Prime numbers and Prime closed geodesics: Similarities and Differences

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1 Similarities

1.1 Prime Number Theorem and Prime Geodesic Theorem

As is well known, Selberg [16] introduced the Selberg zeta function on weakly symmetric spaces and showed, for example, that the Riemann hypothesis holds for a compact Riemann surface of constant negative curvature up to a finite number of exceptional points. Based on this result, Huber [4] derived the prime geodesic theorem as a geometric analogue of the prime number theorem.

- $\pi(T) = {}^\sharp\{p : \text{prime number} \mid p \leq T\}$.

  THEOREM 1.1 (Prime number theorem).

$$\pi(T) \sim \frac{T}{\log T}$$

- M:compact Rieman surface with negative curvature -1
- $\pi(x) = {}^\sharp\{\mathfrak{p} : \text{prime closed geodesic} \mid \ell(\mathfrak{p}) \leq x\}$

  THEOREM 1.2 (Prime geodesic theorem [4]).

$$\pi(x) \sim \frac{e^x}{x} = \frac{T}{\log T} \quad (T = e^x)$$

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1.2 Chebotarev density Theorems in Number theory and Geometry : Finite extensions

Further developments in geometry include the prime geodesic theorem for Riemannian manifolds with variable negative curvature and more generally and the prime orbit theorem for weakly mixing Anosov flows (cf. [13]). Furthermore, the geometric analogue of Dirichlet's theorems for arithmetic progression and its generalizations, the Chebotarev density theorem for finite algebraic extension were developed by many researchers.

Chebotarev density Theorem in Number theory (finite extensions)

- L : finite Galois extension of the number field K with Galois group Γ, α : conjugacy class of an element in Γ
- $\pi(T,\alpha;\ell) = \sharp\{\mathfrak{p} : \text{(unramified) prime ideal in } K | N(\mathfrak{p}) < T, [\mathrm{Frob}_{\mathfrak{p}}] \subset \alpha\}$
- $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$ and $[\mathrm{Frob}_p]$ is the conjugacy class of the Frobenius automorphism $\mathrm{Frob}_p \in \Gamma$.
- If $\Gamma = (\mathbb{Z}/\ell\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(e^{2\pi\sqrt{-1}/\ell})/\mathbb{Q})$, then $\mathrm{Frob}_p : \zeta \mapsto \zeta^p$ with $\zeta = e^{2\pi\sqrt{-1}/\ell}$ and $\mathrm{Frob}_p|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}$ (the case of the Dirichlet density theorem).

  Theorem 1.3 (Chebotarev).

$$\pi(T,\alpha;\ell) \sim \frac{\sharp\alpha}{\sharp\Gamma}\frac{T}{\log T}$$

Chebotarev density Theorem in Geometry (finite extensions)

- M : compact manifold with negative curvature. There exists $1-1$ correspondence
  - Closed geodesics
  - Free homotopy classes of closed curves
  - Conjugacy classes of the elements in $\pi_1(M)$
- Γ: finitely generated discrete group, α : a conjugacy class of an element of Γ, $\Phi : \pi_1(M) \to \Gamma$: surjective homomorphism .

$$\pi(x,\alpha) = \sharp\{\mathfrak{p} : \text{prime closed geodesic} | \ell(\mathfrak{p}) \leq x, \Phi([\mathfrak{p}]) \subset \alpha\}$$

  where $[\mathfrak{p}]$ is the conjugacy class corresponding to $\mathfrak{p}$.

- If Γ is finite group, then

THEOREM 1.4 (Sunada [18], Adachi-Sunada [1], Parry-Pollicott [12]).

$$\pi(x,\alpha) \sim \frac{\#\alpha}{\#\Gamma}\frac{e^{hx}}{hx}.$$

(1) The methods to show necessary analytic properties of zeta or L-functions are *different* in Number theory or Geometry.

(2) The procedures from (1) to obtain density theorems are *similar*.

