ON THE WARING–GOLDBACH PROBLEM: A SURVEY.

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1. Introduction.

When we consider additive representations by some prescribed form, like the sum of seven cubes, we may be primarily interested in determining completely the numbers that can be represented by the form. To this end, we may be theoretically satisfied, if we could show that every sufficiently large number can be represented by the prescribed form. Thus, on Waring's problem, evaluation of $G(k)$ is a subject of absorbing interest, where $G(k)$ denotes the least s such that every sufficiently large number may be written as the sum of s kth powers, for integers $k \geq 2$.

On the Waring–Goldbach problem, we investigate sums of powers of prime numbers, and to introduce the quantity corresponding to $G(k)$ in Waring's problem, we are naturally led to taking some congruence conditions into account.

Hereafter, the letter p, with or without a subscript, denote prime numbers throughout. For a natural number k and a prime p, define $\theta(k,p)$ so that $p^{\theta(k,p)}$ is the highest power of p dividing k, and put $\gamma(k,p) = \theta(k,p) + (p,k,2)$. In other words, we set $\gamma(k,p) = \theta(k,p) + 2$ if k is even and $p = 2$, and $\gamma(k,p) = \theta(k,p) + 1$ otherwise. One may elementarily confirm that whenever $(p-1)|k$ and $p \nmid a$ one has $a^k \equiv 1 \pmod{p^{\gamma(k,p)}}$, and also that $\gamma(k,p)$ is the largest number with the latter property, for each pair of k and p. Accordingly, on putting

$$K(k) = \prod_{(p-1)|k} p^{\gamma(k,p)}, \tag{1}$$

we notice that $a^k \equiv 1 \pmod{K(k)}$ for all integers a coprime to $K(k)$. So if $n = p_1^k + p_2^k + \cdots + p_s^k$ with primes p_j not dividing $K(k)$, then one necessarily has $n \equiv s \pmod{K(k)}$.

Conversely, suppose that $n \not\equiv s \pmod{K(k)}$. Then there is a prime ϖ such that $\varpi | K(k)$ and $n \not\equiv s \pmod{\varpi^{\gamma(k,\varpi)}}$. If this n would be writtens as $n = p_1^k + \cdots + p_s^k$, then at least one of the primes p_j must be ϖ, because $p^k \equiv 1 \pmod{\varpi^{\gamma(k,\varpi)}}$ for all primes $p \neq \varpi$. So this n is the sum of s kth powers of primes, if, and only if, $n - \varpi^k$ is the sum of $(s-1)$ kth powers of primes. Namely, if we intend to represent a natural number, not congruent to s modulo $K(k)$, as the sum of s kth powers of primes, then at least one of the primes utilized in the representation is restricted to one of the prime factors of $K(k)$. Therefore, when we consider representations of n

as the sum of s kth powers of primes, the congruence condition $n \equiv s \pmod{K(k)}$ is necessary, to assure that the problem involves genuinely s prime variables.

Being base on the above observation, Hua [7] defined $H(k)$ to be the least s such that every sufficiently large n satisfying $n \equiv s \pmod{K(k)}$ may be written as the sum of s kth powers of primes. This $H(k)$ may be regarded as a correspondent of $G(k)$ in Waring's problem, and evaluation of $H(k)$ is the central topic on the Waring–Goldbach problem. This article is a short survey of research on $H(k)$.

Before proceeding to the main theme, we make some remarks on $K(k)$ defined at (1). First, when k is odd, the condtion $(p-1)|k$ is fullfilled only with $p = 2$, and $\theta(k, 2) = 0$ and $\gamma(k, 2) = 1$, so we see swiftly $K(k) = 2$ for all odd k.

When k is even, on the other hand, we have $(p-1)|k$ for $p = 2$ and 3 at least, and $\gamma(k, 2) = \theta(k, 2)+2 \geq 3$ and $\gamma(k, 3) = \theta(k, 3)+1 \geq 1$, whence $K(k)$ is divisible by $2^3 \cdot 3^1 = 24$. And if p is a prime greater than 3, and $2p+1$ is not a prime, then we have $K(2p) = 24$ by the definition (1). Since all the primes $p \equiv 1 \pmod 3$, for example, satisfy this assumption, we find that $K(k) = 24$ for infinitely many even k. The following table presents the values of $K(k)$ for even k up to 22.

k	2	4	6	8	10	12	14	16	18	20	22
$K(k)$	24	240	504	480	264	65520	24	16320	28728	13200	552

When the author gave a talk in this conference at RIMS, he presented the values of $K(k)$ only for $k = 2$, 4 and 6, and right after the talk was finished, Professor Toshiki Matsusaka pointed out coincidences of $K(k)$ and a constant appearing in the q-expansion of Eisenstein series $E_k(\tau)$, for $k = 2$, 4 and 6. This indication made us aware of the following relation between $K(k)$ and Bernoulli numbers B_k for even k.

