Certain Continued Fraction Algorithm Converging Simultaneously under $\mathbb{R}$ and $\mathbb{Q}_p$

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Abstract

Let p be a prime number and K be a field with embedding into $\mathbb{R}$ and $\mathbb{Q}_p$. We propose an algorithm that generates continued fraction expansions converging in $\mathbb{Q}_p$ and is expected to simultaneously converge in both $\mathbb{R}$ and $\mathbb{Q}_p$. This algorithm produces finite continued fraction expansions for rational numbers. In cases where $p = 2, 3$ and K is a quadratic field, based on numerical experiments, we conjecture that the continued fraction expansions generated by this algorithm converge in both $\mathbb{R}$ and $\mathbb{Q}_p$. Furthermore, we anticipate that these expansions eventually exhibit periodicity or finiteness.

1 Introduction

Mahler[4] initiated the first attempt at p-adic continued fractions. Schneider [10] and Ruban [8] independently proposed different algorithms during the same period, both contributing significantly to the field of continued fraction expansion algorithms for $\mathbb{Q}_p$ (see for example [7]).

Let J be a representative system modulo p. It is well known that every $u \in \mathbb{Q}_p$ can be written as

$$u = \sum_{n\in\mathbb{Z}} c_n p^n, \quad c_n \in J,$$

where $c_k = 0$ for $k < v_p(u)$. We define

$$\lfloor u \rfloor_p^J := \sum_{n\in\mathbb{Z}_{\le 0}} c_n p^n, \quad \lceil u \rceil_p^J := \sum_{n\in\mathbb{Z}_{<0}} c_n p^n.$$

For the standard representative $J = \{0, 1, \ldots, p-1\}$, we denote $\lfloor \cdot \rfloor_p^J$ and $\lceil \cdot \rceil_p^J$ by $\lfloor \cdot \rfloor_p$ and $\lceil \cdot \rceil_p$ respectively.

Ruban's continued fraction algorithm is applied to $\alpha \in \mathbb{Q}_p$ as outlined below. Starting with $\alpha_0 = \alpha$, we define sequences $\{a_n\}$ and $\{\alpha_n\}$ as follows:

$$a_n = \lfloor \alpha_n \rfloor_p, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n}.$$

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Let K be a field that has an embedding into $\mathbb{R}$ and $\mathbb{Q}_p$ respectively. Assume σ_∞ gives an embedding into $\mathbb{R}$ and σ_p gives an embedding into $\mathbb{Q}_p$. In ([9]), we defined an algorithm aiming to achieve simultaneous rational approximations in both $\mathbb{R}$ and $\mathbb{Q}_p$ for elements of K. This algorithm is a modification of the p-adic continued fraction algorithm presented in [2] and [5]. Let $\alpha \in K$. We denote $\sigma_\infty(\alpha)$ by α_∞ (or alternatively, $(\alpha)_\infty$) and $\sigma_p(\alpha)$ by $\alpha_{\langle p\rangle}$ (or alternatively, $(\alpha)_{\langle p\rangle}$). If K is a real quadratic field for the case of $p = 2$, the continued fraction expansions for the elements in K generated by this algorithm converge in both $\mathbb{R}$ and $\mathbb{Q}_p$ and become eventually periodic. In this note, we propose an algorithm that can be expected to yield simpler simultaneous approximations compared to this algorithm. Let $\alpha \in K$. Starting with $\alpha_0 = \alpha$, we define sequences $\{b_n(\alpha)\}$ and $\{\alpha_n\}$ as follows:

$$b_n(\alpha) = \epsilon_n \lfloor (\epsilon_n \alpha_n)_{\langle p\rangle} \rfloor_p, \quad \alpha_{n+1} = \frac{1}{\alpha_n - b_n(\alpha)}, \tag{1.1}$$

where $\epsilon_n := \begin{cases} 1 & \text{if } (\alpha_n)_\infty > 0, \\ -1 & \text{otherwise} \end{cases}$. The algorithm halts if $\alpha_n - b_n = 0$, resulting in $\alpha_{n+1}, \ldots$ not being defined.

