Solution to the Boltzmann equation whose Fourier transform are integrable

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In this manuscript, the author introduces some recent results (partially with his collaborators) on solutions to the non cut-off Boltzmann equation that are characterized by integrability of their Fourier transform. He will explain the novelty of the result, difficulty in the proof, and possible directions of future research in this topic.

We consider the Cauchy problem on the non-cutoff Boltzmann equation in the whole space

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F), \\ F(0, x, v) = F_0(x, v). \end{cases} \tag{1}$$

Here the unknown $F = F(t, x, v) \geq 0$ denotes the density distribution function of gas particles with position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ and $F_0(x, v)$ is the given initial data. The bilinear collision operator Q is defined as

$$Q(G, F)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \left[G(v'_*) F(v') - G(v_*) F(v) \right] d\sigma dv_*,$$

where the velocity pairs (v, v_*) and (v', v'_*) satisfy

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \; v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^2.$$

We assume that the collision kernel B takes the form

$$\begin{cases} B(v - u, \sigma) = |v - u|^\gamma b(\cos\theta), & \gamma \in (-3, 1], \\ \cos\theta = \dfrac{v - u}{|v - u|} \cdot \sigma, & \theta \in (0, \pi/2], \\ \sin\theta b(\cos\theta) \simeq \theta^{-1-2s} \text{ as } \theta \to 0, & 0 < s < 1. \end{cases} \tag{2}$$

We note that Q has been symmetrized in a standard way such that B is supported in $0 < \theta \leq \pi/2$, cf. [16]. In this manuscript, we are concerned with the hard and moderately soft potential cases with a restriction

$$\gamma > \max\{-3, -3/2 - 2s\}, \tag{3}$$

that is needed only in the use of a technical trilinear estimate (6), cf. [5, Theorem 1.2].

Let $\mu = \mu(v) := (2\pi)^{-3/2}e^{-|v|^2/2}$ be a normalized global Maxwellian as a reference equilibrium. We insert $F(t,x,v) = \mu + \mu^{1/2}f(t,x,v)$ into (1) and derive the reformulated Cauchy problem on $f = f(t,x,v)$ as follows

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{4}$$

where the linearized collision operator L and the nonlinear operator Γ are respectively given by

$$\begin{aligned} Lf &= -\mu^{-1/2}[Q(\mu, \mu^{1/2}f) + Q(\mu^{1/2}f, \mu)], \\ \Gamma(f,f) &= \mu^{-1/2}Q(\mu^{1/2}f, \mu^{1/2}f). \end{aligned}$$

It is well-known that $\ker L$ is spanned by the five functions $\mu^{1/2}$, $v_i\mu^{1/2}$ ($i = 1, 2, 3$), and $|v|^2\mu^{1/2}$. $\mathbf{P}$ denotes the orthogonal projection from L^2_v to $\ker L$, which takes the form

$$\mathbf{P}f(v) = [a + b \cdot v + c(|v|^2 - 3)]\mu^{1/2}(v).$$

Since the works by AMUXY [2, 3, 1, 4] and Gressman-Strain [10, 11], a number of studies on (4) has been carried out. In particular, existence results on the usual or weighted L^2-based Sobolev spaces are numerous, and method and technique developed in such papers are now applied to the study of other kinetic equations, such as the Landau, Vlasov-Poisson-Boltzmann, and Vlasov-Maxwell-Boltzmann equations. The key reason of using such Sobolev spaces is the embedding

$$H^{3/2+\varepsilon}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),$$

where $\varepsilon > 0$ is arbitrary small. Hence, for the physically relevant three dimensional case, the index $3/2$ is somehow a threshold by this method.

