit $\kappa \to 0+$, the well-posedness of $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$ by Bianchin-Bressan [1]. In place of using the Riemann solutions as coordinates, [1] introduces the Center manifold coordinates. It shows that the Glimm solution operator is robust and can be generalized to viscous conservation laws. It remains an open problem to establish the zero dissipation limit for systems with physical viscosity such as compressible Navier-Stokes equations, which are of the form

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \big(\mathbb{B}(\boldsymbol{u})\boldsymbol{u}_x\big)_x.$$

3. Compressible Navier-Stokes equations

Vicous conservation laws in continuum physics are of the form

$$\boldsymbol{u}_t + \nabla_{\boldsymbol{x}} \cdot \mathbb{F}(\boldsymbol{u}) = \nabla_{\boldsymbol{x}} \cdot (\mathbb{G}(\boldsymbol{u})\nabla_{\boldsymbol{x}}\boldsymbol{u}),\ \boldsymbol{x} \in \mathbb{R}^m,\ m \geq 1.$$

Important examples include the compressible Navier-Stokes equations, magento-hydrodynamics, and visco-elasticity equations. The simplest system is the isentropic compressible Navier-Stokes equations (1.2). The second equation in (1.2) has viscosity; while the first equation does not contain dissipative term. This is typical for the physical systems, which is not uniformly parabolic, but dissipative and hyperbolic-parabolic. This makes even the construction of smooth solutions non-trivial. The compressible Navier-Stokes equations have been studied first, e.g. Kanel [13], Itaya [12], and Matsumura-Nishida [15]. There is the general theory by Kawashima-Shizuta, [26], [14]. Since the physical systems are not uniformly parabolic, non-smooth solutions exist. There are constructions of weak solutions for the Navier-Stokes equations, see Lions [19], Hoff [11], Feireisl [8].

As expected, there are well-posedness theory for classical solutions, but not for weak solutions. There have been substantial progresses in the establishment of non-uniqueness of weak solutions for fluid equations using the methodology of convex integration, e.g. Scheffer [25], De Lellis-Szekelyhidi [6], Buckmaster-Vicol [4].

We now explain the well-posedness theory for weak solutions of the system (1.2):

$$\begin{matrix} v_t - u_x = 0, \\ u_t + p(v)_x = (\kappa \frac{u_x}{v})_x. \end{matrix}, \text{ isentropic compressible Navier-Stokes in Lagrangian coordinates}$$

The approach in Liu-Yu [22] is to use the Green's function approach for the construction of weak solutions. To gain control over the time evolution of weak solutions, we express the weak solutions in integral form in terms of the Green's functions. For this, the main steps is the explicit construction of these Green's functions. There are two types of Green's function. The first one is the Green's function $\mathbb{G} = \mathbb{G}(x, t; \alpha, \beta)$ for linearized Navier-Stokes equations:

$$\begin{aligned} &\partial_t \mathbb{G} \begin{pmatrix} 0 & \partial_x \\ \beta^2 \partial_x & \alpha \partial_x^2 \end{pmatrix} \mathbb{G}, \\ &\mathbb{G}(x, 0; \alpha, \beta) = \delta(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &\beta = \sqrt{-p'(v_0)} \text{ sound speed},\ \alpha = \tfrac{\kappa}{v_0} \text{ viscosity.} \end{aligned}$$

The second Green's function $\bar{\mathsf{H}}$ is related to the second equation with viscosity for the study of the effect of dissipation. Thus we consider the Green's function $\bar{\mathsf{H}} = \bar{\mathsf{H}}(x, t; z, t_0; \rho)$ for linear heat equation with dissipation coefficient $\rho(x,t)$:

$$\begin{aligned}&\partial_t\bar{\mathsf{H}}-\partial_x\rho(x,t)\partial_x\bar{\mathsf{H}}=0,\\&\bar{\mathsf{H}}(x,t_0;z,t_0;\rho)=\delta(x-z).\end{aligned}$$