Example 1.1. We give the continued fraction expansion of $\frac{5}{13}$ for some prime numbers by the algorithm.

$$\left[1; -\frac{13}{8}\right] \qquad p = 2,$$

$$\left[2; -\frac{7}{3}, \frac{4}{3}, -\frac{4}{3}\right] \qquad p = 3,$$

$$\left[0; \frac{13}{5}\right] \qquad p = 5.$$

Let $\alpha \in K$. Throughout the following sections, let b_n and α_n be generated using the algorithm (1.1).

2 Fundamental properties

In this chapter, we give fundamental properties for the algorithm (1.1).

From the definition (1.1), it is difficult to observe the following lemma.

Lemma 2.1. *Let $\alpha \in K$. Then, for all $n \in \mathbb{Z}_{>0}$, we have $v_p((\alpha_n)_{\langle p\rangle}) < 0$ and $v_p((\alpha_n)_{\langle p\rangle}) = v_p(b_n(\alpha))$.*

Following lemma gives a sufficient condition for the convergence of continued fractions.

Lemma 2.2 ([6])**.** *Let $b_0, b_1, \ldots \in \mathbb{Z}[\frac{1}{p}]$ be an infinite sequence such that*

$$v_p(b_n b_{n+1}) < 0,$$

for all $n > 0$. Then, the continued fraction $[b_0; b_1, \ldots]$ is convergent to a p-adic number.

From Lemma 2.1 and 2.2, we see that $[b_0(\alpha); b_1(\alpha), \ldots]$ is convergent to a p-adic number. The equality $\alpha_{\langle p \rangle} = [b_0(\alpha); b_1(\alpha), \ldots]$ can be obtained by the standard argument.

As is well known, in the regular continued fraction expansion in real numbers, rational numbers have finite continued fraction expansions. However, the situation is different in p-adic continued fraction expansions (see [3] and [7]). As seen below in the algorithm (1.1), rational numbers have finite expansions.

Theorem 2.3. *Let $\alpha \in \mathbb{Q}$. Then, $\{\alpha_n\}$ is a finite sequence.*

Proof. We assume that $\{\alpha_n\}$ is an infinite sequence. Let $n \in \mathbb{Z}_{>0}$. From Lemma 2.1, we can set

$$\begin{aligned} \alpha_n &= \frac{N_n}{D_n p^{i_n}}, \quad \text{with } (N_n, D_n) = 1,\ D_n > 0 \text{ and } p \nmid N_n D_n, \\ b_n(\alpha) &= \frac{c_n}{p^{i_n}}, \quad \text{with } p \nmid c_n, \end{aligned} \tag{2.1}$$

where N_n, D_n, and c_n are integers.

From Lemma 2.1, we have $i_n > 0$. From the fact that $\alpha_{n+1} = \frac{1}{\alpha_n - b_n(\alpha)}$, we have

$$N_{n+1}(N_n - c_n D_n) = p^{i_n + i_{n+1}} D_{n+1} D_n. \tag{2.2}$$

From (2.1) and (2.2), we have

$$|N_{n+1}| = |D_n|, \tag{2.3}$$

$$p^{i_n + i_{n+1}} |D_{n+1}| = |N_n - c_n D_n|. \tag{2.4}$$

From (1.1), we see that $N_n c_n D_n > 0$. Therefore, from (2.4), we have

$$\begin{aligned} |D_{n+1}| &= \left| \frac{N_n}{p^{i_n+i_{n+1}}} - \frac{c_n D_n}{p^{i_n+i_{n+1}}} \right| \\ &\le \max\left\{ \frac{|N_n|}{p^{i_n+i_{n+1}}}, \frac{|c_n| D_n}{p^{i_n+i_{n+1}}} \right\} \\ &< \max\left\{ \frac{|N_n|}{3}, D_n \right\}. \end{aligned}$$

Hence, considering (2.3), we have

$$\max\left\{ \frac{|N_{n+1}|}{3}, D_{n+1} \right\} < \max\left\{ \frac{|N_n|}{3}, D_n \right\}.$$

Thus, we have

$$\max\{|N_{n+1}|, 3D_{n+1}\} < \max\{|N_n|, 3D_n\}.$$

Therefore, the sequence $\max\{|N_n|, 3D_n\}$ is strictly decreasing, which is a contradiction. ∎