Let us define some norms to describe the main results on a solution to (4) that are characterized by the behaviour of its Fourier transform. Let $\|\cdot\|_{L^p}$ denote the usual L^p-norm. We also use a weighed L^2 space endowed with the norm

$$\|f\|^2_{L^2_\alpha} = \int_{\mathbb{R}^3} |\langle v\rangle^\alpha f(v)|^2 dv, \quad \langle v\rangle := \sqrt{1+|v|^2}$$

for $\alpha \in \mathbb{R}$. The Bochner spaces $L^p_t L^q_v$ are endowed with the norm $\|\|\cdot\|_{L^q_v}\|_{L^p_t}$, $1 \leq p$, $q \leq \infty$. Let $\|\cdot\|_{L^2_{v,D}}$ be the energy dissipation norm defined by [2], which takes the form

$$\begin{aligned} \|f\|^2_{L^2_{v,D}} = &\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B\mu(v_*)(f(v') - f(v))^2 dv dv_* d\sigma \\ &+ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} Bf(v_*)^2(\mu(v')^{1/2} - \mu(v)^{1/2})^2 dv dv_* d\sigma. \end{aligned}$$

L^p_k denotes the space of functions whose Fourier transform is L^p, i.e.,

$$L^p_k = \Big\{ f \in \mathcal{S}'(\mathbb{R}^3) \mid \hat{f} \in L^1_{loc}(\mathbb{R}^3),\ \|f\|^p_{L^p_k} = \int_{\mathbb{R}^3} |\hat{f}(k)|^p dk < \infty \Big\},$$

where $\hat{f}$ is the Fourier transform of f.

Since a solution to (4) depends on (t, x, v), we will use the following mixed norms:

$$\|f\|_{L^p_k L^\infty_T L^2_v} = \Big(\int_{\mathbb{R}^3} \Big(\sup_{0 \le t \le T} \|\hat{f}(t,k,\cdot)\|_{L^2_v} \Big)^p dk \Big)^{\frac{1}{p}},$$

$$\|f\|_{L^p_k L^2_T L^2_{v,D}} = \Big(\int_{\mathbb{R}^3} \Big(\int_0^T \|\hat{f}(t,k,\cdot)\|^2_{L^2_{v,D}} dt \Big)^{\frac{p}{2}} dk \Big)^{\frac{1}{p}}$$

for any $T > 0$ and $1 \le p \le \infty$, with the standard modification of the L^p_k-norm if $p = \infty$. Here, $\hat{}$ acts only on the spatial variable; $\hat{f}(t,k,v) = \mathcal{F}_x[f(t,\cdot,v)](k)$. We also use Fourier multipliers $|\nabla_x|$ and $\langle \nabla_x \rangle$ whose symbols are $|k|$ and $\sqrt{1+|k|^2}$, respectively.

Recently, the author and his collaborators initiated the analysis of the Cauchy problem on non-L^2 based spaces. We introduce the notion

$$\|f\|_{L^1_k} = \begin{cases} \sum_{k \in \mathbb{Z}^3} |\hat{f}(k)|, & (\Omega = \mathbb{T}^3) \\ \int_{\mathbb{R}^3} |\hat{f}(k)| dk & (\Omega = \mathbb{R}^3) \end{cases}$$

Theorem 1 ([7])**.** *Let $\Omega = \mathbb{T}^3$. Let a weight function w_q be defined as*

$$w_q(v) = \begin{cases} 1 & (\gamma + 2s \ge 0), \\ \exp\Big(\frac{q}{4}\langle v \rangle\Big),\ q > 0 & (\gamma + 2s < 0,\ \gamma > \max\{-3, -3/2 - 2s\}). \end{cases}$$

Assume that $f_0(x,v)$ satisfies the conservation laws of mass, momentum, and energy, namely,

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \phi(v) f_0(x,v) dv dx = 0 \tag{5}$$

for $\phi(v) = 1, v, |v|^2$. Then there is $\epsilon_0 > 0$ such that if $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \ge 0$ and

$$\|w_q f_0\|_{L^1_k L^2_v} \le \epsilon_0,$$

then there exists a unique global solution $f(t,x,v)$, $t > 0$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ to the Cauchy problem (4), satisfying that $F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \ge 0$ and

$$\|w_q f\|_{L^1_k L^\infty_T L^2_v} + \|w_q f\|_{L^1_k L^2_T L^2_{v,D}} \lesssim \|w_q f_0\|_{L^1_k L^2_v}$$

for any $T > 0$. Moreover, let

$$\kappa = \begin{cases} 1 & (\gamma + 2s \ge 0), \\ \dfrac{1}{1 + |\gamma + 2s|} & (\gamma + 2s < 0,\ \gamma > \max\{-3, -3/2 - 2s\}), \end{cases}$$

then there is $\lambda > 0$ *such that the solution also enjoys the time decay estimate*

$$\|w_q f(t)\|_{L^1_k L^2_v} \lesssim e^{-\lambda t^\kappa} \|w_q f_0\|_{L^1_k L^2_v}$$

for any $t \geq 0$.

