On the energy identity for the full system of compressible Navier–Stokes equations

Motofumi Aoki
Tohoku University

1 Introduction

This is a survey paper on [2]. The full system of compressible Navier–Stokes equations in $\mathbb{T}^d$ with $d \geq 2$ is written as follows.

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, & t > 0,\ x \in \mathbb{T}^d, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho, \theta) - \mathrm{div}\,\mathbb{S} = 0, & t > 0,\ x \in \mathbb{T}^d, \\ \partial_t(\rho Q(\theta)) + \mathrm{div}\,(\rho Q(\theta) u) - \kappa \Delta \theta = (\mathbb{S} : \nabla u) - p_{th}(\rho, \theta) \mathrm{div}\, u, & t > 0,\ x \in \mathbb{T}^d, \\ (\rho, u, \theta)\big|_{t=0} = (\rho_0, u_0, \theta_0), & x \in \mathbb{T}^d. \end{cases} \tag{1.1}$$

The equations (1.1) consist of the continuity, the motion, and the thermal energy of a fluid. $\rho = \rho(t, x) : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}_+$, $u = (u_1(t, x), \cdots, u_d(t, x)) : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}^d$ and $\theta = \theta(t, x) : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ denote the unknown density of the fluid, the unknown velocity vector and the unknown temperature of the fluid at the point $(t, x) \in (0, T) \times \mathbb{T}^d$. $\rho_0 = \rho_0(x)$ is the initial density, $u_0 = (u_{0,1}(x), \cdots, u_{0,d}(x))$ is the initial velocity vector and $\theta_0 = \theta_0(x)$ is the initial temperature. $\mathbb{T}^d$ denotes the d-dimensional torus $[0, 2\pi]^d$. $\mathbb{S}$ with coefficients μ, λ denotes the viscous stress tensor of fluid such that

$$\begin{cases} \mathbb{S} := \mu\{\nabla u + (\nabla u)^T\} + \lambda\, \mathrm{div}\, u\, \mathbb{I}, \\ \mu > 0,\ \lambda + \dfrac{2}{d}\mu \geq 0, \end{cases}$$

where $\nabla u = (\partial_{x_i} u_j)$, $(\nabla u)^T$ denotes the transpose of ∇u and $\mathbb{I}$ denotes the identity matrix. For the vector-valued functions $u = (u_1, u_2, \cdots, u_d)$ and $v = (v_1, v_2, \cdots, v_d)$, we also $u \otimes v$ by

$$u \otimes v := (u_i v_j)_{1 \leq i, j \leq d}.$$

Here $p = p(\rho, \theta)$ is a scalar function representing the pressure and satisfies a general constitutive relation

$$p(\rho, \theta) := p_e(\rho) + p_{th}(\rho, \theta) = p_e(\rho) + \theta p_\theta(\rho),$$

where p_e means the elastic pressure and p_{th} means the thermal pressure. The thermal energy contribution $Q(\theta)$ satisfies

$$Q(\theta) := \int_0^\theta c_v(z) dz,$$

where $c_v : [0, \infty) \to \mathbb{R}$ and $\inf c_v > 0$. The coefficient κ is positive such that the internal energy flux $\kappa \nabla \theta$ is given by Fourier's law. Here, $(\mathbb{S} : \nabla u)$ stands for a scalar product of matrices:

$$(\mathbb{S} : \nabla u) = \sum_{i,j=1}^{d} S_{ij} \partial_i u_j.$$

We define an energy $E(t)$ by

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^d} \rho(t,x) u^2(t,x) dx + \int_{\mathbb{T}^d} \rho(t,x) h(\rho(t,x)) dx + \int_{\mathbb{T}^d} \rho(t,x) Q(\theta(t,x)) dx, \quad (1.2)$$

where

$$h(\rho) := \int_1^\rho \frac{p_e(z)}{z^2} dz, \quad (1.3)$$

and $p_e(\rho)$ is from the relation of the pressure $p(\rho, \theta) = p_e(\rho) + p_{th}(\rho, \theta)$. Moreover, ρu^2, $\rho h(\rho)$ and $\rho Q(\theta)$ denote the kinetic energy, the elastic potential and the internal energy, respectively.

The existence of a weak solution for (1.1) is proved by Feireisl [7]. However, it is not clear whether the weak solution satisfies energy conservation law or not.

To study the energy conservation law for the compressible Navier–Stokes equations, we first recall several results for the incompressible Navier–Stokes equations and the equations of the viscosity barotropic fluid.

We first consider the incompressible Navier–Stokes equations case.

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, & t > 0, \ x \in \mathbb{R}^d, \\ \operatorname{div} u = 0, & t > 0, \ x \in \mathbb{R}^d, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^d, \end{cases} \quad (1.4)$$

where $u = (u_1(t,x), \cdots, u_d(t,x))$ and $p = p(t,x)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(t,x) \in (0,T) \times \mathbb{R}^d$, and $u_0 = (u_{0,1}(x), \cdots, u_{0,d}(x))$ is the given initial velocity vector.

The global existence of weak solutions in $L^\infty(0,T;L^2)\cap L^2(0,T;\dot{H}^1)$ was proved by Leray [13] for the whole space and Hopf [9] for arbitrary bounded domains. They proved that weak solutions satisfy the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(t)\|_2^2 ds \le \|u_0\|_2^2. \tag{1.5}$$

The weak solution above is called Leray–Hopf weak solution. In the case of three or higher dimensions, however, it is a famous open problem whether Leray–Hopf's solution is unique or not. By assuming additional conditions, partial results are known. The uniqueness provided

$$u \in L^p(0,T;L^q), \quad \frac{2}{p}+\frac{d}{q}<1, \quad 2\le p<\infty, \quad d<q\le\infty, \tag{1.6}$$

was proved by Prodi [17] for whole space and by Serrin [18] for arbitrary domain. Ladyzhenskaya [12] showed the regularity in the above class. The conditions (1.6) are nowadays called the Serrin class (or Ladyzhenskaya–Prodi–Serrin class) (see also Serrin [19]). Masuda [15], Sohr [21], Giga [8] and Kozono–Sohr [11] generalized to the critical case $2/p+d/q=1$.