3 Numerical experiments

We demonstrate in Table 1 that for $1 < n \leq 200$, the continued fraction expansion of $\sqrt{n}$, obtained using Algorithm (1.1) with $p = 2$ satisfies the condition $\sqrt{n} = 2^m\sqrt{k}$, where $m, k \in \mathbb{Z}_{\geq 0}$, and k is not the square of an integer. Additionally, $(\sqrt{k})_{\langle 2\rangle} \in \mathbb{Q}_2$ and $(\sqrt{k})_{\langle 2\rangle} \equiv 1 \mod 8$. We show in Table 2 that for $1 < n \leq 200$, the length of the periodic part of the continued fraction expansion of $\sqrt{n}$, obtained using Algorithm (1.1) with $p = 3$ satisfies the condition $\sqrt{n} = 3^m\sqrt{k}$, where $m, k \in \mathbb{Z}_{\geq 0}$, and k is not the square of an integer and $k \equiv 1 \mod 3$. In both tables, the column labeled with ∞ shows the convergent value in real numbers of its continued fraction expansion. Interestingly, for some numbers, the convergent value in real numbers does not match the original number but corresponds to its conjugate. We note that for periodic continued fraction expansions, their convergence in the complex number field can be determined (see [1]).

Table 1: Continued fraction expansion of $\sqrt{n}$ with $n > 0$ and $p = 2$

$\sqrt{n}$	continued fraction expansion	∞
$\sqrt{17}$	$[1; 5/8, -1/2, -7/4, 3/2, \overline{-1/2}]$	$\sqrt{17}_\infty$
$\sqrt{33}$	$[1; 9/16, -1/4, -5/4, 7/4, \overline{-1/2, -1/4}]$	$\sqrt{33}_\infty$
$\sqrt{41}$	$[1; 3/4, -1/2, -3/8, \overline{-5/4, -1/2, -1/2, -5/4, 13/8}]$	$\sqrt{41}_\infty$
$\sqrt{57}$	$[1; 5/4, -1/2, -3/4, -7/8, 5/4, 3/4, \overline{-1/2, -3/2, 21/16, -3/2, -1/2, -5/4}]$	$\sqrt{57}_\infty$
$\sqrt{65}$	$[1, 17/32, -1/8, -1/4, -1/2, -3/4, 3/2, \overline{1/2, 1/2, -1/2, -1/2}]$	$\sqrt{65}_\infty$
$\sqrt{68}$	$[0; 1/2, -5/4, -3/2, \overline{1/2}]$	$\sqrt{68}_\infty$