When $\Omega = \mathbb{T}^3$, the space

$$A(\mathbb{T}^3) = \{f \in \mathcal{S}'(\mathbb{T}^3) \mid \|f\|_{L^1_k} = \sum_{k\in\mathbb{Z}^3} |\hat{f}(k)| < \infty\}$$

is called the Wiener space [9]. This space is a Banach algebra because

$$\|fg\|_{L^1_k} = \sum_k |\hat{f} * \hat{g}(k)| \leq \sum_k \sum_\ell |\hat{f}(k-\ell)||\hat{g}(\ell)| \leq \|f\|_{L^1_k}\|g\|_{L^1_k},$$

where $*$ stands for the convolution. By this property, which we cannot have for L^2, we may consider $A(\mathbb{T}^3)$ is somehow close to L^∞. Actually, for $1/2 < \beta \leq 1$ it holds

$$C^{1,\beta}(\mathbb{T}^3) \subset A(\mathbb{T}^3), \quad C^{1,1/2}(\mathbb{T}^3) \not\subset A(\mathbb{T}^3).$$

Here, $C^{n,\beta}(\mathbb{T}^3)$ is the set of C^n-functions on $\mathbb{T}^3$ whose all the α-th derivative with $|\alpha| = n$ is β-Hölder continuous.

The significance of [7] would be in the fact that $A(\mathbb{T}^3)$ is a very easy-to-use space in the analysis. Indeed, not only the Boltzmann equation on $\mathbb{T}^3$, we also studied

- initial-boundary value problems on $[-1,1] \times \mathbb{T}^2$ with specular reflection and diffuse conditions,
- Landau equation (both on $\mathbb{T}^3$ and on $[-1,1] \times \mathbb{T}^2$ with boundary conditions),

where the unified strategy is applicable. In particular, the study of a Cauchy problem of the non cut-off Boltzmann and Landau equations with boundary conditions was unprecedented, even for simple conditions like above.

The basic strategy of the proof of Theorem 1 is twofold: First, for the estimate of the non-linear part, we combine the trilinear estimate by AMUXY [4] and analysis on the Fourier side to have

$$\left| \left(\widehat{\Gamma}(f,g)(k), \hat{h}(k) \right)_{L^2_v} \right| \leq C \int_{\hat{\Omega}_\ell} \|\hat{f}(k-\ell)\|_{L^2_v} \|\hat{g}(\ell)\|_{L^2_{v,D}} \|\hat{h}(k)\|_{L^2_{v,D}} d\Sigma(\ell). \tag{6}$$

Here $\hat{\Omega}$ is equal to $\mathbb{Z}^3$ if $\Omega = \mathbb{T}^3$ and $\mathbb{R}^3$ if $\Omega = \mathbb{R}^3$, and $d\Sigma$ is the counting measure on $\mathbb{Z}^3$ if $\Omega = \mathbb{T}^3$ and the Lebesgue measure if $\Omega = \mathbb{R}^3$. Thus this estimate is useful for the both cases. Second, if (5) is assumed, then the Fourier zero mode is 0, hence

$$\sum_{k\in\mathbb{Z}^3} |k||\mathbf{P}\hat{f}(k)| = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k||\mathbf{P}\hat{f}(k)|$$

and we can divide the right hand side with k before taking summation. On the phase space, this corresponds to using the Poincaré inequality, and the macroscopic part exhibits dissipation effect. Contrary to the first remark, this cannot be expected for the case $\Omega = \mathbb{R}^3$. These strategy are applicable to the Landau equation because of the same structure of the linear and non-linear terms (cf. [12]), and to initial-boundary value problems by employing appropriate energy functional taking into account the effect from the boundary.