Compared with uniqueness, we can find more general conditions for the energy conservation law. Shinbrot [20] gave the following condition.

$$\begin{aligned} & u \in L^p(0,T;L^q), \quad \frac{1}{p}+\frac{3}{q}=1, \quad 3\le q\le 4, \\ \text{or}\quad & u \in L^p(0,T;L^q), \quad \frac{2}{p}+\frac{2}{q}=1, \quad q\ge 4. \end{aligned} \tag{1.7}$$

The most important case is characterized by $p=q=4$, and let us explain how to derive the $L^4(0,T;L^4)$ integrability. To this end, we multiply the equation (1.4) by u and integrate time and space variables. Since $u\in L^4(0,T;L^4)$, we can estimate the nonlinear term $(u\cdot\nabla)u$,

$$\left|\int_0^T \langle (u\cdot\nabla)u, u\rangle ds\right| \le \|u\|_{L^4(0,T;L^4)}^2 \|u\|_{L^2(0,T;\dot{H}^1)},$$

which can justify

$$\int_0^t \langle (u\cdot\nabla)u, u\rangle ds = 0,$$

where $\langle\cdot,\cdot\rangle$ denotes L^2 inner product. Therefore we formally obtain

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds = \|u_0\|_2^2, \quad \text{a.e.}\, t\in(0,T),$$

for almost any $t \in (0,T)$. In the rigorous proof, we take a mollification of u. Later Taniuchi [22] also studies a sufficient condition for energy conservation law on the general dimensional case. The most general results are those in

$$\begin{aligned} & L^{p,w}(0,T;B^{\frac{2}{p}+\frac{2}{q}-1}_{q,\infty}), \quad \frac{1}{p}+\frac{2}{q}<1, \quad 1\le p<q\le\infty, \\ \text{or} \quad & L^{p}(0,T;B^{\frac{5}{2p}+\frac{3}{q}-\frac{3}{2}}_{q,\infty}), \quad \frac{1}{p}+\frac{2}{q}\ge 1, \quad 0<p\le 3, 1\le q\le\infty, \end{aligned}$$

by Cheskidov–Luo [5]. Here, $B^s_{p,q}$ denotes the inhomogeneous Besov spaces.

We next consider the barotropic compressible Navier–Stokes equations case. The motion of viscosity barotropic fluid is described by the following.

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & t>0,\ x\in\mathbb{T}^d, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) - \operatorname{div}\mathbb{S} = 0, & t>0,\ x\in\mathbb{T}^d, \\ (\rho,u)\big|_{t=0} = (\rho_0,u_0), & x\in\mathbb{T}^d. \end{cases} \tag{1.8}$$

Lions [14] first showed the existence of weak solutions for (1.8) on the bounded domain. However, it is also unknown whether the weak solutions of (1.8) satisfy the energy conservation law.

We mention previous studies on energy conservation laws for the compressible Navier–Stokes equations. Yu [23] proved the energy identity for the isentropic Navier–Stokes equations if the velocity u belongs to $L^p(0,T;L^q(\mathbb{T}^3))$ with $2/p+2/q\le 5/6$ and $q\ge 6$. Here, the isentropic Navier–Stokes equations are one of the barotropic fluids such that the pressure $p(\rho)$ satisfies

$$p(\rho)=\rho^\gamma, \quad \gamma>\frac{d}{2}.$$

Akramov–Dębiec–Skipper–Wiedemann [1] showed the energy identity for (1.8) when $u\in B^\alpha_{3,\infty}((0,T)\times\mathbb{T}^3)\cap L^2(0,T;W^{1,2})$, $\rho,\rho u\in B^\beta_{3,\infty}((0,T)\times\mathbb{T}^3)$ with $\alpha+2\beta>1$, $2\alpha+\beta>1$, $0\le\alpha,\beta\le 1$ and the pressure is continuous with respect to the density. Furthermore, Nguyen–Nguyen–Tang [16] considered energy conservation law if weak solutions satisfy

$$\begin{aligned} & u\in L^p(0,T;L^q), \quad \frac{2}{p}+\frac{2}{q}=1, \quad q\ge 4, \\ & 0<c_1\le\rho\le c_2<\infty, \quad \rho\in L^\infty(0,T;B^{\frac{1}{2}}_{\frac{12}{5},\infty}(\mathbb{T}^3)), \quad p(\cdot)\in C^2((0,\infty)), \end{aligned}$$

for some constants c_1, c_2.

Our purpose of this paper that we show the energy conservation law for the equations (1.1) with (1.7). Let us introduce a weak solution of (1.1) based on an idea of Feireisl [7].

Definition (Weak solution). $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is bounded domain. Let $T > 0$ and $\gamma > \dfrac{d}{2}$. A measurable function (ρ, u, θ) on $(0,T) \times \Omega$ is called a weak solution of (1.1) on $(0,T)$ if

1. ρ, u, θ satisfy for $r > \dfrac{2d}{d+2}$,
   $\rho \in L^\infty(0,T;L^\gamma)$, $u \in L^2(0,T;W_0^{1,2})$, $\rho u \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})$, $\mathbb{S} \in L^2(0,T;L^2)$, $p_e(\rho) \in L^1(0,T;L^1)$, $\rho Q(\theta) \in L^\infty(0,T;L^1) \cap L^2(0,T;L^2 \cap L^r)$, $\mathcal{K}(\theta) \in L^1(0,T;L^1)$, $\log\theta \in L^2(0,T;L^2)$.