$\sqrt{73}$	$[1; 7/4, -7/4, 1/4, 1/2, 5/8, 3/4, 31/16, -157/128, 31/16, -5/4, -3/2, \overline{3/16, 7/4, -1/2, -1/2, 1/2, 13/8, -15/8, 3/2, -1/2, -3/16, -7/4, 1/2, 1/2\ -1/2, -13/8, 15/8, -3/2, 1/2}]$	$\sqrt{73}_\infty$
$\sqrt{89}$	$[1; 1/4, -1/2, -3/2, 3/2, -1/2, -3/2, 5/4, 3/2, -5/8, -1/2, -1/2, 1/2, 3/4, \overline{-5/4}]$	$\sqrt{89}_\infty$
$\sqrt{97}$	$[1; 19/16, -1/2, -15/8, -3/2, -1/4, \overline{-3/2, 1/8, 7/4}]$	$-\sqrt{97}_\infty$
$\sqrt{105}$	$[1; 3/4, -3/4, -3/2, \overline{1/2, 7/4, -9/8, 7/4, 1/2, 1/2}]$	$\sqrt{105}_\infty$
$\sqrt{113}$	$[1; -13/8, 1/2, 39/32, 1/4, 3/2, -1/4, -3/2, \overline{-3/2, -1/2, -3/2, 7/4, -7/4, 3/2, 1/2, 3/2, -7/4, 7/4}]$	$\sqrt{113}_\infty$
$\sqrt{129}$	$[1; 33/64, -1/16, -1/4, -1/4, -3/4, 3/2, \overline{1/4, 1/2, -1/2, -1/2}]$	$\sqrt{129}_\infty$
$\sqrt{132}$	$[0; 1/2, -9/8, -1/2, -1/2, -3/2, \overline{1/4, 1/2}]$	$\sqrt{132}_\infty$
$\sqrt{137}$	$[1; 7/4, -3/2, 23/16, -3/2, -1/2, \overline{-1/2, -5/4, 1/2, 13/8, 5/4, -1/2, -1/2, 5/4, 13/8, -3/2, 3/2, 1/2, 29/16, -11/8, 3/2}]$	$\sqrt{137}_\infty$
$\sqrt{145}$	$[1; 13/8, -7/4, 7/16, 29/16, 9/8, 5/4, -1/2, 5, \overline{-9/8, -1/2, -3/4, 1/2, 3/2}]$	$\sqrt{145}_\infty$
$\sqrt{153}$	$[1; 1/4, -5/4, -3/4, 3/4, \overline{1/2, 1/2, 5/16, 1/2, 1/2, -5/4, -1/2, 1/2, 5/4, -1/2, -1/2, -5/16, -1/2, -1/2, 5/4, 1/2, -1/2}]$	$\sqrt{153}_\infty$
$\sqrt{161}$	$[1; 21/16, -1/8, -9/8, -1/2, -3/2, 1/4, \overline{5/4, 1/2, 5/4, -7/4}]$	$\sqrt{161}_\infty$
$\sqrt{164}$	$[0; 1/2, -3/2, -3/2, \overline{1/2, 5/4, -13/8, 5/4, 1/2}]$	$\sqrt{164}_\infty$
$\sqrt{177}$	$[1; 7/8, -1/2, -1/4, -3/2, 3/2, \overline{1/2, 3/2, -3/2, 7/4, -1/4, -1/2}]$	$\sqrt{177}_\infty$
$\sqrt{185}$	$[1; 5/4, -3/2, 5/4, 3/2, 21/32, -1/2, -1/2, 1/2, 1/4, 1/2, 3/2, -1/4, -11/8, \overline{-5/8, -1/2, -1/2, -5/8, 5/8, 1/2, 1/2, 5/8}]$	$\sqrt{185}_\infty$
$\sqrt{193}$	$[1; 59/32, -1/4, -1/4, -3/2, 1/4, 3/2, 1/4, 1/2, -5/4, -3/2, 1/2, 5/4, \overline{1/2, 7/4, -3/4}]$	$-\sqrt{193}_\infty$