2 Difference

2.1 Diffence for extention groups (Galois group) of infinite extesions

- **Geometry** : Discrete, finitely generated : countable, i.e. noncompact in discrete topology
- **Number Theory** : uncountable, totally disconnected, compact in Krull topology

- (Yasutaka Ihara) There would be connection. It is important to consider countable group in Number theory : Comments in his Kodaira Prize Lecture, cf. MSJ Memoire vol 18, On Congruence Monodromy Problems, 2008. 230p.

2.2 Chebotarev density Theorems in Geometry : Infinite extensions

Abelian extensions

- $\Gamma = \mathbb{Z}^{2g} = H_1(M,\mathbb{Z})$

  THEOREM 2.1 (Phillips-Sarnak [14]). *Let M be a compact Riemann surface with constant negative curvature -1. For $\alpha \in H_1(M,\mathbb{Z})$,*

  $$\pi(x,\alpha) \sim \frac{(g-1)^g e^x}{x^{1+g}}\left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\dots\right) \quad \text{as} \quad x\to\infty.$$

  where g is the genus of M and satisfies $g=\frac{1}{2}\mathrm{rank}\, H_1(M,\mathbb{Z})$.

- The leading term is also obtained by K.-Sunada [9]. This theorem is generalized to the case of variable negative curvature or (more generally) weakly mixing Anosov flows on compact manifolds by many people [10], [11], [15].

- Proof consists of
  - the Selberg trace formula,
  - Floquet Bloch Theory (based on Fourier analysis of $\mathbb{Z}^{2g} \simeq H_1(M, \mathbb{Z})$),
  - Perturbative analysis of eigenvalues of the twisted Laplacian on the flat line bundle (i.e. local system) associated to unitary character of Γ.

Abelian vs. Nilpotent

- *Why nilpotent?*
  - Abelian : $\Gamma = \pi_1/[\pi_1, \pi_1] = H_1(M, \mathbb{Z})$ with $\pi_1 = \pi_1(M)$ (Hurwicz)
  - Nilpotent : $\Gamma = \pi_1/[\pi_1, [\pi_1, \pi_1]]$

- *Difficulty*
  - Abelian case :
    * Discrete abelian group : Type I
    * (Tractable) "Fourier Analysis" is available (Bloch theory)
    * Nilpotent Lie group : Type I
  - Nilpotent case :
  - non abelian infinite discrete group (including the Heisenberg group): non type I (type II_1)
  - Complete understanding of representation theory of non type I groups is beyond the ability of human being
  - Our strategy:
    Relate finite dim. rep's of discrete nilpotent group Γ to infinite dim. rep. of simply connected Lie group which contains Γ as a lattice. (G ia called Malcev completion of Γ.)

Nilpotent extension : Conjecture

CONJECTURE 2.2. [6] Let M be a compact manifold with negative curvature and Γ be finitely generated discrete nilpotent group and α be a conjugacy class of an element in the center of Γ.

$$\pi(x, \alpha) \sim \frac{Ce^{hx}}{x^{1+d/2}} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right)$$

where d is an exponent of (polynomial) volume growth and C can be written as a combination of "the volume of Jacobi torus" and a special value of spectral zeta function $\zeta_H(d/2)$ associated with a hypo-elliptic operator H coming from irreducible representations related Γ. $c_1, c_2, \dots$ are also written as geometric quantities related to Chen's iterated integrals.

Nilpotent extension : Examples

EXAMPLE 2.3 (Discrete Heisenberg group $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$).

$$(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{Z}$.(If we replace them with $x, y, z \in \mathbb{R}$, then the group is the Heisenberg Lie group $G = \mathrm{Heis}_3(\mathbb{R})$. Let $\mathrm{Lie}(\mathrm{Heis}_3(\mathbb{R}))$ be the Lie algebra of $G = \mathrm{Heis}_3(\mathbb{R})$.