2. ρ, u satisfy the equation of continuity in the distribution sense, i.e.
   $$\int_0^t \{\langle \rho, \partial_\tau \Phi\rangle\} + \langle \rho u, \nabla\Phi\rangle\} d\tau = \langle \rho(t), \Phi(t)\rangle - \langle \rho_0, \Phi_0\rangle,$$
   for any test function $\Phi \in \mathcal{D}([0,T) \times \Omega)$ with $\Phi > 0$.

3. ρ, u satisfy the momentum equation in the distribution sense, i.e.
   $$\begin{aligned}\int_0^t \{\langle \rho u, \partial_\tau \phi\rangle + \langle \rho u \otimes u, \nabla\phi\rangle + \langle p(\rho,\theta), \operatorname{div}\phi\rangle\} d\tau - \int_0^t \langle \mathbb{S}, \nabla\phi\rangle d\tau \\ = \langle \rho u(t), \phi(t)\rangle - \langle \rho_0 u_0, \phi(0)\rangle,\end{aligned}$$
   for any test function $\phi \in \mathcal{D}([0,T) \times \Omega)^d$.

4. ρ, u, θ satisfy the internal equation in the distribution sense, i.e.
   $$\begin{aligned}&\int_0^t \{\langle \rho Q(\theta), \partial_\tau \psi\rangle + \langle \rho Q(\theta) u, \nabla\psi\rangle - \langle Q(\theta), \Delta\psi\rangle\} d\tau \\ &= \langle \rho(t) Q(\theta)(t), \psi(t)\rangle - \langle \rho_0 Q(\theta_0), \psi(0)\rangle - \int_0^t \{\langle (\mathbb{S}, \nabla u), \psi\rangle - \langle \theta \partial_\theta p \operatorname{div} u, \psi\rangle\} d\tau,\end{aligned}$$
   for any test function $\psi \in \mathcal{D}([0,T) \times \Omega)$.

5. ρ, ρu and $\rho Q(\theta)$ satisfy the initial conditions
   $$\begin{cases} \langle \rho(t,x), \eta(x)\rangle & \to \langle \rho_0, \eta(x)\rangle, \\ \langle \rho(t,x) u(t,x), \eta(x)\rangle & \to \langle \rho_0 u_0, \eta(x)\rangle, \qquad t \to 0, \\ \langle \rho(t,x) Q(\theta)(t,x), \eta(x)\rangle & \to \langle \rho_0 Q(\theta_0), \eta(x)\rangle, \end{cases}$$
   for any test function $\eta \in \mathcal{D}(\Omega)$.

In this paper, we study sufficient conditions of energy conservation law for (1.1).

Theorem 1.1. (A-Iwabuchi, [2]) *Let* $d = 2, 3$, $\Omega = \mathbb{T}^d$. *Suppose that* $T < \infty$, (ρ, u, θ) *is a weak solution of* (1.1) *on* $(0, T)$. *Assume that*

$$0 < c_1 \leq \rho \leq c_2 < \infty, \quad u \in L^\infty(0, T; L^2), \quad p_e(\,\cdot\,) \in C^1((0, \infty)) \tag{1.9}$$

for some constants c_1, c_2. *In the case when* $d = 3$, *we additionally assume that*

$$\begin{aligned} & u \in L^p(0, T; L^q), \quad \frac{1}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq 4, \\ \text{or} \quad & u \in L^p(0, T; L^q), \quad \frac{2}{p} + \frac{2}{q} = 1, \quad q \geq 4. \end{aligned} \tag{1.10}$$

Then the energy (1.2) *conserves, i.e.,* $E(t) = E(0)$ *for all* $0 < t < T$.

It is also possible to apply the condition (1.10) to the barotropic compressible Navier–Stokes equations (1.8).

Corollary 1.2. (A-Iwabuchi, [2]) *Let* $d = 2, 3$, $\Omega = \mathbb{T}^d$. *Suppose that* $T < \infty$, (ρ, u) *is a weak solution of* (1.8) *on* $(0, T)$, *and*

$$0 < c_1 \leq \rho \leq c_2 < \infty, \quad u \in L^\infty(0, T; L^2), \quad p(\,\cdot\,) \in C^1((0, \infty))$$

for some constants c_1, c_2. *In the case when* $d = 3$, *we additionally assume* (1.10). *Then the energy equality holds.*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^d} \rho(t,x) u^2(t,x) dx + \int_{\mathbb{T}^d} \rho(t,x) h(\rho(t,x)) dx \\ & + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u(t,x) + (\nabla u)^T(t,x)|^2 dx d\tau + \lambda \int_0^t \int_{\mathbb{T}^d} |\operatorname{div} u(t,x)|^2 dx d\tau \\ = & \frac{1}{2} \int_{\mathbb{T}^d} \rho_0(x) u_0^2(x) dx + \int_{\mathbb{T}^d} \rho_0(x) h(\rho_0(x)) dx \end{aligned}$$

for all $0 < t < T$, *where*

$$h(\rho) := \int_1^\rho \frac{p(z)}{z^2} dz.$$

Remarks.

1. ((1.10) in $d = 2$). In the case when $d = 2$, we have by the Gagliardo-Nirenberg's inequality and the Hölder's inequality that

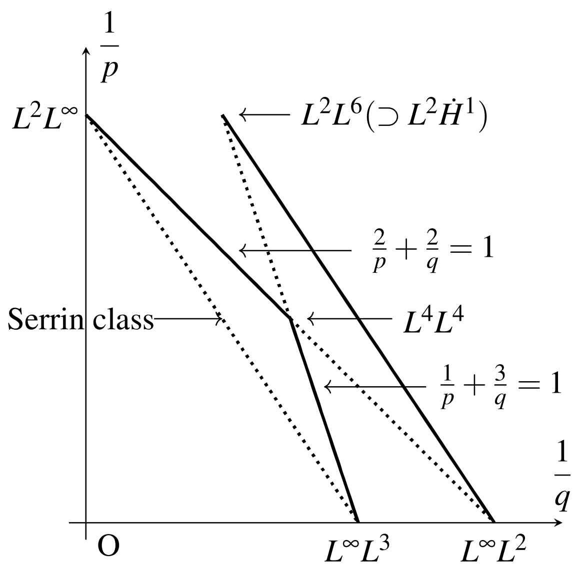
$$\|u\|_{L^p(0,T;L^q)} \leq C \|u\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{p}} \|u\|_{L^2(0,T;W^{1,2})}^{\frac{2}{p}},$$

when $2/p + 2/q = 1$ and $2 \leq q < \infty$. We do not need the additional assumption (1.10).