For the construction of the weak solution for the initial value problem,

$$\begin{cases}v_t-u_x=0,\\u_t+p(v)_x=(\kappa\frac{u_x}{v})_x,\\(v,u)|_{t=0}=(1+v_0^*,u_0^*).\end{cases}$$

we use the iterations

$$\begin{cases}V_t^0-U_x^0=0,\\U_t^0-\beta_0^2V_x^0=\kappa U_{xx}^0,\\(V^0,U^0)(x,0)=(v_0^*,u_0^*)(x),\end{cases}$$

$$\begin{cases}V_t^{k+1}-U_x^{k+1}=0,\\U_t^{k+1}+p(1+V^k)_x=\kappa\Big(U_x^{k+1}/(1+V^k)\Big)_x,\\(V^k,U^k)(x,0)=(v_0^*,u_0^*)(x),\ k\ge1.\end{cases}$$

For the first step of the interation, we have the Integral representation using Green's function $\mathbb{G}$:

$$\begin{cases}V_t^0-U_x^0=0,\\U_t^0-\beta_0^2V_x^0=\kappa U_{xx}^0,\\(V^0,U^0)(x,0)=(v_0^*,u_0^*)(x),\end{cases}$$

$$\begin{pmatrix}V^0\\U^0\end{pmatrix}(x,t)=\int_{\mathbb{R}}\mathbb{G}(x-y,t)\begin{pmatrix}v_0^*\\u_0^*\end{pmatrix}(y)dy.$$

For the subsequent steps of the iteration, we have the Integral representation in terms of the Green's function $\bar{\mathsf{H}}$ with the heat conductivity coefficient $\kappa/(1+V^k)$ depending on the function from the previous step:

$$\begin{cases}V_t^{k+1}-U_x^{k+1}=0,\\U_t^{k+1}+p(1+V^k)_x=\kappa\Big(U_x^{k+1}/(1+V^k)\Big)_x,\\(V^k,U^k)(x,0)=(v_0^*,u_0^*)(x),\ k\ge1.\end{cases}$$

$$\begin{aligned}U^{k+1}(x,t)=&\int_{\mathbb{R}}\bar{\mathsf{H}}(x,t;y,0;\kappa/(1+V^k))u_0^*(y)dy\\&+\int_0^t\int_{\mathbb{R}}\bar{\mathsf{H}}_y(x,t;y,\tau;\kappa/(1+V^k))p(1+V^k(y,\tau))dyd\tau.\end{aligned}$$

$$V^{k+1}(x,t)=v_0^*(x)+\int_0^tU_x^{k+1}(x,\tau)d\tau.$$

We now describe the explicit construction of these Green's functions. Consider the first Green's function $\mathbb{G}$

$$\begin{aligned}&\partial_t\mathbb{G}(x,t;\alpha,\beta)=\begin{pmatrix}0&\partial_x\\\beta^2\partial_x&\alpha\partial_x^2\end{pmatrix}\mathbb{G}(x,t;\alpha,\beta),\\&\mathbb{G}(x,0;\alpha,\beta)=\delta(x)\begin{pmatrix}1&0\\0&1\end{pmatrix},\\&\beta=\sqrt{-p'(v_0)}\text{ sound speed},\ \alpha=\tfrac{\kappa}{v_0}\text{ viscosity}.\end{aligned}$$

There are two parts of the Green's function $\mathbb{G}=\mathbb{G}^*+\mathbb{G}^\sharp$. The regular part $\mathbb{G}^\sharp$ represents the large time behavior and the singular part $\mathbb{G}^*$ represents the small time behavior. The singular part $\mathbb{G}^*=\mathbb{G}^{*,1}+\mathbb{G}^{*,2}$ resolves the delta initial datum in two ways: $\mathbb{G}^{*,1}$ focuses on the propagation of the delta function for the component of specific volume v as a consequence of the absence of dissipation parameter in the first equation of the Navier-Stokes equation; while $\mathbb{G}^{*,2}$ resolves the delta function in the usual heat equation fashion due to the appearance of the viscosity in the second equation:

Theorem 3.1.