A nonzero rational number r is written uniquely as $r = a/b$ by coprime integers a and b with $a > 0$, and let us call here the a and b, respectively, the numerator and the denominator of r. Then a couple of known results on Bernoulli numbers imply that $K(k)$ is the numerator of $2k/B_k$ for each even k. In fact, for even $k \geq 2$, Clausen–von Staudt's theorem shows that the denominator of $|B_k|$ is $\prod_{(p-1)|k} p$, while we know that any prime p with $(p-1) \nmid k$ does not divide the denominator of B_k/k, thanks to Kummer. So we find that the numerator of $2k/B_k$ is

$$2 \cdot \prod_{(p-1)|k} p^{\theta(k,p)} \cdot \prod_{(p-1)|k} p \;=\; \prod_{(p-1)|k} p^{\gamma(k,p)} \;=\; K(k).$$

In particular, since the $2k/B_k$ are integers for $k = 2$, 4, 6, 8, 10 and 14, we have $K(k) = 2k/|B_k|$ for these k.

2. History in the twentieth century.

In 1920's, Hardy and Littlewood applied the circle method and made legendary contribution to additive theory of numbers in their series of "partitio numerorum" papers (see Bibliography of [16]). Before the work of Hardy and Littlewood, it must be difficult even to imagine a plan of attack on additive problems involving primes. In order to establish conclusions on such problems by the Hardy–Littlewood method, however, it had been required to assume some unproved hypothesis on zeros of Dirichlet L-functions until 1937.

In 1937, Vinogradov [19] proved unconditionally that every sufficiently large odd number can be written as the sum of three primes. This celebrated result of Vinogradov on the ternary Goldbach problem may be represented as $H(1) \leq 3$. The point of Vinogradov's work [19] is a method of estimating exponential sums over primes, and the method made a great impact on research of additive problems involving primes. In 1938, the next year of publication of Vinogradov's paper [19], Hua [4, 6] presented various results in additive prime number theory, including $H(2) \leq 5$ and

$$H(k) \leq 2\left[\frac{\log(b/2) + \log(1 - 2/k)}{\log k - \log(k-1)}\right] + 2k + 7, \tag{2}$$

where $b = 2^{k-1}$ for $4 \leq k < 15$ and $b = k^3(\log k + 1.25 \log\log k^2)$ for $k \geq 15$, and $[z]$ denotes the largest integer $\leq z$. The latter result implies that

$$H(k) \leq 6k \log k + 2k \log\log k + O(k).$$

In the same volume of the journal as [4], Hua [5] established the mean value estimate that is well-known as Hua's lemma (see Theorem 4 of Hua [7], or Lemma 2.5 of Vaughan [16]), so he was quite likely able to show the bound

$$H(k) \leq 2^k + 1 \quad \text{for } k \geq 1, \tag{3}$$

in 1938 or thereabouts. The proof of (3) is contained in Hua's book [7], and this result (3) is still the best at present for $k \leq 3$. According to the footnote of "Preface originally intended for the Russian edition" of [7], the manuscript of the original Russian edition of [7] was sent to the editorial department of special publications of the Institute of Mathematics in 1941, but the publication was delayed as a result of the war. The original Russian edition of [7] was finally published in 1947.

In this book [7], Hua showed the bound (2) with $b = 2k^2(2\log k + \log\log k + 3)$ for $k > 12$ (see Theorem 14 in Chapter IX of [7]), which implies that

$$H(k) \leq 4k \log k + 2k \log\log k + O(k). \tag{4}$$

Chapter IX of [7] also includes expositions of Davenport's method which was developed around 1940, and at the end of the chapter, Hua mentioned that Davenport's method can provide the bounds "$H(4) \leq 15$, $H(5) \leq 25$ and so forth". In fact, one may obtain the bounds

$$H(4) \leq 15, \quad H(5) \leq 25, \quad H(6) \leq 37, \quad H(7) \leq 55, \quad H(8) \leq 75, \tag{5}$$

by Davenport's method (see Ch. 6 of Vaughan [16], Theorems 6.8 and 6.9, and Exercise 1, together with the expotision in the next section on a relation between upper bounds of $G(k)$ and $H(k)$). Presumably scholars in this research area had recognized all the results above by early 1940's.