If $\Omega = \mathbb{R}^3$, the above-mentioned method to capture dissipation from the macroscopic part does not work. In this case, using the L^p_k-norm as an auxiliary norm, we have the following existence result:

Theorem 2 ([8], Hard potentials)**.** *Assume* (2) *and* (3). *Let* $\Omega = \mathbb{R}^3$, $\gamma + 2s \geq 0$, $3/2 < p \leq \infty$ *and*

$$\sigma = 3\Big(1 - \frac{1}{p}\Big) - 2\varepsilon$$

with $\varepsilon > 0$ *arbitrary small. There is* $\epsilon_1 > 0$ *such that if* $F_0(x, v) = \mu + \mu^{\frac{1}{2}} f_0(x, v) \geq 0$ *and*

$$\|f_0\|_{L^1_k L^2_v} + \|f_0\|_{L^p_k L^2_v} \leq \epsilon_1,$$

then the Cauchy problem (4) *admits a unique global mild solution* $f(t, x, v)$, $t > 0$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, *which satisfies* $F(t, x, v) = \mu + \mu^{\frac{1}{2}} f(t, x, v) \geq 0$ *as well as the following estimates:*

$$\begin{aligned} &\|(1+t)^{\frac{\sigma}{2}} f\|_{L^1_k L^\infty_T L^2_v} + \|(1+t)^{\frac{\sigma}{2}} (\mathbf{I} - \mathbf{P}) f\|_{L^1_k L^2_T L^2_{v,D}} \\ &\qquad + \Big\|(1+t)^{\frac{\sigma}{2}} \frac{|\nabla_x|}{\langle \nabla_x \rangle}(a, b, c)\Big\|_{L^1_k L^2_T} \leq C\|f_0\|_{L^1_k L^2_v} + C\|f_0\|_{L^p_k L^2_v} \end{aligned} \tag{7}$$

and

$$\|f\|_{L^p_k L^\infty_T L^2_v} + \|(\mathbf{I} - \mathbf{P}) f\|_{L^p_k L^2_T L^2_{v,D}} + \Big\|\frac{|\nabla_x|}{\langle \nabla_x \rangle}(a, b, c)\Big\|_{L^p_k L^2_T} \leq C\|f_0\|_{L^p_k L^2_v}$$

for any $T > 0$. *Here* C *is independent of* T.

For the soft potential case, we need the following weight function:

$$w_{\ell,q}(v) = \begin{cases} \langle v \rangle^{\ell|\gamma+2s|}, & \ell > 0,\ q = 0 \quad \text{(polynomial weight)}, \\ e^{q\langle v \rangle/4}, & \ell = 0,\ q > 0 \quad \text{(exponential weight)}, \\ \langle v \rangle^{\ell|\gamma+2s|} e^{q\langle v \rangle/4}, & \ell > 0,\ q > 0 \quad \text{(mixed weight)}. \end{cases} \tag{8}$$

Theorem 3 ([8], Soft potentials)**.** *Assume* (2) *and* (3). *Let* $\Omega = \mathbb{R}^3$, $\gamma + 2s < 0$, $3/2 < p \leq \infty$ *and*

$$\sigma = 3\Big(1 - \frac{1}{p}\Big) - 2\varepsilon$$

with $\varepsilon > 0$ arbitrary small. Also let $\ell \geq 0$, $q \geq 0$, $w_{\ell,q}$ be any of the weights in (8), and suppose

$$\Big(j > \frac{\sigma}{2r}\Big) \wedge \Big(q = 0 \Rightarrow \ell > \frac{\sigma}{2r}\Big), \tag{9}$$

where

$$0 < r < \frac{p'\varepsilon}{3+p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then there is $\epsilon_2 > 0$ such that if $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \geq 0$ and

$$\|w_{\ell+j,q} f_0\|_{L^1_k L^2_v} + \|w_{\ell+j,q} f_0\|_{L^p_k L^2_v} \leq \epsilon_2,$$

then the Cauchy problem (4) admits a unique global mild solution $f(t,x,v)$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, which satisfies $F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \geq 0$ as well as the following estimates:

$$\begin{aligned} &\|(1+t)^{\sigma/2} w_{\ell,q} f\|_{L^1_k L^\infty_T L^2_v} + \|(1+t)^{\sigma/2} w_{\ell,q} (\mathbf{I} - \mathbf{P}) f\|_{L^1_k L^2_T L^2_{v,D}} \\ &\qquad + \Big\| \frac{|\nabla_x|}{\langle \nabla_x \rangle} (1+t)^{\sigma/2} (a,b,c) \Big\|_{L^1_k L^2_T} \leq C \|w_{\ell+j,q} f_0\|_{L^1_k L^2_v} + C \|w_{\ell+j,q} f_0\|_{L^p_k L^2_v} \end{aligned} \tag{10}$$

and

$$\|w_{\ell,q} f\|_{L^p_k L^\infty_T L^2_v} + \|w_{\ell,q} (\mathbf{I} - \mathbf{P}) f\|_{L^p_k L^2_T L^2_{v,D}} + \Big\| \frac{|\nabla_x|}{\langle \nabla_x \rangle} (a,b,c) \Big\|_{L^p_k L^2_T} \leq C \|w_{\ell+j,q} f_0\|_{L^p_k L^2_v}$$

for any $T > 0$. Here C is independent of T.