$$\mathbb{G}^{*,1}(x,t)=e^{-\frac{\beta^2}{\alpha}t}\delta(x)\begin{pmatrix}1&0\\0&0\end{pmatrix}+O(1)e^{-\sigma^* t-\sigma_0|x|}.$$

$$\mathbb{G}^{*,2}(x,t)=O(1)e^{-\sigma^* t-\sigma_0|x|}+\begin{cases}\frac{1}{\sqrt{4\pi\alpha t}}e^{-\frac{x^2}{4\alpha t}}\begin{pmatrix}0&0\\0&1\end{pmatrix},\ 0<t<1;\\ 0,\ t\geq 1,\end{cases}$$

The regular part $\mathbb{G}^\sharp$ is the viscous version of the acoustic waves of Euler equations:

Theorem 3.2.

$$\begin{aligned}&\mathbb{G}^\sharp(x,t)=O(1)te^{-\sigma_0|x|},\ \textit{for } 0<t<1;\\
&\mathbb{G}^\sharp(x,t)=\Big(\frac{e^{-\frac{(x+\beta t)^2}{2\alpha t}}}{\sqrt{2\pi\alpha t}}+O(1)\frac{e^{-\frac{(x+\beta t)^2}{C_* t}}}{t}\Big)\begin{pmatrix}\frac{1}{2}&-\frac{1}{2\beta}\\-\frac{\beta}{2}&\frac{1}{2}\end{pmatrix}\\
&+\Big(\frac{e^{-\frac{(x-\beta t)^2}{2\alpha t}}}{\sqrt{2\pi\alpha t}}+O(1)\frac{e^{-\frac{(x-\beta t)^2}{C_* t}}}{t}\Big)\begin{pmatrix}\frac{1}{2}&\frac{1}{2\beta}\\\frac{\beta}{2}&\frac{1}{2}\end{pmatrix}\\
&+\frac{\alpha}{4\beta}\Big(\frac{e^{-\frac{(x+\beta t)^2}{2\alpha t}}}{\sqrt{2\pi\alpha t}}\Big)_x\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{\alpha}{4\beta}\Big(\frac{e^{-\frac{(x-\beta t)^2}{2\alpha t}}}{\sqrt{2\pi\alpha t}}\Big)_x\begin{pmatrix}-1&0\\0&1\end{pmatrix}\\
&+O(1)t^{-\frac{3}{2}}\Big(e^{-\frac{(x+\beta t)^2}{C_* t}}+e^{-\frac{(x-\beta t)^2}{C_* t}}\Big)+O(1)e^{-\sigma_0|x|-\sigma_* t},\\
&\textit{for } t>1.\end{aligned}$$

The above explicit construction of the Green's function $\mathbb{G}$ was first done by Zeng [28], with later improvements. Fourier transform is used and the explicit inversion of the Fourier transform is calculated using the complex analysis. For the study of solutions in $L_2(x)$, as usually done in the construction of smooth solutions, there is no need to invert the Fourier transform, as the transform is isometric in the $L_2(x)$ norm and the analysis on the Fourier side suffices.

Because of the hyperbolic-parabolic nature of the Navier-Stokes equations, an initial discontinuity propagates into the solution at later time along the particle path. In the Lagrangian coordinates, this means that the discontinuity propagates along $x=constant$. Thus we consider the second Green's function $\bar{\mathsf{H}}(x,t)$ with the discontinuity sets the lines $x=constant$. There are two steps in the construction of $\bar{\mathsf{H}}$. The first step, the main one, is to consider the case when the heat conductivity coefficient $\mu(x)$ is a BV function of x only. Thus let $\mu=\mu(x)$ be of small variation and discontinuity set D:

$$\begin{cases}\mathsf{H}_t=\big(\mu\mathsf{H}_x\big)_x,\\ \mathsf{H}(x,0)=\delta(x),\end{cases}\tag{3.1}$$

The second step is to use this to construct the heat kernel when the heat conductivity coefficient is of the general form $\rho=\rho(x,t)$:

$$\begin{cases}\bar{\mathsf{H}}_t=\big(\rho\bar{\mathsf{H}}_x\big)_x,\\ \bar{\mathsf{H}}(x,0)=\delta(x),\end{cases}\tag{3.2}$$