Table 2: Continued fraction expansion of $\sqrt{n}$ with $n > 0$ and $p = 3$

$\sqrt{n}$	length of periodic part	∞	$\sqrt{n}$	length of periodic part	∞
$\sqrt{7}$	2	$\sqrt{7}_\infty$	$\sqrt{109}$	26	$\sqrt{109}_\infty$
$\sqrt{10}$	2	$-\sqrt{10}_\infty$	$\sqrt{112}$	6	$\sqrt{112}_\infty$
$\sqrt{13}$	4	$\sqrt{13}_\infty$	$\sqrt{115}$	14	$-\sqrt{115}_\infty$
$\sqrt{19}$	2	$\sqrt{19}_\infty$	$\sqrt{117}$	4	$\sqrt{117}_\infty$
$\sqrt{22}$	8	$\sqrt{22}_\infty$	$\sqrt{118}$	16	$\sqrt{118}_\infty$
$\sqrt{28}$	2	$\sqrt{28}_\infty$	$\sqrt{124}$	6	$\sqrt{124}_\infty$
$\sqrt{31}$	4	$\sqrt{31}_\infty$	$\sqrt{127}$	38	$\sqrt{127}_\infty$
$\sqrt{34}$	158	$\sqrt{34}_\infty$	$\sqrt{130}$	790	$\sqrt{130}_\infty$
$\sqrt{37}$	4	$\sqrt{37}_\infty$	$\sqrt{133}$	12	$\sqrt{133}_\infty$
$\sqrt{40}$	2	$-\sqrt{40}_\infty$	$\sqrt{136}$	6	$\sqrt{136}_\infty$
$\sqrt{43}$	10	$\sqrt{43}_\infty$	$\sqrt{139}$	2	$\sqrt{139}_\infty$
$\sqrt{46}$	34	$\sqrt{46}_\infty$	$\sqrt{142}$	10	$\sqrt{142}_\infty$
$\sqrt{52}$	2	$-\sqrt{52}_\infty$	$\sqrt{145}$	36	$\sqrt{145}_\infty$
$\sqrt{55}$	2	$\sqrt{55}_\infty$	$\sqrt{148}$	4	$-\sqrt{148}_\infty$
$\sqrt{58}$	26	$-\sqrt{58}_\infty$	$\sqrt{151}$	14	$\sqrt{151}_\infty$
$\sqrt{61}$	6	$-\sqrt{61}_\infty$	$\sqrt{154}$	6	$\sqrt{154}_\infty$
$\sqrt{63}$	2	$\sqrt{63}_\infty$	$\sqrt{157}$	12	$\sqrt{157}_\infty$
$\sqrt{67}$	4	$\sqrt{67}_\infty$	$\sqrt{160}$	6	$-\sqrt{160}_\infty$
$\sqrt{70}$	2	$\sqrt{70}_\infty$	$\sqrt{163}$	2	$\sqrt{163}_\infty$
$\sqrt{73}$	6	$\sqrt{73}_\infty$	$\sqrt{166}$	6	$-\sqrt{166}_\infty$
$\sqrt{76}$	8	$\sqrt{76}_\infty$	$\sqrt{171}$	2	$\sqrt{171}_\infty$
$\sqrt{79}$	260	$\sqrt{79}_\infty$	$\sqrt{172}$	50	$\sqrt{172}_\infty$
$\sqrt{82}$	1	$\sqrt{82}_\infty$	$\sqrt{175}$	18	$-\sqrt{175}_\infty$
$\sqrt{85}$	6	$-\sqrt{85}_\infty$	$\sqrt{178}$	2	$\sqrt{178}_\infty$
$\sqrt{88}$	10	$-\sqrt{88}_\infty$	$\sqrt{181}$	27	$-\sqrt{181}_\infty$
$\sqrt{90}$	1	$\sqrt{90}_\infty$	$\sqrt{184}$	354	$-\sqrt{184}_\infty$
$\sqrt{91}$	2	$-\sqrt{91}_\infty$	$\sqrt{187}$	10	$-\sqrt{187}_\infty$
$\sqrt{94}$	30	$\sqrt{94}_\infty$	$\sqrt{190}$	4	$\sqrt{190}_\infty$
$\sqrt{97}$	4	$\sqrt{97}_\infty$	$\sqrt{193}$	12	$\sqrt{193}_\infty$
$\sqrt{103}$	2	$\sqrt{103}_\infty$	$\sqrt{198}$	8	$\sqrt{198}_\infty$
$\sqrt{106}$	10	$-\sqrt{106}_\infty$	$\sqrt{199}$	26	$\sqrt{199}_\infty$

We show in Table 3 that for prime $3 < p \leq 100$, the smallest value of n with $1 < n$ for which the periodic continued fraction expansion of $\sqrt{n}$ using Algorithm (1.1) cannot be detected within 2000 steps.

Table 3: Smallest n with undetectable periodic CF of $\sqrt{n}$ in 2000 steps

prime number	n	prime number	n
5	11	47	2
7	8	53	5
11	5	59	3
13	3	61	5
17	2	67	6
19	5	71	2
23	3	73	2
29	5	79	2
31	2	83	3
37	3	89	2
41	2	97	2
43	6		

From these numerical experiments, we give a following conjecture.

Conjecture 3.1.

1. *Let K be a quadratic field that has an embedding into $\mathbb{R}$ and $\mathbb{Q}_2$ respectively. Let $\alpha \in K \backslash \mathbb{Q}$ and $\{\alpha_n\}$ be the sequence obtained by applying Algorithm (1.1) with $p = 2$ to α. Then, $\{\alpha_n\}$ becomes eventually periodic. The continued fraction expansion converges to α or its conjugate in $\mathbb{R}$.*

2. *Let K be a quadratic field that has an embedding into $\mathbb{R}$ and $\mathbb{Q}_3$ respectively. Let $\alpha \in K \backslash \mathbb{Q}$ and $\{\alpha_n\}$ be the sequence obtained by applying Algorithm (1.1) with $p = 3$ to α. Then, $\{\alpha_n\}$ becomes eventually periodic. The continued fraction expansion converges to α or its conjugate in $\mathbb{R}$.*

3. *Let K be a quadratic field that has an embedding into $\mathbb{R}$ and $\mathbb{Q}_p$ with $p > 3$ respectively. Then, there exists $\alpha \in K \backslash \mathbb{Q}$ such that $\{\alpha_n\}$ does not become eventually periodic, where $\{\alpha_n\}$ be the sequence obtained by applying Algorithm (1.1).*

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