In mid-1980's, Vaughan [15] and Thanigasalam [13, 14] independently refined Davenport's method, and in particular Thanigasalam [14] established the bounds

$$H(5) \leq 23,\ H(6) \leq 33,\ H(7) \leq 47,\ H(8) \leq 63,\ H(9) \leq 83,\ H(10) \leq 103. \tag{6}$$

And no further improvement on $H(k)$ appeared in the twentieth century.

3. A traditional phenomenon

In this section, we observe a traditional relation between the results on $H(k)$ mentioned in the preceding section and research on $G(k)$ in Waring's problem.

Let n be a large natural number, and set $2P = n^{1/k}$. Write $e(\alpha) = e^{2\pi i\alpha}$, and for a natural number k and an interval $I \subset [1, 2P]$, define the exponential sums

$$f_{k,I}(\alpha) = \sum_{x\in I} e(x^k\alpha) \qquad \text{and} \qquad g_{k,I}(\alpha) = \sum_{p\in I} e(p^k\alpha). \tag{7}$$

Especially we write $f_k(\alpha) = f_{k,(P,2P]}(\alpha)$ and $g_k(\alpha) = g_{k,(P,2P]}(\alpha)$.

Until mid-1980's, upper bounds of $G(k)$ were quite often shown by evaluating integrals of the form

$$\int_0^1 f_k(\alpha)^t \big(f_{k,I_1}(\alpha)\cdots f_{k,I_u}(\alpha)\big)^2 e(-n\alpha)\, d\alpha, \tag{8}$$

with suitable intervals I_j. Since $\displaystyle\int_0^1 e(m\alpha)\, d\alpha = 0$ for nonzero integers m, and the integral is 1 for $m = 0$, we see that the integral at (8) is equal to the number of representaions of n in the form

$$n = x_1^k + \cdots + x_t^k + y_1^k + \cdots + y_{2u}^k,$$

with $P < x_j \leq 2P$ $(1 \leq j \leq t)$ and $y_j,\ y_{u+j} \in I_j$ $(1 \leq j \leq u)$. So if we could show that the integral (8) is positive for all large n, then we would obtain the bound $G(k) \leq t + 2u$. Once we could make it, we may usually establish an asymptotic formula for the integral $\displaystyle\int_0^1 |f_k(\alpha)|^{2t'} \big|f_{k,I_1}(\alpha)\cdots f_{k,I_u}(\alpha)\big|^2\, d\alpha$ as well, provided that $2t' \geq t$. Thus, for $t' = [(t+1)/2]$ in particular, we would have the best possible upper bound of the later integral, which takes the shape

$$\int_0^1 |f_k(\alpha)|^{2t'} \big|f_{k,I_1}(\alpha)\cdots f_{k,I_u}(\alpha)\big|^2\, d\alpha \ll P^{2t'-k}X, \tag{9}$$

where $X = \big(f_{k,I_1}(0) \cdots f_{k,I_u}(0)\big)^2$. Since $|f_{k,I_j}(\alpha)|^2 = f_{k,I_j}(\alpha) f_{k,I_j}(-\alpha)$, the integral on the left hand side of (9) is equal to the number of x_j and y_j satisfying

$$\begin{gathered} x_1^k + \cdots + x_{t'}^k + y_1^k + \cdots + y_u^k = x_{t'+1}^k + \cdots + x_{2t'}^k + y_{u+1}^k + \cdots + y_{2u}^k, \\ P < x_j \le 2P \ \ (1 \le j \le 2t'), \quad y_j,\ y_{u+j} \in I_j \ \ (1 \le j \le u), \end{gathered} \tag{10}$$

by orthogonality. Then trivially we may replace all the "f"s appearing in (9) by "g"s, namely we have

$$\int_0^1 |g_k(\alpha)|^{2t'} \big|g_{k,I_1}(\alpha) \cdots g_{k,I_u}(\alpha)\big|^2 \, d\alpha \ll P^{2t'-k} X, \tag{11}$$

because the integral on the left hand side of (11) is the number of x_j and y_j satisfying (10) with the additional constraint that all the x_j and y_j are primes. We next see that by virtue of (11) we can successfully evaluate the integral

$$\int_0^1 g_k(\alpha)^{2t'+1} \big(g_{k,I_1}(\alpha) \cdots g_{k,I_u}(\alpha)\big)^2 e(-n\alpha) \, d\alpha. \tag{12}$$