Again, the discontinuity set consists of lines $x=constant$. $\bar{\mathsf{H}}$ is constructed from H by iterations, using the representation from Duhamel's principle:

$$\begin{aligned}\bar{\mathsf{H}}(x,t)&=\mathsf{H}(x,t-t_0;z,\tilde{\mu})\\
&-\int_{t_0}^t\int_{\mathbb{R}\backslash D}\Big(\rho(y,\tau)-\rho(y,T)\Big)\mathsf{H}_y(x,t-\tau;y,\tilde{\mu})\bar{\mathsf{H}}_y(y,\tau)dyd\tau\text{ for all }t,T>t_0;\end{aligned}$$

$$\begin{aligned}\bar{\mathsf{H}}_x(x,t)&=\mathsf{H}_x(x,t-t_0;z,\tilde{\mu})\\
&+\int_{t_0}^t\int_{\mathbb{R}\backslash D}\int_\tau^T\rho_\sigma(y,\sigma)\mathsf{H}_{xy}(x,t-\tau;y,\tilde{\mu})\bar{\mathsf{H}}_y(y,\tau)d\sigma dyd\tau\Big|_{T=t}.\end{aligned}$$

We now explain the construction of the heat kernel H of (3.1) using the Laplace transform $\mathbb{L}$ with respect to the time variable t. Approximate $\mu(x)$ by step functions. The main idea is to use the Laplace transform to express the solution as sum of integrals along the random paths. Each integral can be explicitly inverted as heat kernel for constant coefficient. We start with the simple case when the heat conductivity coefficient is a constant κ_0:

$$\begin{cases} u_t - \kappa_0 u_{xx} = 0, \\ u(x,0) = \delta(x), \end{cases}$$

This is the classical heat kernel

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa_0 t}} e^{-\frac{x^2}{4\kappa_0 t}}.$$

When the Laplace transform is applied, it becomes

$$\mathbb{L}[u](x,s) = \begin{cases} \dfrac{e^{-\sqrt{s/\kappa_0}x}}{2\sqrt{s/\kappa_0}} \text{ for } x > 0, \text{ Laplace wave train towards the right,} \\ \frac{e^{\sqrt{s/\kappa_0}x}}{2\sqrt{s/\kappa_0}} \text{ for } x < 0, \text{ Laplace wave train towards the left.} \end{cases}$$

For the general conductivity coefficient $\mu(x)$, we approximate it by a step function $\mu_k(x)$: $\mu_k(x) = \kappa_j$ for $x_j < x < x_{j+1}$. To construct the heat kernel $\mathsf{H}(x,t;y,\mu_k)$, with source $x_0 < y < x_1$, and use the Laplace transform

$$\begin{cases} (\partial_t - \partial_x \mu_k(x)\partial_x)\mathsf{H}(x,t;y,\mu_k) = 0, \\ \mathsf{H}(x,0;y,\mu_k) = \delta(x-y), \\ \mathsf{H}(x,t;y,0),\ \mu(x)\partial_x \mathsf{H}(x,t;y,\mu_k): \text{ continous in } x. \end{cases}$$

$$\mathbb{L}[\mathsf{H}](x,s;y,\mu^k) = \\ \begin{cases} e^{\sqrt{s/\kappa_j}(x-x_{j+1})}\mathsf{U}_{j+1}(y,s) + e^{-\sqrt{s/\kappa_j}(x-x_j)}\mathsf{S}_j(y,s), \text{ for } j \neq 0, \\ e^{\sqrt{s/\kappa_0}(x-x_1)}\mathsf{U}_1(y,s) + e^{-\sqrt{s/\kappa_j}(x-x_0)}\mathsf{S}_0(y,s) + \dfrac{e^{-\sqrt{s/\kappa_0}|x-y|}}{2\sqrt{s/\kappa_0}}, \text{ for } j = 0. \end{cases}$$

Set $\Delta_j \equiv \frac{x_j - x_{j-1}}{\sqrt{\kappa_{j-1}}}$. For $j \leq -1$ or $j \geq 2$,

$$\begin{cases} e^{-\sqrt{s}\Delta_{j+1}}\mathsf{U}_{j+1} + \mathsf{S}_j = \mathsf{U}_j + e^{-\sqrt{s}\Delta_j}\mathsf{S}_{j-1}, \\ \sqrt{\kappa_j}\big(e^{-\sqrt{s}\Delta_{j+1}}\mathsf{U}_{j+1} - \mathsf{S}_j\big) = \sqrt{\kappa_{j-1}}\big(\mathsf{U}_j - e^{-\sqrt{s}\Delta_j}\mathsf{S}_{j-1}\big); \end{cases}$$