Note (7) and (10) give decay rate of the solutions to (4) in the hard and soft potential cases, respectively. The rate $(1+t)^{-\sigma/2}$ is the same. This decay rate is almost optimal, in the sense that it coincides with that of a solution to the linearized problem, disregarding arbitrary small $\varepsilon > 0$.

We also remark that, if $3/2 < p < 2$, a local solution to (4) is not yet constructed in $(L^1 \cap L^p)_k$ because the space does not provide enough regularity of functions in the argument. In particular, since we could not find an appropriate Hilbert space that contain $(L^1 \cap L^p)_k$, the known method to construct an approximate family of solutions via the Hahn-Banach theorem does not apply. In these cases, we assume that first we construct a local solution in $(L^1 \cap L^2)_k$, where we can construct one, and then assume initial datum is small in $(L^1 \cap L^p)_k$. The above-mentioned theorems (and the following ones too, since they are consequences of the above) should be understood in this way.

The key ideas to the proof of Theorems 2 and 3 are following: First, since dissipation effect from $\mathbf{P}f$ (whose $L^2_{v,D}$-norm is equivalent to (a,b,c) above) because of the symbol $|k|/\langle k \rangle$, we have introduced the ansatz that

$$\|f\|_{L^1_k L^\infty_T L^2_v} + \|(1+t)^{\frac{\sigma}{2}} \mathbf{P} f\|_{L^1_k L^\infty_T L^2_v} \leq \delta, \tag{11}$$

where $\delta > 0$ is a small constant that will be chosen later. This means The energy term and the time-weighted energy of the macroscopic part are assumed to be small. The first component is to make the argument concise, so not an essential assumption, but the smallness of $\|(1+t)^{\frac{\sigma}{2}}\mathbf{P}f\|_{L^1_k L^\infty_T L^2_v}$ is essential. By assuming this, we have

$$\|\mathbf{P}f\|_{L^1_k L^2_T L^2_{v,D}} \leq C\|\mathbf{P}f\|_{L^1_k L^2_T L^2_v} \leq C\delta\Big(\int_0^T (1+t)^{-\sigma}dt\Big)^{\frac{1}{2}} \leq C\delta, \tag{12}$$

which gives the control of macroscopic dissipation. Estimates of the energy and microscopic dissipation terms can be shown by usual energy method, hence combining these estimates we get the lowest order a priori estimate. We remark that by this ansatz we need to assume $p > 3/2$, which guarantees the uniform boundedness of the time integral with respect to upper bound T. This is the only part where we need $p > 3/2$, hence if we can introduce a new idea for energy method, we may remove this restriction.

Second, in order to verify (11) so that we can close all the estimates, we carry out time-weighted energy estimates with the help of L^p_k estimates. These are employed to control the low-frequency terms. Not like the case on the torus, the conservation laws do not guarantee that there are no low-frequency terms, because all we can say is

$$\int_{|k|\leq 1} \hat{f}(k)dk = \int_{0<|k|\leq 1} \hat{f}(k)dk$$

even if $\hat{f}(0) = 0$, which can be assured for solutions to the Boltzmann equation around Maxwellian by the conservation laws. In the hard cases, we have the remainder term

$$\int_{\mathbb{R}^3} \Big(\int_0^T (1+t)^{\sigma-1}\|\hat{f}\|^2_{L^2_v}dt\Big)^{1/2} dk. \tag{13}$$