Let $\mathfrak{M}$ be the union of the sets $\{\alpha \in [0,1) \,:\, |q\alpha - a| \le (\log P)^A P^{-k}\}$ for all the integers q and a satisfying $1 \le q \le (\log P)^A$, $0 \le a \le q$ and $(q,a) = 1$, where A is a positive constant determined in terms of k, t and u. And put $\mathfrak{m} = [0,1) \backslash \mathfrak{M}$. Then, on one hand we may routinely compute the contribution of $\mathfrak{M}$ to the integral (12) by the Siegel–Walfisz theorem, and obtain an estimate of the form

$$\int_{\mathfrak{M}} g_k(\alpha)^{2t'+1} \big(g_{k,I_1}(\alpha) \cdots g_{k,I_u}(\alpha)\big)^2 e(-n\alpha) \, d\alpha \gg P^{2t'+1-k} X (\log P)^{-2t'-2u-1},$$

provided that n is sufficiently large and $n \equiv 2t' + 1 + 2u \pmod{K(k)}$. On the way of this computation, we meet an object called the singular series, and the latter congruence condition is required to assure that the singular series is positive.

On the other hand, after Vinogradov, we have $\sup_{\alpha \in \mathfrak{m}} |g_k(\alpha)| \ll P(\log P)^{-2t'-2u-2}$, by assigning a suitably large value to A, so we see that

$$\begin{aligned} &\int_{\mathfrak{m}} g_k(\alpha)^{2t'+1} \big(g_{k,I_1}(\alpha) \cdots g_{k,I_u}(\alpha)\big)^2 e(-n\alpha) \, d\alpha \\ &\qquad \ll \sup_{\alpha \in \mathfrak{m}} |g_k(\alpha)| \int_0^1 |g_k(\alpha)|^{2t'} \big|g_{k,I_1}(\alpha) \cdots g_{k,I_u}(\alpha)\big|^2 \, d\alpha \\ &\qquad \ll P(\log P)^{-2t'-2u-2} \cdot P^{2t'-k} X \ll P^{2t'+1-k} X (\log P)^{-2t'-2u-2}. \end{aligned} \tag{13}$$

Therefore, by comparing the contributions of $\mathfrak{M}$ and $\mathfrak{m}$, we find that the integral (12) is positive, whenever n is large and congruent to $2t' + 1 + 2u$ modulo $K(k)$. Since the integral (12) expresses the number of representaions of n in the form

$$n = p_1^k + \cdots + p_{2t'+1}^k + \varpi_1^k + \cdots + \varpi_{2u}^k,$$

with primes p_j and ϖ_j satisfying $P < p_j \le 2P$ $(1 \le j \le 2t'+1)$ and $\varpi_j, \varpi_{u+j} \in I_j$ $(1 \le j \le u)$, we conclude that $H(k) \le 2t' + 1 + 2u = 2[(t+1)/2] + 1 + 2u$.

The above observation describes that if the bound $G(k) \le s$ could be proved via evaluation of an integral of the shape (8) (which means $t + 2u = s$), then one has the bound $H(k) \le s'$, where s' is the least odd integer exceeding s, or $s' = 2[(s+1)/2] + 1$. As a matter of fact, all the bounds (2), (5) and (6) were derived by this principle, and the bound (3) also was derived quite similarly from Hua's lemma.