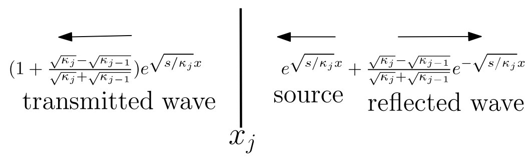
for $j = 1$,

$$\begin{cases} e^{-\sqrt{s}\Delta_2}\mathsf{U}_2 + \mathsf{S}_1 = \mathsf{U}_1 + e^{-\sqrt{s}\Delta_1}\mathsf{S}_0 + \frac{e^{-\sqrt{s/\kappa_0}\,(x_1-y)}}{2\sqrt{s/\kappa_0}}, \\ \sqrt{\kappa_1}\big(e^{-\sqrt{s}\Delta_2}\mathsf{U}_2 - \mathsf{S}_1\big) = \sqrt{\kappa_0}\big(\mathsf{U}_1 - e^{-\sqrt{s}\Delta_1}\mathsf{S}_0 - \frac{e^{-\sqrt{s/\kappa_0}\,(x_1-y)}}{2\sqrt{s/\kappa_0}}\big); \end{cases}$$

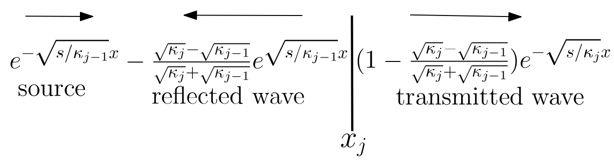
for $j = 0$,

$$\begin{cases} \frac{e^{\sqrt{s/\kappa_0}\,(x_0-y)}}{2\sqrt{s/\kappa_0}} + e^{-\sqrt{s}\Delta_1}\mathsf{U}_1 + \mathsf{S}_0 = \mathsf{U}_0 + e^{-\sqrt{s}\Delta_0}\mathsf{S}_{-1}, \\ \sqrt{\kappa_0}\big(\frac{e^{\sqrt{s/\kappa_0}\,(x_0-y)}}{2\sqrt{s/\kappa_0}} + e^{-\sqrt{s}\Delta_1}\mathsf{U}_1 - \mathsf{S}_0\big) = \sqrt{\kappa_{-1}}\big(\mathsf{U}_0 - e^{-\sqrt{s}\Delta_0}\mathsf{S}_{-1}\big). \end{cases}$$

The weak formulation of the hear kernel implies the continuity of heat and heat flux, which implies the transition and reflection coefficients as depicted in the following gifures.



Source on the right



Source on the left

This induces an iteration of a 3 points finite difference scheme

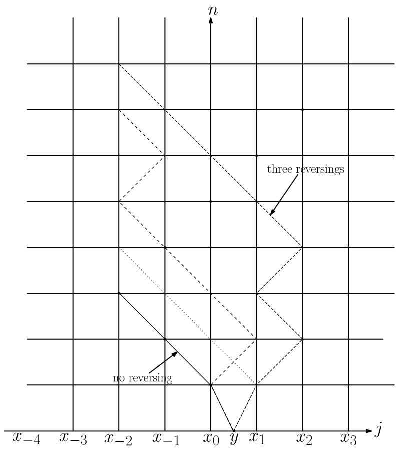
$$\begin{cases} e^{-\sqrt{s}\Delta_{j+1}}\mathsf{U}_{j+1} + \mathsf{S}_j = \mathsf{U}_j + e^{-\sqrt{s}\Delta_j}\mathsf{S}_{j-1}, \\ \sqrt{\kappa_j}\big(e^{-\sqrt{s}\Delta_{j+1}}\mathsf{U}_{j+1} - \mathsf{S}_j\big) = \sqrt{\kappa_{j-1}}\big(\mathsf{U}_j - e^{-\sqrt{s}\Delta_j}\mathsf{S}_{j-1}\big); \end{cases}$$