2. (Best choice of indices in (1.10) in $d=3$). In the case when $d=3$, the condition $u \in L^4(0,T;L^4)$ is the best in (1.10), and the other cases follow from the following inequalities of Gagliardo–Nirenberg type and Hölder interpolation.

$$\|u\|_{L^4(0,T;L^4)} \le \|u\|_{L^2(0,T;W^{1,2})}^{1-\frac{p}{2(p-2)}} \|u\|_{L^p(0,T;L^q)}^{\frac{p}{2(p-2)}}, \qquad \frac{1}{p}+\frac{3}{q}=1, \quad \frac{1}{4}\le\frac{1}{q}\le\frac{1}{3},$$
$$\|u\|_{L^4(0,T;L^4)} \le \|u\|_{L^\infty(0,T;L^2)}^{1-\frac{p}{4}} \|u\|_{L^p(0,T;L^q)}^{\frac{p}{4}}, \qquad \frac{2}{p}+\frac{2}{q}=1, \qquad \frac{1}{q}\le\frac{1}{4}.$$

The figure implies the relation of the sufficient conditions of (1.10) for $L^pL^q := L^p(0,T;L^q)$ spaces:



3. (Comparison with the known result of incompressible case). Compared with the previous studies on incompressible fluids, our result corresponds to that of Shinbrot [20].

4. (Comparison with Nguyen–Nguyen–Tang [16]). We have obtained the energy identity without the positive regularity assumption for the density and the pressure by Nguyen–Nguyen–Tang [16], where they suppose $\rho \in L^\infty(0,T;B^{\frac{1}{2}}_{2,\infty})$ when $d=2$ and $\rho \in L^\infty(0,T;B^{\frac{1}{2}}_{\frac{12}{5},\infty})$ when $d=3$. Moreover we only need $p \in C^1(0,\infty)$, while they impose $p \in C^2(0,\infty)$.

5. (Comparison with Akramov–Dębiec–Skipper–Wiedemann [1]). Akramov–Dębiec–Skipper–Wiedemann [1] showed the energy identity for the compressible Navier–Stokes equations such that $u \in B^{\alpha}_{3,\infty}((0,T)\times\mathbb{T}^3)\cap L^2(0,T;W^{1,2})$, $\rho, \rho u \in B^{\beta}_{3,\infty}((0,T)\times$

$\mathbb{T}^3$), $0 \le c_1 \le \rho \le c_2 < \infty$ with $\alpha + 2\beta > 1, 2\alpha + \beta > 1, 0 \le \alpha, \beta \le 1$ and $p \in C([c_1, c_2])$. Our theorem does not require the positive regularity assumption in the Besov space, while we assume $p \in C^1([c_1, c_2])$.

We finally mention the energy equality for compressible Navier–Stokes equations describing the motion of the ideal gas. Taking the pressure as $p(\rho, \theta) = \rho\theta$ and the internal energy $Q(\theta) = \theta$ in (1.1), we write the motion of the ideal gas as follows.

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, & t > 0,\ x \in \mathbb{T}^d, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla(\rho\theta) - \mathrm{div}\,\mathbb{S} = 0, & t > 0,\ x \in \mathbb{T}^d, \\ \partial_t(\rho\theta) + \mathrm{div}\,(\rho\theta u) - \Delta\theta = (\mathbb{S} : \nabla u) - \rho\theta \mathrm{div}\, u, & t > 0,\ x \in \mathbb{T}^d, \\ (\rho, u, \theta)\big|_{t=0} = (\rho_0, u_0, \theta_0), & x \in \mathbb{T}^d. \end{cases} \tag{1.11}$$

We can prove a similar result under the assumption for (1.11) as in Theorem 1.1;

Corollary 1.3. (A-Iwabuchi, [2]) *Let $d = 2, 3$ and $\Omega = \mathbb{T}^d$. Suppose that $T < \infty$, (ρ, u, θ) is a weak solution of* (1.11) *on* $(0, T)$. *Assume that*

$$0 < c_1 \le \rho \le c_2 < \infty, \quad u \in L^\infty(0, T; L^2),$$

for some constants c_1, c_2. In the case when $d = 3$ we additionally assume (1.10). *Then the energy equality holds, i.e.,*

$$\frac{1}{2}\int_{\mathbb{T}^d} \rho(t,x)u^2(t,x)dx + \int_{\mathbb{T}^d} \rho(t,x)\theta(t,x)dx = \frac{1}{2}\int_{\mathbb{T}^d} \rho_0(x)u_0^2(x)dx + \int_{\mathbb{T}^d} \rho_0(x)\theta_0(x)dx$$

for all $0 < t < T$.