To control the low-frequency part of this integral (i.e. integration over $\{|k| \leq 1\}$), we use the interpolation

$$\begin{aligned}(1+t)^{\sigma-1} &= (1+t)^{\sigma(1-\theta)}|k|^{2(1-\theta)} \cdot (1+t)^{(-1-\varepsilon)\theta}|k|^{-2(1-\theta)} \\ &\leq \eta^2(1+t)^{\sigma}|k|^2 + C_\eta^2(1+t)^{-1-\varepsilon}|k|^{-2\frac{1-\theta}{\theta}},\end{aligned} \tag{14}$$

where $\theta = (\sigma+1+\varepsilon)^{-1} \in (0,1)$. The reason why we need $|k|^2$ is that, since the macroscopic part is applied the operator $|\nabla_x|/\langle\nabla_x\rangle$, whose symbol $|k|/\langle k\rangle$. This behaves like $|k|$ on $\{|k| \leq 1\}$, hence we need $|k|^2$ (that will be square rooted). Substituting (14) into (13), we will estimate the remainder as follows:

$$\begin{aligned}\int_{|k|\leq 1} \Big(\int_0^T (1+t)^{-1-\varepsilon}|k|^{-2\frac{1-\theta}{\theta}}\|\hat{f}\|^2_{L^2_v}dt\Big)^{1/2} dk &\leq C\int_{|k|\leq 1} |k|^{-\frac{1-\theta}{\theta}}\|\hat{f}\|_{L^\infty_T L^2_v}dk \\ &\leq C\|f\|_{L^p_k L^\infty_T L^2_v}\Big(\int_{|k|\leq 1} |k|^{-p'\frac{1-\theta}{\theta}}dk\Big)^{1/p'} \\ &\leq C\|f_0\|_{L^p_k L^\infty_T L^2_v},\end{aligned}$$

because one can compute

$$-p'\frac{1-\theta}{\theta} = -3 + p'\varepsilon > -3. \tag{15}$$

Notice this calculation clearly demonstrates we need L^p_k, $p > 1$ to control the low-frequency term. Also note that arbitrary small $\varepsilon > 0$ is needed here for the boundedness of

$$\int_0^\infty (1+t)^{-1-\varepsilon} dt.$$

This loss of ε seems to come from the use of interpolation technique. It would be an interesting problem to remove this, which gives optimal decay rate.

Rather recently, the author has proved the following generalization of Theorems 2 and 3 and faster decay of the microscopic part in L^1_k:

Theorem 4 ([13], Hard potentials)**.** *In addition to the assumptions as in Theorem 2, assume*

$$\epsilon_{0,\alpha} = \|f_0\|_{L^1_k L^2_v} + \|f_0\|_{L^p_k L^2_v} + \||\nabla_x|^\alpha f_0\|_{L^1_k L^2_v} \le \epsilon_1$$

for some $\alpha \ge 0$. Then, taking $\epsilon_1 > 0$ smaller if necessary, the solution obtained in Theorem 2 satisfies

$$\|(1+t)^{\frac{\sigma+\alpha}{2}} |\nabla_x|^\alpha f\|_{L^1_k L^\infty_T L^2_v} + \|(1+t)^{\frac{\sigma+\alpha}{2}} |\nabla_x|^\alpha (\mathbf{I}-\mathbf{P}) f\|_{L^1_k L^2_T L^2_{v,D}} + \left\|(1+t)^{\frac{\sigma+\alpha}{2}} \frac{|\nabla_x|^{\alpha+1}}{\langle \nabla_x \rangle}(a,b,c)\right\|_{L^1_k L^2_T} \le C\epsilon_{0,\alpha} \tag{16}$$

for any $T > 0$. Moreover, if $\epsilon_{0,\alpha+1} \le \epsilon_1$ is further assumed, then it holds

$$(1+T)^{\frac{\sigma+\alpha+1}{2}} \||\nabla_x|^\alpha (\mathbf{I}-\mathbf{P}) f(T)\|_{L^1_k L^2_v} \le C\epsilon_{0,\alpha+1} \tag{17}$$

for a.e. $T > 0$. Here C in (16) *and* (17) *are independent of T.*

Theorem 5 ([13], Soft potentials)**.** *In addition to the assumptions as in Theorem 3, assume*

$$\epsilon^w_{0,\alpha} = \|w_{\ell,q} f_0\|_{L^1_k L^2_v} + \|w_{\ell+j,q} f_0\|_{L^p_k L^2_v} + \|w_{\ell+j,q} |\nabla_x|^\alpha f_0\|_{L^1_k L^2_v} \le \epsilon_2$$