which is solved by the random walks, see figure below:

$$\mathsf{U}_j = \sum_{n=0}^{\infty}\mathsf{U}_j^n,\ \mathsf{S}_j = \sum_{n=0}^{\infty}\mathsf{S}_j^n,$$

$$\begin{pmatrix}\mathsf{U}_j^{n+1}\\ \mathsf{S}_j^{n+1}\end{pmatrix} = e^{-\sqrt{s}\Delta_j}\boldsymbol{R}_j\begin{pmatrix}\mathsf{U}_{j-1}^n\\ \mathsf{S}_{j-1}^n\end{pmatrix} + e^{-\sqrt{s}\Delta_{j+1}}\boldsymbol{L}_j\begin{pmatrix}\mathsf{U}_{j+1}^n\\ \mathsf{S}_{j+1}^n\end{pmatrix},$$

$$\boldsymbol{R}_j \equiv \begin{pmatrix} 0 & \frac{\sqrt{\kappa_{j-1}}-\sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}}+\sqrt{\kappa_j}} \\ 0 & 1-\frac{\sqrt{\kappa_{j-1}}-\sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}}+\sqrt{\kappa_j}}\end{pmatrix},\ \boldsymbol{L}_j \equiv \begin{pmatrix} 1-\frac{\sqrt{\kappa_{j-1}}-\sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}}+\sqrt{\kappa_j}} & 0 \\ -\frac{\sqrt{\kappa_{j-1}}-\sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}}+\sqrt{\kappa_j}} & 0\end{pmatrix}$$



Some of random paths, for $x_{-2} < x < x_{-1},\ x_0 < y < x_1$.

The coefficients $\boldsymbol{R}_j$ and $\boldsymbol{L}_j$ are functions of κ_{j-1}, κ_j. We obtain factor $e^{-\sqrt{s}\Delta_{j+1}}/\sqrt{s}$, which corresponds to classical heat kernel. Sum up the contribution from each random path, we obtain explicit expression in terms of classical heat kernels:

$$\mathsf{H}(x,t;y,\mu^k) = \frac{1}{\sqrt{4\pi\kappa_0 t}}\Big(1+O(1)\|\mu^k\|_{BV}\Big)e^{-\frac{\left(\int_x^y \frac{d\tau}{\sqrt{\mu^k(\tau)}}\right)^2}{4\kappa_0 t}}.$$

The explicit construction yields the other estimates needed for the control of singularity of the differentials of the hear kernel.

$$\begin{cases}\partial_t\bar{\mathsf{H}} - \partial_x\rho(x,t)\partial_x\bar{\mathsf{H}} = 0,\\ \bar{\mathsf{H}}(x,t_0) = \delta(x-z)\end{cases}$$

Theorem 3.3.

$$\begin{aligned}
&|\bar{\mathsf{H}}(x,t;z,t_0;\rho)| \le O(1)\frac{e^{-\frac{(x-z)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}},\\
&|\bar{\mathsf{H}}_x(x,t;z,t_0;\rho)| \le O(1)\frac{e^{-\frac{(x-z)^2}{C_*(t-t_0)}}}{t-t_0},\\
&|\bar{\mathsf{H}}_z(x,t;z,t_0;\rho)| \le O(1)\frac{e^{-\frac{(x-z)^2}{C_*(t-t_0)}}}{(t-t_0)},\\
&\textstyle\int_{t_0}^{\tau_1}\bar{\mathsf{H}}_x(x,t,z,t_0;\rho)dt \le C_\flat e^{-\frac{|x-z|}{C_\flat}},\\
&\int_{t_0}^{\tau_1}\bar{\mathsf{H}}_{xxz}(x,t,z,t_0;\rho)dt \le C_\flat\Big(\frac{e^{-\frac{(x-z)^2}{C_\flat(\tau_1-t_0)}}}{(\tau_1-t_0)} + e^{-\frac{|x-z|}{C_\flat}}\Big).
\end{aligned}$$