We mention the existing results of the Cauchy problem for (1.11) and discuss the difference between the class of the Cauchy problem and the class of energy conservation law. There are many results of the Cauchy problem of (1.11) in the scaling critical space $(\rho_0 - 1, u_0, \theta_0) \in (\dot{B}^{\frac{d}{p}}_{p,1}, \dot{B}^{-1+\frac{d}{p}}_{p,1}, \dot{B}^{-2+\frac{d}{p}}_{p,1})$. Chikami–Danchin [6] discussed the unique solvability in the case when $1 \le p < d$, and Chen–Miao–Zhang [4] proved the ill-posedness result in the case when $p > d$. In the two-dimensional case, Iwabuchi and Ogawa [10] proved the ill-posedness for the initial data $(\rho_0, u_0, \theta_0) \in (\dot{B}^{\frac{2}{p}}_{p,q}, \dot{B}^{-1+\frac{2}{p}}_{p,q}, \dot{B}^{-2+\frac{2}{p}}_{p,q})$ with $1 \le p \le \infty$ and $1 \le q < \infty$. Recently, the author and Iwabuchi [3] proved the ill-posedness for the initial data $(\rho_0, u_0, \theta_0) \in (\dot{B}^1_{3,1}, \dot{B}^0_{3,1}, \dot{B}^{-1}_{3,1})$ in three-dimensional case. The relation between the class of the Cauchy problem and the class of energy conservation law seems conflicting. It is known that the following inclusion relations hold:

$$C([0,T); \dot{B}^1_{d,1}) \subset L^\infty(0,T; L^\infty),\ C([0,T); \dot{B}^0_{d,1}) \subset L^\infty(0,T; L^d), \quad d = 2, 3.$$

By Corollary 1.3, $\rho \in L^\infty(0,T;L^\infty)$, $u \in L^\infty(0,T;L^2)$, $(d=2)$, $u \in L^\infty(0,T;L^3)$, $(d=3)$ are sufficient conditions that the ideal gas satisfies the energy conservation law. On the other hand, Iwabuchi–Ogawa [10] and the author and Iwabuchi [3] proved the ill-posedness of the Cauchy problem for (1.11) in $\rho \in C([0,T);\dot{B}^1_{d,1})$ and $u \in C([0,T);\dot{B}^0_{d,1})$. There is indeed no relation between L^1 and $\dot{B}^{-1}_{d,1}$. However, except for this point, these results imply that the equations (1.11) have different aspects under similar conditions: a positive result that the equations (1.11) satisfy the energy conservation law and a negative result that the Cauchy problem of (1.11) is ill-posed.

2 Preliminaries

To prove Theorem 1.1, we introduce a mollifier in spatial variables and a lemma.

Definition. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be such that η is radially symmetric and

$$\operatorname{supp}\eta \subset B_1(0), \quad 0 \le \eta \le 1, \quad \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

For $\varepsilon > 0$ we set

$$\eta_\varepsilon(x) = \varepsilon^{-d}\eta(\varepsilon^{-1}x), \quad x \in \mathbb{T}^d,$$

and define $u_\varepsilon(x)$ by

$$u_\varepsilon(x) = (\eta_\varepsilon * u)(x) := \int_{\mathbb{T}^d} \eta_\varepsilon(x-y)u(y)dy, \quad x \in \mathbb{T}^d.$$

Lemma 2.1. ([16]) *Let $d \ge 2, 1 \le p,q \le \infty$.*

1. *There exists $C > 0$ such that for every $\varepsilon > 0$ and every $f \in L^p(0,T;L^q(\mathbb{T}^d))$*

$$\|f_\varepsilon\|_{L^p(0,T;L^\infty)} \le C\varepsilon^{-\frac{d}{q}}\|f\|_{L^p(0,T;L^q)},$$
$$\|\nabla f_\varepsilon\|_{L^p(0,T;L^\infty)} \le C\varepsilon^{-1-\frac{d}{q}}\|f\|_{L^p(0,T;L^q)}.$$

2. *There exists $C > 0$ such that for every $\varepsilon > 0$ and every $f \in L^p(0,T;L^q(\mathbb{T}^d))$*

$$\|\nabla f_\varepsilon\|_{L^p(0,T;L^q)} \le C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q)}.$$

Moreover, if $p,q < \infty$, then

$$\limsup_{\varepsilon\to 0} \varepsilon\|\nabla f_\varepsilon\|_{L^p(0,T;L^q)} = 0, \tag{2.12}$$

provided that $f \in L^p(0,T;L^q(\mathbb{T}^d))$.

3. *For every $g \in L^\infty(0,T;L^\infty)$ with $\inf g > 0$, there exists $C > 0$ such that for every $f \in L^p(0,T;L^q)$ and every $\varepsilon > 0$*

$$\left\| \nabla \frac{f_\varepsilon}{g_\varepsilon} \right\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1} \|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}.$$

Moreover, if $p, q < \infty$, then

$$\limsup_{\varepsilon \to 0} \varepsilon \left\| \nabla \frac{f_\varepsilon}{g_\varepsilon} \right\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0, \tag{2.13}$$

provided that $f \in L^p(0,T;L^q) \cap L^\infty(0,T;L^\infty)$, $g \in L^\infty(0,T;L^\infty)$ and $\inf\limits_{x \in \mathbb{T}^d} g > 0$.

Lemma 2.2. ([2]) *Let $d \geq 2$, $p, p_1, q, q_1 \in [1,\infty)$, $p_2, q_2 \in (1,\infty]$, $1/p = 1/p_1 + p_2, 1/q = 1/q_1 + 1/q_2$. Then there exists $C > 0$ such that for every $f \in L^{p_1}(0,T;W^{1,q_1})$ and $g \in L^{p_2}(0,T;L^{q_2})$*

$$\|(fg)_\varepsilon - f_\varepsilon g_\varepsilon\|_{L^p(0,T;L^q)} \leq C\varepsilon \|f\|_{L^{p_1}(0,T;W^{1,q_1})} \|g\|_{L^{p_2}(0,T;L^{q_2})}.$$

Moreover,

$$\limsup_{\varepsilon \to 0} \varepsilon^{-1} \|(fg)_\varepsilon - f_\varepsilon g_\varepsilon\|_{L^p(0,T;L^q)} = 0, \tag{2.14}$$

provided that $f \in L^{p_1}(0,T;W^{1,q_1})$ and $g \in L^{p_2}(0,T;L^{q_2})$.

3 Proof of Theorem 1.1

We can also show Corollary 1.2 in the same way as Theorem 1.1 and omit details. For simplicity, we may consider only the case when $u \in L^4(0,T;L^4)$ (see Remark below Corollary 1.2).