Vinogradov showed upper bounds of $G(k)$ that were best at the time for large k in the era from 1935 to 1959, but instead of $f_k(\alpha)^t$ in (8), he utilized exponential sums of the form

$$\sum_{p, x_1, \dots, x_t} e(p^k(x_1^k + \cdots + x_t^k)\alpha),$$

with suitable conditions on p and x_j. The use of such an exponential sum means that we can obtain no meaningful information on the number of prime solutions of an equation like (10) (see the trivial relation between (9) and (11)). Consequently, from a bound of $G(k)$ shown by using such a technique, we may extract no information on $H(k)$. For example, Vinogradov showed in 1947 that $\limsup_{k\to\infty} \dfrac{G(k)}{k \log k} \le 3$ (see [16], Ch. 5, and Ch. 12 also), but as regards $H(k)$, still now we cannot remove the equality sign even from the estimate $\limsup_{k\to\infty} \dfrac{H(k)}{k \log k} \le 4$ that follows form (4).

4. History after 2000.

In the beginning of the current century, Kawada and Wooley [8] showed

$$H(4) \le 14 \qquad \text{and} \qquad H(5) \le 21. \tag{14}$$

The former result is the first instance that an even number becomes the best known upper bound of $H(k)$. One of the key ingredients of this paper [8] is treatment of Weyl sums over primes on minor arcs.

Now recall the exponential sums $f_k(\alpha)$ and $g_k(\alpha)$ defined in the line following (7). For a real number α, let q and a be integers satisfying

$$1 \le q \le P^{k/2}, \quad (q, a) = 1 \quad \text{and} \quad |q\alpha - a| \le P^{-k/2}.$$

Dirichlet's theorem ([16], Lemma 2.1) assures the existence of such q and a. Then, as for the classical Weyl sum $f_k(\alpha)$, we know the bound of the form

$$f_k(\alpha) \ll P^{1-\sigma(k)+\varepsilon} + \frac{w_k(q)P}{1 + P^k|\alpha - a/q|}, \tag{15}$$

for any fixed $\varepsilon > 0$. Here we omit the definition of $w_k(q)$, and mention instead that $w_k(q)$ is a multiplicative function of q, and satisfies $q^{-1/2} \le w_k(q) \ll q^{-1/k}$. Weyl's inequality ([16], Lemma 2.4) yields (15) with $\sigma(k) = 2^{1-k}$ for $k \ge 2$ indeed.

From the bound (15), Kawada and Wooley [8] derived the bound

$$g_k(\alpha) \ll P^{1-c\cdot\sigma(k)+\varepsilon} + \frac{w_k(q)^{1/2}P(\log P)^4}{(1+P^k|\alpha - a/q|)^{1/2}}, \quad \text{with } c = \frac{1}{4}, \tag{16}$$

provided that $\sigma(k)$ is not too large, and it is the case for $k \geq 4$ in reality.

This bound (16) was exploited in [8] to estimate integrals of the shape

$$\int_{\mathfrak{m}} g_k(\alpha)^v S(\alpha)\, d\alpha,$$

where v is a natural number, and $S(\alpha)$ is essentially a product of certain Weyl sums over primes. In practical applications, we take v to be either 1 or 2. The integral on the leftmost side of (13) may be expressed as the above shape for example, and it is vital to estimate such integrals to derive bounds of $H(k)$.

Traditional approach described at (13) is based on the inequality

$$\int_{\mathfrak{m}} g_k(\alpha)^v S(\alpha)\, d\alpha \;\leq\; \sup_{\alpha\in\mathfrak{m}} |g_k(\alpha)|^v \int_0^1 |S(\alpha)|\, d\alpha, \tag{17}$$

with $v = 1$. In this context, Kawada and Wooley [8] revealed that the last term on the right hand side of the inequality at (16) is harmless in actural applications. Namely, making use of (16), they proved inequalities of the form

$$\int_{\mathfrak{m}} g_k(\alpha)^v S(\alpha)\, d\alpha \;\ll\; \left(P^{1-c\cdot\sigma(k)+\varepsilon}\right)^v \int_0^1 |S(\alpha)|\, d\alpha \;\; + \text{“admissible error”}. \tag{18}$$

The last estimate (18) means, in a sense, that $\sup_{\alpha\in\mathfrak{m}} |g_k(\alpha)|$ can be replaced by $P^{1-c\cdot\sigma(k)+\varepsilon}$ within the traditional approach (17), although the order of magnitude of $\sup_{\alpha\in\mathfrak{m}} |g_k(\alpha)|$ should be as large as $P(\log P)^{-A/2}$ in truth. The bounds at (14) were obtained because of this advantage.