- Dilatation $\delta_t\colon \mathrm{Lie}(\mathrm{Heis}_3(\mathbb{R})) \to \mathrm{Lie}(\mathrm{Heis}_3(\mathbb{R}))$.

  $\mathrm{Lie}(\mathrm{Heis}_3(\mathbb{R}))$ is generated by

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  with the commutater relation $[X, Y] = XY - YX = Z$ and δ_t is defined by

$$\delta_t(X) = tX, \delta_t(Y) = tY, \delta_t(Z) = t^2 Z$$

- Exponent of volume growth of $\mathrm{Heis}_3(\mathbb{Z}) = d = 1 + 1 + 2 = 4$.
- The hypo-elliptic operator $H = -\frac{d^2}{du^2} + u^2$ (the harmonic oscillator), $d = 4$ and $\zeta_H(2) = 3\zeta(2)/16$ where ζ is the Riemann zeta function and $\zeta(2) = \pi^2/6$ by Euler.

EXAMPLE 2.4 (Engel group (discrete) E_4 and (Lie) $E_4(\mathbb{R})$).

- $\mathrm{Lie}(\mathrm{Heis}_3(\mathbb{R})) = \langle X, Y.Z \mid [X, Y] = Z\rangle$
- $\mathrm{Lie}(E_4(\mathbb{R})) = \langle W, X, Y, Z \mid [W, X] = Y, [W, Y] = Z\rangle$
- $E_4 =$ some lattice of $E_4(\mathbb{R})$
- Dilatation $\delta_t\colon \mathrm{Lie}(E_4(\mathbb{R})) \to \mathrm{Lie}(E_4(\mathbb{R}))$

$$\delta_t(W) = tW, \delta_t(X) = tX, \delta_t(Y) = t^2 Y, \delta_t(Z) = t^3 Z$$

- Exponent of volume growth of $E_4 = d = 1 + 1 + 2 + 3 = 7$.
- The hypo-elliptic operator $H = -\frac{d^2}{du^2} + u^4$ (the quartic oscillator), $d = 7$ and $\zeta_H(7/2)$ can be written in terms of the Bessel functions (Voros et al [19].).

Nilpoten extension : Results

THEOREM 2.5 (K.[6], [7]). *If* $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$*,* α *is the conjugacy class of central element and* M *is compact Riemann surface with constant negative curvature* -1*, then the above conjecture holds.*

$$\pi(x,\alpha) \sim \frac{Ce^x}{x^3}\left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\dots\right)$$

PROPOSITION 2.6 (K. [6], [7]). *If* α *does not come from central elements, then we have*

$$\pi(x,\alpha) \sim \frac{Ce^x}{x^2}\left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\dots\right)$$

THEOREM 2.7 (Main result [8]). *Conjecture 2.2 holds for compact Riemann surface with constant negative curvature* -1*.*

2.3 Chebotarev Theorems for infinite extension : Positive density

Chebotarev Theorems for infinite extension : Positive density

- L/K: infinite extension unramified outside a finite set S of primes of K
- Galois group Γ : compact $\Rightarrow \exists!$ Haar measure μ on Γ
- C : a subset of Γ stable under conjugation and whose boundary has Haar measure zero. Then, the cardinality $\pi(x, C)$ of the set of primes $\mathfrak{p}$ of $K \setminus S$ such that the Frobenius conjugacy class $\mathrm{Fr}_{\mathfrak{p}} \subset X$ has positive density.
- Then we have

  THEOREM 2.8.
  $$\pi(x,C) \sim \frac{\mu(C)}{\mu(G)}\frac{x}{\log x}.$$
- This reduces to the finite case when L/K is finite (the Haar measure is then just the counting measure).

Chebotarev Theorems for infinite extension : Zero density

- A naive analogy to the above situations in geometry is break down. For example, if we consider the distribution of the p-Frobenius conjugate class belonging to the inverse image of the surjective homomorphism

  $$\Phi : G_{\mathbb{Q}} \to G_{\mathbb{Q}}^{\mathrm{ab}}$$

  from the absolute Galois group $G_{\mathbb{Q}}$ to its abelian quotient $G_{\mathbb{Q}}^{\mathrm{ab}} := G_{\mathbb{Q}}/\overline{[G_{\mathbb{Q}}, G_{\mathbb{Q}}]}$, then, by almost trivial reasons, the counting problem doesn't make sense.