We first consider the continuity equation and the momentum equation. For $x \in \mathbb{T}^d$, let test function be the mollifier $\eta_\varepsilon(x - \cdot)$. Multiplying the test function by the continuity equation and the momentum equation and differentiating by t gives

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho u)_\varepsilon = 0, \\ \partial_t (\rho u)_\varepsilon + \operatorname{div}(\rho u \otimes u)_\varepsilon + \nabla p(\rho,\theta)_\varepsilon - \operatorname{div} \mathbb{S}_\varepsilon = 0, \end{cases} \tag{3.15}$$

for all $(t,x) \in (0,T) \times \mathbb{T}^d$. The multiplication by $\rho_\varepsilon^{-1}(\rho u)_\varepsilon$ of the second equation of (3.15) yields

$$\begin{aligned} 0 = \int_s^t \Bigg\{ & \left\langle \partial_\tau (\rho u)_\varepsilon, \frac{(\rho u)_\varepsilon}{\rho_\varepsilon} \right\rangle + \left\langle \operatorname{div}\,(\rho u \otimes u)_\varepsilon, \frac{(\rho u)_\varepsilon}{\rho_\varepsilon} \right\rangle \\ & + \left\langle \nabla p(\rho,\theta)_\varepsilon, \frac{(\rho u)_\varepsilon}{\rho_\varepsilon} \right\rangle - \left\langle \operatorname{div} \mathbb{S}_\varepsilon, \frac{(\rho u)_\varepsilon}{\rho_\varepsilon} \right\rangle \Bigg\} d\tau \\ =: & (A) + (B) + (C) + (D). \end{aligned} \tag{3.16}$$

We here extract the important terms by the lemma below.

Lemma 3.1. *The equality* (3.16) *is equivalent to*

$$\begin{aligned}&\frac{1}{2}\int_s^t\int_{\mathbb{T}^d}\partial_\tau\Big\{\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon}\Big\}dxd\tau+\int_s^t\int_{\mathbb{T}^d}\partial_\tau(\rho_\varepsilon h(\rho_\varepsilon))dxd\tau\\&-\int_s^t\langle\{p_{th}(\rho,\theta)\}_\varepsilon,\operatorname{div}u_\varepsilon\rangle d\tau+\int_s^t\langle\mathbb{S}_\varepsilon:\nabla u_\varepsilon\rangle d\tau+R_\varepsilon(t,s)=0,\end{aligned}\tag{3.17}$$

where h is defined by (1.3), *and the error term $R_\varepsilon(t,s)$ satisfies*

$$\lim_{\varepsilon\to0}\sup_{s\in(0,t)}|R_\varepsilon(t,s)|=0\text{ for all }t.$$

Proof. We write that

$$\begin{aligned}(A)=&\int_s^t\Big\langle\partial_\tau(\rho u)_\varepsilon,\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\rangle d\tau\\=&\frac{1}{2}\int_s^t\int_{\mathbb{T}^d}\partial_\tau\Big\{\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon}\Big\}dxd\tau+\frac{1}{2}\int_s^t\Big\langle\partial_\tau\rho_\varepsilon,\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon^2}\Big\rangle d\tau.\end{aligned}\tag{3.18}$$

By integration by parts and the mollified continuity equation,

$$\begin{aligned}(B)=&-\int_s^t\Big\langle(\rho u\otimes u)_\varepsilon,\nabla\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\rangle d\tau\\=&-\int_s^t\Big\langle(\rho u\otimes u)_\varepsilon-u_\varepsilon\otimes(\rho u)_\varepsilon,\nabla\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\rangle d\tau\\&+\frac{1}{2}\int_s^t\Big\langle\operatorname{div}\{(\rho_\varepsilon u_\varepsilon)-(\rho u)_\varepsilon\},\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon^2}\Big\rangle d\tau-\frac{1}{2}\int_s^t\Big\langle\partial_\tau\rho_\varepsilon,\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon^2}\Big\rangle d\tau\\=:&(B_1)+(B_2)-\frac{1}{2}\int_s^t\Big\langle\partial_\tau\rho_\varepsilon,\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon^2}\Big\rangle d\tau.\end{aligned}$$

We notice that the last term above line is canceled with the last term in (3.18). We prove that $(B_1),(B_2)$ tends to 0 as $\varepsilon\to0$. By Hölder's inequality,

$$\begin{aligned}|(B_1)|=&\Big|\int_s^t\Big\langle(\rho u\otimes u)_\varepsilon-u_\varepsilon\otimes(\rho u)_\varepsilon,\nabla\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\rangle d\tau\Big|\\\le&\Big\|\nabla\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\|_{L^4(0,T;L^4)}\|(\rho u\otimes u)_\varepsilon-u_\varepsilon\otimes(\rho u)_\varepsilon\|_{L^{\frac43}(0,T;L^{\frac43})}.\end{aligned}$$

By the assumption $u\in L^4(0,T;L^4)$, we can apply the inequality of Lemma 2.1 3 and the convergence in Lemma 2.2, and obtain

$$\begin{aligned}&\Big\|\nabla\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\Big\|_{L^4(0,T;L^4)}\le C\varepsilon^{-1}\|u\|_{L^4(0,T;L^4)},\\&\limsup_{\varepsilon\to0}\varepsilon^{-1}\|(\rho u\otimes u)_\varepsilon-u_\varepsilon\otimes(\rho u)_\varepsilon\|_{L^{\frac43}(0,T;L^{\frac43})}=0,\end{aligned}$$

which yields

$$\limsup_{\varepsilon\to 0}\sup_{s\in(0,t)}|(B_1)|=0.$$

We also show the convergence for (B_2). Therefore we extract the important part of $(A)+(B)$ with the error estimate that

$$\lim_{\varepsilon\to 0}\sup_{s\in(0,t)}\left|(A)+(B)-\frac{1}{2}\int_s^t\int_{\mathbb{T}^d}\partial_\tau\left\{\frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon}\right\}dxd\tau\right|=0.$$