for some $\alpha \ge 0$. Also, the conditions (9) *on the exponent of the weight functions are replaced with*

$$\Big(j > \frac{\sigma+\alpha}{2r}\Big) \wedge \Big(q = 0 \Rightarrow \ell > \frac{\sigma+\alpha}{2r}\Big),$$

where

$$0 < r < \frac{p'\varepsilon}{3+(\alpha+1)p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then, taking $\epsilon_2 > 0$ smaller if necessary, the solution obtained in Theorem 3 satisfies

$$\|(1+t)^{\frac{\sigma}{2}}|\nabla_x|^\alpha w_{\ell,q} f\|_{L^1_k L^\infty_T L^2_v} + \|(1+t)^{\frac{\sigma}{2}}|\nabla_x|^\alpha w_{\ell,q}(\mathbf{I}-\mathbf{P})f\|_{L^1_k L^2_T L^2_{v,D}} + \left\|(1+t)^{\frac{\sigma}{2}}\frac{|\nabla_x|^{\alpha+1}}{\langle\nabla_x\rangle}(a,b,c)\right\|_{L^1_k L^2_T} \le C\epsilon^w_{0,\alpha} \quad (18)$$

for any $T > 0$. Moreover, if $\epsilon^w_{0,\alpha+1} \le \epsilon_2$ is further assumed, it holds

$$(1+T)^{\frac{\sigma+\alpha+1}{2}}\||\nabla_x|^\alpha(\mathbf{I}-\mathbf{P})f(T)\|_{L^1_k L^2_v} \le C\epsilon^w_{0,\alpha+1} \quad (19)$$

for a.e. $T > 0$. Here C (18) and (19) are independent of T.

The main statement of these theorems are (17) and (19). These inequalities show that the microscopic part of the solutions always decay faster than the solution itself with the rate $(1+t)^{-1/2}$ for any order of differentiation $\alpha \ge 0$. In order to do that, we first need to deduce (16) and (18) for each case.

The strategy of the proof of (16) and (18) are rather similar to those of (7) and (10). Indeed, when we need to apply the multiplication of $|k|^\alpha$ to the non-linear terms (the linear terms can be dealt with pretty much the same way), we use

$$|k|^\alpha \le C(|k-\ell|^\alpha + |\ell|^\alpha).$$

As we can observe form the convolution, in the product of functions on the Fourier side, we apply this multiplier to one of them and the other is of zeroth order. Since such terms are already proved to be small, we can apply similar energy method.

We remark that the interpolation technique works uniform in $\alpha \ge 0$. Indeed, for general $\alpha > 0$ we have to estimate

$$\int_{\mathbb{R}^3} |k|^\alpha \Big(\int_0^T (1+t)^{\sigma+\alpha-1}\|\hat{f}\|^2_{L^2_v} dt\Big)^{1/2} dk.$$

For this term we use the interpolation

$$\begin{aligned}(1+t)^{\sigma+\alpha-1} &= (1+t)^{(\sigma+\alpha)(1-\theta)}|k|^{2(1-\theta)} \cdot (1+t)^{(-1-\varepsilon)\theta}|k|^{-2(1-\theta)} \\ &\le \eta^2(1+t)^{\sigma+\alpha}|k|^2 + C^2_\eta(1+t)^{-1-\varepsilon}|k|^{-2\frac{1-\theta}{\theta}}, \quad (20)\end{aligned}$$

where $\theta = (\sigma+\alpha+1+\varepsilon)^{-1} \in (0,1)$. (20) generalizes (14). Then the remainder on $\{|k| \le 1\}$ is bounded by

$$\begin{aligned}\int_{|k|\le 1} |k|^\alpha \Big(\int_0^T (1+t)^{-1-\varepsilon}|k|^{-2\frac{1-\theta}{\theta}}\|\hat{f}\|^2_{L^2_v} dt\Big)^{1/2} dk &\le C\int_{|k|\le 1} |k|^\alpha |k|^{-\frac{1-\theta}{\theta}}\|\hat{f}\|_{L^\infty_T L^2_v} dk \\ &\le C\|f\|_{L^p_k L^\infty_T L^2_v}\Big(\int_{|k|\le 1} |k|^{p'(\alpha-\frac{1-\theta}{\theta})} dk\Big)^{1/p'} \\ &\le C\|f_0\|_{L^p_k L^\infty_T L^2_v},\end{aligned}$$

because, as in (15), one can compute

$$p'\Big(\alpha - \frac{1-\theta}{\theta}\Big) = -3 + p'\varepsilon > -3.$$

Note this exponent of $|k|$ is uniform in α, even though θ is not.