Shortly after [8], Harman [2] proved the inequality (16) with $c = 1/3$ for $k \geq 5$. Kumchev [10] also established (16) with $c = 1/3$ for $k \geq 4$, and he further replaced the function $w_k(q)$ by q^{-1} within (16). Taking advantage of the refined value $c = 1/3$, Kumchev [9] showed the bound $H(7) \leq 46$.

In 2014, Lilu Zhao [20] came up with an ingenious idea in this area. He wrote he was inspired by the work of Heath-Brown and Tolev [3] (see Zhao [20], the proof of Lemma 3.1). In the case $v = 1$, Zhao's method starts with observing that

$$\begin{aligned}\left|\int_{\mathfrak{m}} g_k(\alpha)S(\alpha)\, d\alpha\right| &= \left|\sum_{P<p\leq 2P} \int_{\mathfrak{m}} e(p^k\alpha)S(\alpha)\, d\alpha\right| \\ &\leq \sum_{P<p\leq 2P}\left|\int_{\mathfrak{m}} e(p^k\alpha)S(\alpha)\, d\alpha\right| \leq \sum_{P<x\leq 2P}\left|\int_{\mathfrak{m}} e(x^k\alpha)S(\alpha)\, d\alpha\right|,\end{aligned}$$

and then by applying Cauchy's inequality,

$$\ll \Big(\sum_{P<x\le 2P} 1 \Big)^{1/2} \Big(\sum_{P<x\le 2P} \Big| \int_{\mathfrak{m}} e(x^k\alpha) S(\alpha)\, d\alpha \Big|^2 \Big)^{1/2}$$

$$\ll P^{1/2} \Big(\sum_{P<x\le 2P} \int_{\mathfrak{m}} \int_{\mathfrak{m}} e(x^k(\alpha-\beta)) S(\alpha)\overline{S(\beta)}\, d\alpha d\beta \Big)^{1/2}$$

$$= P^{1/2} \Big(\int_{\mathfrak{m}} \int_{\mathfrak{m}} f_k(\alpha-\beta)\, S(\alpha)\overline{S(\beta)}\, d\alpha d\beta \Big)^{1/2}, \tag{19}$$

where $\overline{S(\beta)}$ denotes the complex conjugate of $S(\beta)$. Next we apply the estimate (15) to $f_k(\alpha-\beta)$. Then, in the situation where the second term on the right hand side of (15) is harmless ultimately as in (18), we have

$$\int_{\mathfrak{m}} \int_{\mathfrak{m}} f_k(\alpha-\beta)\, S(\alpha)\overline{S(\beta)}\, d\alpha d\beta \ll P^{1-\sigma(k)+\varepsilon} \int_{\mathfrak{m}} \int_{\mathfrak{m}} \big| S(\alpha) S(\beta) \big| d\alpha d\beta \ + \text{“a. e.”}$$

$$\ll P^{1-\sigma(k)+\varepsilon} \Big(\int_{\mathfrak{m}} \big| S(\alpha) \big|\, d\alpha \Big)^2 \ + \text{“a. e.”},$$

where "a. e." stands for "admissible error." Thus we deduce from (19) that

$$\int_{\mathfrak{m}} g_k(\alpha) S(\alpha)\, d\alpha \ll P^{1/2} \Big(P^{1-\sigma(k)+\varepsilon} \Big(\int_{\mathfrak{m}} \big| S(\alpha) \big|\, d\alpha \Big)^2 \ + \text{“a. e.”} \Big)^{1/2}$$

$$\ll P^{1-\sigma(k)/2+\varepsilon} \int_{\mathfrak{m}} \big| S(\alpha) \big|\, d\alpha \ + \text{“a. e.”},$$

whence we have (18) with $c = 1/2$, when $v = 1$.