We turn to consider (C) and write

$$(C)=\int_s^t\left\{\left\langle\nabla(p_e(\rho))_\varepsilon,\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\right\rangle+\left\langle\nabla(p_{th}(\rho,\theta))_\varepsilon,\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\right\rangle\right\}d\tau=(C_1)+(C_2).$$

We extract the second term and the third term of (3.17) from $(C_1),(C_2)$, respectively. We first estimate (C_1) and write

$$\begin{aligned}(C_1)&=\int_s^t\left\langle\nabla(p_e(\rho))_\varepsilon,\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\right\rangle d\tau\\&=\int_s^t\left\{\left\langle\nabla\Big\{(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\Big\},\frac{(\rho u)_\varepsilon}{\rho_\varepsilon}\right\rangle+\left\langle\frac{\nabla p_e(\rho_\varepsilon)}{\rho_\varepsilon},(\rho u)_\varepsilon\right\rangle\right\}d\tau\\&=(C_{1,1})+(C_{1,2}).\end{aligned}$$

Using the fundamental theorem of calculus, integration by parts and the mollification of the continuity equation, we can see that $(C_{1,2})$ becomes the second term of (3.17):

$$(C_{1,2})=\int_s^t\int_{\mathbb{T}^d}\partial_\tau(\rho_\varepsilon h(\rho_\varepsilon))\;dxd\tau.$$

We next show that $(C_{1,1})$ converges to 0. We approximate $(\rho u)_\varepsilon$ by $\rho_\varepsilon u_\varepsilon$ and apply integration by parts, and then have that

$$\begin{aligned}|(C_{1,1})|\le&\left|\int_s^t\left\langle\nabla\Big\{(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\Big\},\frac{(\rho u)_\varepsilon-\rho_\varepsilon u_\varepsilon}{\rho_\varepsilon}\right\rangle d\tau\right|\\&+\left|\int_s^t\left\langle\Big\{(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\Big\},\operatorname{div}u_\varepsilon\right\rangle d\tau\right|\\\le&\left\|\nabla\Big\{(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\Big\}\right\|_{L^\infty(0,T;L^\infty)}\left\|\frac{(\rho u)_\varepsilon-\rho_\varepsilon u_\varepsilon}{\rho_\varepsilon}\right\|_{L^1(0,T;L^1)}\\&+\|(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\|_{L^2(0,T;L^2)}\|\operatorname{div}u_\varepsilon\|_{L^2(0,T;L^2)}.\end{aligned}\tag{3.19}$$

In this paper, we especially deal with the second term of the right-hand side of (3.19). It suffices to show that

$$\|(p_e(\rho))_\varepsilon-p_e(\rho_\varepsilon)\|_{L^2(0,T;L^2)}\to 0.\tag{3.20}$$

By $c_1 \le \rho \le c_2$ from the assumption (1.9), the mean value theorem and η_ε having the unit mass, we have

$$|(p_e(\rho))_\varepsilon(\tau,x) - p_e(\rho_\varepsilon)(\tau,x)| \le 2 \sup_{c\in(c_1,c_2)} |p_e'(c)| \left| \int_{\mathbb{T}^d} \Big(\rho(\tau,x-y) - \rho(\tau,x)\Big)\eta_\varepsilon(y)dy \right|.$$

By $\rho \in L^\infty(0,T;L^\infty) \subset L^2(0,T;L^2)$ and the continuity of the translation in $L^2(0,T;L^2)$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \|(p_e(\rho))_\varepsilon - p_e(\rho_\varepsilon)\|_{L^2(0,T;L^2)} \\
\le& 2 \limsup_{\varepsilon\to 0} \sup_{c\in[c_1,c_2]} |p_e'(c)| \left(\int_0^T \sup_{y\in B_\varepsilon(0)} \|\rho(\tau,\cdot - y) - \rho(\tau,\cdot)\|_{L^2(\mathbb{T}^d)}^2 d\tau \right)^{\frac{1}{2}} = 0,
\end{aligned}$$

which proves (3.20). We also show the convergence of the first term of the right-hand side of (3.19), then we obtain

$$\limsup_{\varepsilon\to 0} \sup_{s\in(0,t)} |(C_{1,1})| = 0.$$

As for (C_2), we write

$$(C_2) = \int_s^t \left\{ \left\langle \nabla\{p_{th}(\rho,\theta)\}_\varepsilon, \frac{(\rho u)_\varepsilon - \rho_\varepsilon u_\varepsilon}{\rho_\varepsilon} \right\rangle + \left\langle \nabla\{p_{th}(\rho,\theta)\}_\varepsilon, u_\varepsilon \right\rangle \right\} d\tau.$$

The second term on the right-hand side is nothing but the third term of (3.17), and we show the first term above converges to 0 as $\varepsilon \to 0$. By Hölder's inequality, Lemma 2.1 (1), and the assumption of $\rho \in L^\infty(0,T;L^\infty)$, we get

$$\begin{aligned}
&\left| \int_s^t \left\langle \nabla\{p_{th}(\rho,\theta)\}_\varepsilon, \frac{(\rho u)_\varepsilon - \rho_\varepsilon u_\varepsilon}{\rho_\varepsilon} \right\rangle d\tau \right| \\
&\le \|\nabla\{p_{th}(\rho,\theta)\}_\varepsilon\|_{L^2(0,T;L^2)} \left\| \frac{(\rho u)_\varepsilon - \rho_\varepsilon u_\varepsilon}{\rho_\varepsilon} \right\|_{L^2(0,T;L^2)} \\
&\le C\varepsilon^{-1} \|p_{th}(\rho,\theta)\|_{L^2(0,T;L^2)} \|(\rho u)_\varepsilon - \rho_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2)}.
\end{aligned}$$