With the sharp estimates of $\mathbb{G}$ and $\bar{\mathsf{H}}$, we show that the iterations

$$\begin{cases} V_t^{k+1} - U_x^{k+1} = 0, \\ U_t^{k+1} + p(1+V^k)_x = \kappa\Big(U_x^{k+1}/(1+V^k)\Big)_x, \\ (V^k, U^k)(x,0) = (v_0^*, u_0^*)(x),\ k \geq 1. \end{cases}$$

converge in the following norms:

$$\begin{cases} |||f|||_\infty \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_\infty, \\ |||f|||_{BV} \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{BV}, \\ |||f|||_{L^1} \equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{L^1}, \\ |||f|||_{\mathsf{D}} \equiv \sup_{\sigma\in(0,t_\sharp)} \sup_{z\in\mathsf{D}} \left| \dfrac{f(\cdot,\sigma)\Big|_{z-}^{z+}}{v_0^*(\cdot)\Big|_{z-}^{z+}} \right|, \\ \mathsf{D} : \text{discontinuity points in the solution.} \end{cases}$$

Theorem 3.4. *Suppose that the initial data has small norm $\delta \ll 1$. Then there exists a weak solution (v,u) for the Navier-Stokes equations local in time and satisfies*

$$\|v(\cdot,t)-1\|_{L^1} + \|v(\cdot,t)\|_{BV} + \|\sqrt{t}u_x(\cdot,t)\|_\infty \leq 2C_\sharp\delta.$$

Let (u^a,v^a) and (u^b,v^b) be two solutions, then $\tilde{v} \equiv v^b - v^a,\ \tilde{u} \equiv u^b - u^a$ satisfy

$$\begin{cases} \partial_t\tilde{v} - \partial_x\tilde{u} = 0, \\ \partial_t\tilde{u} - \kappa\partial_x\dfrac{\partial_x\tilde{u}}{v^a} = \partial_x\Big(-p(v^b)+p(v^a)+\kappa\big(\dfrac{1}{v^b}-\dfrac{1}{v^a}\big)\partial_x u^b\Big), \text{ or,} \end{cases}$$
$$\tilde{u}(x,t) = \int_{\mathbb{R}} \bar{\mathsf{H}}(x,t;y,0;\kappa/v^a)(u_0^b-u_0^a)(y)dy$$
$$-\int_0^t\int_{\mathbb{R}} \bar{\mathsf{H}}_y(x,t;y,\tau;\kappa/v^a)\Big(-p(v^b)+p(v^a)+\kappa\big(\tfrac{1}{v^b}-\tfrac{1}{v^a}\big)\partial_y u^b\Big)(y,\tau)dyd\tau.$$

Theorem 3.5. *The solution depends continuously on the initial data in $L_1(x)$.*

Global well-posedness and decay of the solutions are shown for polyatomic gases $p(\tau) = A\tau^{-\gamma},\ 1<\gamma<e$:

Theorem 3.6. *Suppose that the initial value is a small perturbation of a constant state $(v,u)+(v_0,0)$ and that $(v,u)(x,0)$ has small total variation and small $L_1(x)$ norm. Then there exists a weak solution $(v,u)(x,t)$ global in time and satisfies optimal time-asymptotic decay rate $t^{-1/2}$:*

$$\begin{aligned} ||(v_0-1)(\cdot,t)||_{L_1} + ||\sqrt{t+1}(v-1)(\cdot,t)||_{L_\infty} + ||(v-1)(\cdot,t)||_{BV} \\ + ||u(\cdot,t)||_{L_1} + ||\sqrt{t+1}u(\cdot,t)||_{L_\infty} \\ + ||\sqrt{t+1}u_x(\cdot,t)||_{L_\infty} + ||u(\cdot,t)||_{BV} \leq C\delta,\ t>0. \end{aligned}$$

The result is generalized to the whole Navier-Stokes equations by Wang-Yu-ZHang [27], including the energy equation. The paper also introduces a function space for the uniqueness of solutions.