When $v \ge 2$, the above argument for $v = 1$ yields the estimate

$$\int_{\mathfrak{m}} |g_k(\alpha)|^v |S(\alpha)|\, d\alpha = \int_{\mathfrak{m}} g_k(\alpha) g_k(-\alpha) |g_k(\alpha)|^{v-2} |S(\alpha)|\, d\alpha$$

$$\ll P^{1-\sigma(k)/2+\varepsilon} \int_{\mathfrak{m}} |g_k(\alpha)|^{v-1} \big| S(\alpha) \big|\, d\alpha \ + \text{“a. e.”}, \tag{20}$$

and by Hölder's inequlity, we have

$$\int_{\mathfrak{m}} |g_k(\alpha)|^{v-1} \big| S(\alpha) \big|\, d\alpha \ll \Big(\int_{\mathfrak{m}} |g_k(\alpha)|^v \big| S(\alpha) \big|\, d\alpha \Big)^{1-1/v} \Big(\int_{\mathfrak{m}} \big| S(\alpha) \big|\, d\alpha \Big)^{1/v}.$$

Therefore we deduce from (20) that

$$\int_{\mathfrak{m}} |g_k(\alpha)|^v |S(\alpha)|\, d\alpha \ll \big(P^{1-\sigma(k)/2+\varepsilon} \big)^v \int_{\mathfrak{m}} \big| S(\alpha) \big|\, d\alpha \ + \text{“a. e.”}.$$

Hence we have (18) with $c = 1/2$ for all natural numbers v. By this superiority, Zhao [20] obtained the bounds $H(4) \le 13$ and $H(6) \le 32$.

As is obvious, in the above argument of Zhao, we may replace $g_k(\alpha)$ by the exponential sum $\sum_{x\in\mathcal{A}} e(x^k\alpha)$ with any subset $\mathcal{A}$ of integers in $(P, 2P]$, provided that the second term on the right hand side of (15) is really harmless.

Two years before the work of [20], Wooley [17] had made astonishing progress of Vinogradov's mean value theorem, by proving the estimate

$$(21) \qquad \int_{[0,1)^k} \Bigg| \sum_{1\le x\le P} e(\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k) \Bigg|^{2s} d\alpha_1 \ldots d\alpha_k \ll P^{2s-k(k+1)/2+\varepsilon},$$

for $s \ge k(k+1)$. Before this work [17], this estimate (21) had been known for s substantially larger than $k^2 \log k$, so Wooley [17] reduced the restriction on s by a factor of $\log k$, roughly. As a straightforward consequence of this progress, one may obtain Hua's bound (2) with $b = 2k(k+1)$, which means that one can erase the term $2k \log\log k$ from the right hand side of (4). This immediate effect is concerned with the value of $\sigma(k)$ at (15). Wooley [17, Theorem 1.5] brought (15) with $\sigma(k) = 1/(2k(k-1))$, relaxing the previous restrection $\sigma(k) \ll (k^2 \log k)^{-1}$ for larger k. This improvement on $\sigma(k)$ moreover affects the iterative procedure of the diminishing range method of Vaughan [15], and by pursueing the argument in these lines, Kumchev and Wooley [11] proved that

$$H(k) \le (4k-2)\log k + k - 7,$$

for larger k. They implied that this bound is indeed valid for k exceeding 64, according to their computations. This result is the first improvement of (4) appeared in the literature. Kumchev and Wooley [11] also presented the best upper bounds of $H(k)$ for $8 \le k \le 20$ at the time, and subsequently they [12] refined all these bounds.

Actually, almost the same time as [11], Bourgain, Demeter and Guth [1] made the final progress on Vinogradov's mean value theorem by establishing the estimate (21) for $s \ge k(k+1)/2$ and $k \ge 4$ (see also Wooley [18]). This restriction on s is best possible, and permits no further improvement. By incorporating the effect stemming from the progress made by Bourgain, Demeter and Guth [1], Kumchev and Wooley [12] proved that

$$H(k) \le (4k-2)\log k - (2\log 2 - 1)k - 3,$$

and that $H(k) \le s(k)$ for $7 \le k \le 20$, where $s(k)$ is given by the following table.

k	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$s(k)$	45	57	69	81	93	107	121	134	149	163	177	193	207	223

These results of Kumchev and Wooley [12] are the best known upper bounds of $H(k)$ for all $k \ge 7$, as of March 2024. We close this survey by recording the currently best upper bounds of $H(k)$ for $k \le 6$:

$$
\begin{array}{ll}
H(1) \leq 3 \quad \text{(Vinogradov [19])}, & H(2) \leq 5 \quad \text{(Hua [4])}, \\
H(3) \leq 9 \quad \text{(Hua [7])}, & H(4) \leq 13 \text{ (Zhao [20])}, \\
H(5) \leq 21 \text{ (Kawada and Wooley [8])}, & H(6) \leq 32 \text{ (Zhao [20])}.
\end{array}
$$

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