- Answer 1: Extension $\overline{\mathbb{Q}}/\mathbb{Q}$ is totally ramified. The Frobenius conjugacy class is not well defined.
- Answer 2: Even if we modify the definition of Frobenius conjugacy class with considering inertia group, it is trivial since $\Phi^{-1}(e) = \emptyset$ (e is identity element in $G_{\mathbb{Q}}^{\text{ab}}$). (cf. Geometric case : $\pi(x, 0) \sim e^x/x^{g+1}$ by Phillips-Sarnak)
- (Explanation in the case of Dirichlet) If we first fix ℓ, then we find p satisfying $p \equiv 1 \pmod{\ell}$. However, if we first fix p and $\ell > p$, then $p \not\equiv 1 \pmod{\ell}$. This kind of situation happens in the above problem, since $G_{\mathbb{Q}}^{\text{ab}} \simeq \widehat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$ has arbitarily large quotient group.

Lang-Trotter conjecture

- The Lang-Trotter conjecture is known as a substantially meaningful Chebotarev-type density theorem for infinite extensions of density zero. It is considered as a difficult question since it gives more detailed information of the distribution than the Sato-Tate conjecture in some cases.
- E : elliptic curve defined by $Y^2 - Y = X^3 - X^2$
  $f = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 = \sum_{n=1}^{\infty} a_n q^n$: modular form associated to E, weight 2, level 11.
  $P(x) := \sharp\{p | a_p = 0\} = \sharp\{p | \text{reduction } E \text{ at } p \text{ is supersingular elliptic}\}$
- **Lang-Trotter conjecture** cf. [17] : $P(x) \sim c_E \frac{x^{1/2}}{\log x}$
- **Serre** [17]: $P(x) = O(x/(\log x)^{3/2-\varepsilon}$ (unconditional)
  $P(x) = O(x^{3/4})$ (under GRH)
- Proof 1: Approximation by effective Chebotarev for finite extensions. (indirect but applicable beyond supersingular case)
- **Elkies** [3]: $P(x) = O(x^{3/4})$ (uncondition)
- Proof 2: based on Kaneko's work [5]. Simple but it seems to be valid only for supersingular case
- (cf. Geometric case : directly consider infinite extension over universal covering.)
- (Another formulation [17]) ℓ positive prime number $\neq p$ $\rho_\ell : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$: ℓ-adic rep. defined by ℓ^n-division points of E.
- a_p is the trace of $\rho_\ell(\mathrm{Frob}_p)$
- $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is compact profinite ($\risingdotseq$ pronilopotent) $\Leftarrow$ hopefully relate our analysis on nilpotent groups in geometry

Other examples of the Chebotarev density theorem for infinite extensions of density zero

- I have asked experts about another example of Chebotarev-type density Theorem for infinte extension with zero density.

- **N. Kurokawa** : **Bateman-Horn Conjecture**(cf. [2])

  It implies several results including Twin prime conjecture, Green-Tao Theorem, ...etc.

- **P. Sarnak** : Watch the video of Talk 1 of the following Serre's Lecture series.

  It is important to notice that *motivated* or *non motivated* question. Lang-Trotter conjecture is *motivated.*

- Minerva Lectures 2012 - J.P. Serre Talk 1: Equidistribution :

  `https://www.youtube.com/watch?v=RxI3BemTjfk`

  Talk 2: How to use linear algebraic groups :

  `https://www.youtube.com/watch?v=5IWogUgYoZI&t=43s`

  Talk 3: Counting solutions mod p and letting p tend to infinity :

  `https://www.youtube.com/watch?v=vyVbMmm73hg&t=191s`

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