References

[1] Bianchini, S.; Bressan, A. Vanishing viscosity solutions of nonlinear hyperbolic systems. Ann. of Math. 161, (2005), 223-342.
[2] Bressan, A. Contractive metrics for nonlinear hyperbolic systems, *Indiana Univ. Math. J.* 37 (1988), 409-421.
[3] A remark on the uniqueness of solutions to hyperbolic conservation laws A Bressan, C De Lellis - arXiv preprint arXiv:2305.17203, 2023 - arxiv.org.
[4] Buckmaster, T.; Vicol, V. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math. (2)* 189 (2019), no. 1, 101-144.
[5] Dafermos, C. M. Polygonal approximations of solutions of the initial value problem for a conservation law. *J. Math. Anal. Appl.* 38 (1972), 33-41.
[6] De Lellis C., Szkelyhidi L. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* 195 (2010), no. 1, 225-260.
[7] DiPerna, R. J.; Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math.* (2) 130 (1989), no. 2, 321-366.
[8] Feireisl, E. Dynamics of Viscous Compressible Fluids. Oxford Lecture Series in Mathematics and its Applications, 26. (2004).
[9] Glimm, J. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.* 18 (1965), 697-715.
[10] Glimm, James; Lax, Peter D. Decay of solutions of systems of nonlinear hyperbolic conservation laws. *Memoirs of the American Mathematical Society*, No. 101 American Mathematical Society, Providence, R.I. 1970 xvii+112 pp.

[11] Hoff, D. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. JDE, 120 (1995) 215-254.
[12] Itaya , N., On the Cauchy problem for the e system of fundamental equations describing the movement of compressible viscous fluid. *Kodai M ath. Sem . Rep.*, 23 (1971) 60-120.
[13] K a n e l, Y . I. O n a model system of equations for one-dimensional gas motion, Diff. Eq. 4 (1968) 721-734.
[14] Kawashima, S. Large-time behavior of solutions to hyperbolic-parabolic systems of conservations lows and applications, *Prof. Roy. Soc. Edinburgh Sect. A* 106 (1987), 169-194.
[15] Matsumura, A., and Nishida, T. The initial value problem for the equations of motion of viscous and heat conductive gases. *J. Math. Kyoto Univ.* 20-1 (1980) 67-104.
[16] Nash, J. Le probleme de Cauchy pour les equations differentielles d'un fluide general. *Bull. Soc. Math. France* 90 (1962) 487-497.
[17] Lax, P.D. Hyperbolic systems of conservation laws. II. Comm. Pure Appl. Math. 10 (1957), 537-566.
[18] Leray, J. Sur le mouvement d'un liquide visqueux emplissant l'espace. (French) *Acta Math.* 63 (1934), no. 1, 193-248.
[19] Lions, P.L. Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models. Oxford Lecture Series in Mathematics and its Applications, 10. (1996).
[20] Liu, T.-P.; Yang, T. A new entropy functional for a scalar conservation law. Comm. Pure Appl. Math. 52 (1999), no. 11, 1427-1442.
[21] Liu, T.-P. Shock Waves. Graduate Studies in Mathematics, 215. American Mathematical Society, Providence, RI, (2021) xvii+437 pp.
[22] Liu, T.-P., Yu, S.-H.: Navier–Stokes equations in gas dynamics: Green's function, singularity, and well-posedness. *Comm. Pure Appl. Math.* 75(2), (2022) 223-348.
[23] Liu, Tai-Ping; Zeng, Y. Compressible Navier-Stokes equations with zero heat conductivity. *J. of Diff. Equ.* Vol. 153, No.2 (1999), 225-291.
[24] Shizuta, Y. and Kawashima, S. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.* 14 (1985), 249-275.
[25] Scheffer, V. An inviscid flow with compact support in space-time. *J. Geom. Anal.* 3(4) (1993), 343-401.
[26] Shizuta, Y. and Kawashima, S.. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.* 14 (1985), 249-275.
[27] Wang, H., Yu, S.-H. and Zhang, X. Global well-posedness of compressible Navier–Stokes equations with $BV \cup L^1$ initial data. *Arch. Ration. Mech. Anal.* 245 (2022), 375-477.
[28] Zeng, Y. L_1 asymptotic behavior of compressible, isentropic, viscous 1-D flow. textitComm. Pure Appl. Math. 47 (1994), 1053-1082.

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