To prove faster decay (17) and (19), we employ the strategy from [15], where we deduce a priori estimates for $(\mathbf{I}-\mathbf{P})f$. In the hard potential case one can deduce

$$\begin{aligned}
&\|(\mathbf{I}-\mathbf{P})f(T)\|_{L^1_k L^2_v} \\
&\le Ce^{-\frac{\lambda}{2}T}\|(\mathbf{I}-\mathbf{P})f_0\|_{L^1_k L^2_v} + C\int_{\mathbb{R}^3} |k| \Big(\int_0^T e^{-\lambda(T-t)} \|\mathbf{P}\hat{f}(t,k)\|^2_{L^2_{v,D}} dt\Big)^{1/2} dk \\
&+ C\int_{\mathbb{R}^3} \Big(\int_0^T e^{-\lambda(T-t)} \Big(\int_{\mathbb{R}^3} \|\hat{f}(k-\ell)\|_{L^2_v} \|\hat{f}(\ell)\|_{L^2_{v,D}} d\ell\Big)^2 dt\Big)^{1/2} dk
\end{aligned}$$

by applying $(\mathbf{I}-\mathbf{P})$ to (4) then carrying out the usual energy estimate. Substituting the decay estimates of the solution (16), one can calculate that the decay rates of the second and third terms are $(1+t)^{-\frac{\sigma+1}{2}}$ and $(1+t)^{-\sigma}$. Since we assume $p > 3/2$, it holds $\sigma > (\sigma+1)/2$. Therefore we have $(1+t)^{-1/2}$-faster decay.

Open problems

Here we catalogue some open possible problems related with the above-mentioned theorems.

First, concerned with the problem on $\mathbb{T}^3$, we may consider whether we can weaken the condition $f \in A(\mathbb{T}^3)$. Although this space is not contained in C^1, it contains $C^{1,\beta}$ $(1/2 < \beta \le 1)$, so somehow we are taking smooth data. It would be interesting to find a solution in wider classes that just contained in C but does not exhibit differentiability.

Second, concerned with the problem on $[-1,1]\times\mathbb{T}^2$, generalizing the boundary conditions is an interesting problem. Since what were dealt with are very basic ones, still there a lot to be done in this direction. Also, considering the problem on $[-1,1]\times\mathbb{R}^2$ would be a problem too. Since the domain is bounded in one direction, a solution in this case behaves like that on $[-1,1]\times\mathbb{T}^2$ (if exists), but such properties are not yet fully revealed.

Third, concerned with the solution in $(L^1\cap L^p)_k$, there are some possibilities:

- Can we construct a local (and hence a global, given we have already proved a priori estimates) solution if $3/2 < p < 2$? The author guesses that this may require the analysis of the equation on Sobolev or Besov spaces of negative order. Analysis in [14] would be helpful, but not yet sure.

- Can we remove the restriction $3/2 < p \leq \infty$, which came from the use of (11) and (12)? So far this seems difficult, because $\mathbf{P}f$ lacks dissipativity on $\mathbb{R}^3$. Thoroughly new method would be required for this problem.

- Related with the second item, can we remove $\varepsilon > 0$ and recover optimal decay of the solution? ε is needed for (14). One possibility would be the use of more localized spaces, such as the locally uniform L^p or Fourier-Besov spaces, instead of L^p_k as an auxiliary space.

- Can we use some spaces other than L^1_k as a base space (in view of decay rate)? The difficulty of replacing L^1_k with others lies in inequality

$$\|fg\|_{L^p_k} \leq \|f\|_{L^q}\|g\|_{L^r}$$

  with $1/p = 1/q + 1/r - 1$, i.e., L^p_k estimates are of convolution type. If we want either q or r is equal to p, then the possible choice is to take at least one of them 1. Some finer estimates would be required.

Finally, for possible problems on smoothing effects of (4), consult [6].

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