It follows from $\rho \in L^\infty(0,T;L^\infty)$, $u \in L^2(0,T;W^{1,2})$ and Lemma 2.2 that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} \|(\rho u)_\varepsilon - \rho_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2)} = 0.$$

Therefore, we conclude

$$\limsup_{\varepsilon\to 0} \sup_{s\in(0,t)} \left| (C) - \int_s^t \int_{\mathbb{T}^d} \Big(\partial_t(\rho_\varepsilon h(\rho_\varepsilon)) - \{p_{th}(\rho,\theta)\}_\varepsilon \operatorname{div} u_\varepsilon \Big)\, dx d\tau \right| = 0.$$

Finally, we consider (D). Since $\mathbb{S}_\varepsilon \in L^2(0,T;W^{1,2})$, we can prove the convergence of (D) in the same way of (C_2) and then conclude

$$\limsup_{\varepsilon\to 0} \sup_{s\in(0,t)} \left| (D) - \int_s^t \left\langle \mathbb{S}_\varepsilon : \nabla u_\varepsilon \right\rangle \Big\} d\tau \right| = 0.$$

Therefore we complete the proof of (3.17) with the error estimate. ❑

We prove Theorem 1.1 with the help of Lemma 3.1. We consider the thermal equation by choosing a test function as a constant function. It follows from the definition of the weak solution and $\partial_t 1 = \partial_x 1 = 0$ that

$$\int_{\mathbb{T}^d} \rho Q(\theta)(t,x)dx = \int_{\mathbb{T}^d} \rho Q(\theta)(s,x)dx + \int_s^t \Big\{ \langle \mathbb{S} : \nabla u \rangle - \langle p_{th}(\rho,\theta)\, , \operatorname{div} u \rangle \Big\} d\tau. \quad (3.21)$$

By adding (3.17) to (3.21), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^d} \Big\{ \frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon} \Big\}(t,x)dx + \int_{\mathbb{T}^d} \rho_\varepsilon h(\rho_\varepsilon)(t,x)dx + \int_{\mathbb{T}^d} \rho Q(\theta)(t,x)dx \\
=&\frac{1}{2}\int_{\mathbb{T}^d} \Big\{ \frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon} \Big\}(s,x)dx + \int_{\mathbb{T}^d} \rho_\varepsilon h(\rho_\varepsilon)(s,x)dx + \int_{\mathbb{T}^d} \rho Q(\theta)(s,x)dx \\
&+ \int_s^t \langle \{p_{th}(\rho,\theta)\}_\varepsilon, \operatorname{div} u_\varepsilon \rangle d\tau - \int_s^t \langle p_{th}(\rho,\theta), \operatorname{div} u \rangle d\tau \\
&- \int_s^t \langle \mathbb{S}_\varepsilon : \nabla u_\varepsilon \rangle d\tau + \int_s^t \langle \mathbb{S} : \nabla u \rangle d\tau - R_\varepsilon(t,s).
\end{aligned} \quad (3.22)$$

We start by taking the limit as $s \to 0$ for each $\varepsilon > 0$. The weak continuity of ρu gives the pointwise convergence of $(\rho u)_\varepsilon$ to ρu, and the Lebesgue dominated convergence theorem implies that

$$\frac{1}{2}\int_{\mathbb{T}^d} \frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon}(s,x)dx \to \frac{1}{2}\int_{\mathbb{T}^d} \frac{(\rho_0 u_0)_\varepsilon^2}{(\rho_0)_\varepsilon}(x)dx, \quad \text{as } s \to 0 \text{ for each } \varepsilon > 0.$$

Similarly, we also have from the weak continuity of ρ that

$$\int_{\mathbb{T}^d} \rho_\varepsilon h(\rho_\varepsilon)(s,x)dx \to \int_{\mathbb{T}^d} (\rho_0)_\varepsilon h((\rho_0)_\varepsilon)(x)dx, \quad \text{as } s \to 0 \text{ for each } \varepsilon > 0.$$

The convergence of the third term $\rho Q(\theta)(s,x)$ as $s \to 0$ follows from the weak continuity of $\rho Q(\theta)$ due to the definition of the weak solutions. As for the integrals, the well-definedness is assured by the definition of the weak solution and it is possible to take the limit as $s \to 0$ due to the integrability, and we will apply Lemma 3.1 to the error term $R_\varepsilon(t,s)$. Finally, we take the limit as $\varepsilon \to 0$. The integrability of $\rho u^2, \rho h(\rho), p_{th}(\rho,\theta), \operatorname{div} u, \mathbb{S}, \nabla u$

and an elemental property of the mollifier imply that

$$\frac{1}{2}\int_{\mathbb{T}^d} \frac{(\rho u)_\varepsilon^2}{\rho_\varepsilon}(\tau, x)dx \to \frac{1}{2}\int_{\mathbb{T}^d} \rho u^2(\tau, x)dx, \qquad \text{for } \tau = 0, t,$$
$$\int_{\mathbb{T}^d} \rho_\varepsilon h(\rho_\varepsilon)(\tau, x)dx \to \int_{\mathbb{T}^d} \rho h(\rho)(\tau, x)dx, \qquad \text{for } \tau = 0, t,$$
$$\int_0^t \langle \{p_{th}(\rho, \theta)\}_\varepsilon, \operatorname{div} u_\varepsilon\rangle d\tau - \int_0^t \langle (p_{th}(\rho, \theta), \operatorname{div} u)\rangle d\tau \to 0,$$
$$\int_0^t \langle \mathbb{S}_\varepsilon : \nabla u_\varepsilon\rangle d\tau - \int_0^t \langle \mathbb{S} : \nabla u\rangle d\tau \to 0,$$

as $\varepsilon \to 0$, which proves the energy equality in Theorem 1.1. ❑

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Motofumi Aoki

Mathematical Institute

Tohoku University

Sendai 980-8578

JAPAN

E-mail address: `motofumi.aoki.c2@tohoku.ac.jp`