ON LEHMER'S PROBLEM AND RELATED PROBLEMS

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Abstract. We show that if $N \pm 1 = M\varphi(N)$ with $N \neq 15, 255$ composite, then $M < 15.76515 \log\log\log N$ and $M < 16.03235 \log\log \omega(N)$, together with similar results for the unitary totient function, Dedekind function, and the sum of unitary divisors.

1. Introduction

As usual, let $\varphi(N)$ denote the Euler totient function of N. Clearly, $\varphi(p) = p-1$ for any prime p.

Lehmer [13] conjectured that there exists no composite number N such that $\varphi(N)$ divides $N-1$ and showed that such an integer must be an odd squarefree integer with at least seven prime factors. In other words, if $\varphi(N) \mid (N-1)$ and N is composite, then N is odd and $\omega(N) = \Omega(N) \geq 7$, where $\omega(N)$ and $\Omega(N)$ respectively denote the number of distinct and not necessarily distinct prime factors of N.

For such an integer N,

1. Cohen and Hagis [5] showed that $\omega(N) \geq 14$ and $N > 10^{20}$,
2. Renze's notebook [22] shows that $\omega(N) \geq 15$ and $N > 10^{26}$,
3. Pinch claims that $N > 10^{30}$ at his research page [17].
4. Burcsi, Czirbusz, and Farkas [3] proved that if $3 \mid N$, then $\omega(N) \geq 4 \times 10^7$ and $N > 10^{3.6\times 10^8}$.
5. Burek and Żmija [4] showed that $N \leq 2^{2^r} - 2^{2^{r-1}}$ if $\varphi(n)$ divides $N-1$ and $2 \leq \omega(N) \leq r$.

Pomerance [18] showed that the number of such integers $N \leq x$ is $O(x^{1/2} \log^{3/4} x)$ and $N \leq r^{2^r}$ if $2 \leq \omega(N) \leq r$ additionally. Luca and Pomerance [14] showed that the number of such integers $N \leq x$ is at most $x^{1/2}/\log^{1/2+o(1)} x$.

For integers N such that $N-1 = M\varphi(N)$ with M a large integer, stronger results are known. Hagis [10] proved that if $N-1 = 3\varphi(N)$, then $\omega(N) \geq 1991$ and $N > 10^{8171}$. For integers $N = M\varphi(N)+1$, $M \geq 4$, Grytczuk and Wójtowicz [9] showed that $\omega(N) \geq 3049^{M/4} - 1509$ if $3 \mid N$ and $\omega(N) \geq 143^{M/4} - 1$ otherwise.

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Subbarao [25] considered the problem analogous to Lehmer's problem involving φ^*, the unitary analogue of φ. So φ^* is defined by

$$\varphi^*(N) = \prod_{p^e || N} (p^e - 1), \tag{1.1}$$

where the product is over all prime powers unitarily dividing N. We call the value $\varphi^*(N)$ the *unitary totient* of an integer N. Subbarao conjectured that $\varphi^*(N)$ divides $N-1$ if and only if N is a prime power. This conjecture is still unsolved. However, Subbarao and Siva Rama Prasad [26] showed that N must have at least eleven distinct prime factors if N is not a prime power and $\varphi^*(N)$ divides $N-1$. Moreover, Siva Rama Prasad, Goverdhan, and Al-Aidroos [19] proved that for integers $N = M\varphi^*(N) + 1$ with $M \geq 4$,

1. $\omega(N) > (800000)^{M/4} - 499883$ and $N > (k_1 M \beta_1^M)^{\beta_1^M}$ if $15 \mid N$,
2. $\omega(N) > (597515)^{M/4} - 298668$ and $N > (k_2 M \beta_2^M)^{\beta_2^M}$ if $3 \mid N, 5 \nmid N$,
3. $\omega(N) > (1889)^{M/4} - 468$ and $N > (k_3 M \beta_3^M)^{\beta_3^M}$ if $3 \nmid N, 5 \mid N$, and
4. $\omega(N) > (608)^{M/4} - 3$ and $N > (k_4 M \beta_4^M)^{\beta_4^M}$ otherwise,

where $(\beta_1, \beta_2, \beta_3, \beta_4) = (23.4, 23.38, 6.1, 4.9)$ and $k_j = (\log \beta_j)/3$ for $j = 1, 2, 3, 4$.

We prove the following upper bounds for M.

Theorem 1. *Let N_1 denote the product of prime factors p dividing N exactly once here and hereafter. If $M\varphi^*(N) = N \pm 1$, then $M < 19.44947 \log\log\log N_1$ for $N_1 \geq 23$ or $N_1 = 19$. Moreover, if $M\varphi(N) = N \pm 1$, then $M < 15.76515 \log\log\log N$ for $N \geq 19$.*

Theorem 2. *If $M\varphi^*(N) = N \pm 1$ and $\omega(N_1) \geq 4$, then $M < 19.77911 \log\log \omega(N_1)$. Moreover, if $M\varphi(N) = N \pm 1$ and $\omega(N) \geq 4$, then $M < 16.03235 \log\log \omega(N)$.*

As Lehmer [13] observed, we see that $M\varphi(N) = N \pm 1$ and $\omega(N) \leq 3$, then N must be prime or $N = 15, 255$. Hence, if $M\varphi(N) = N \pm 1$ with $N \neq 15, 255$ composite, then $M < 15.76515 \log\log\log N$ and $M < 16.03235 \log\log \omega(N)$.

Subbarao [25] also studies similar problems for Dedekind function $\psi(N) = N \prod_{p^e || N} p^{e-1}(p+1)$ and the sum $\sigma^*(N) = \prod_{p^e || N}(p^e + 1)$ of unitary divisors of N. Clearly, $\sigma^*(N) = N + 1$ if and only if N is a prime power. Moreover, if $\psi(N) = aN + b$ and $\gcd(b, N) = 1$ with a, b integers, then N must be squarefree and $\sigma^*(N) = \psi(N) = aN + b$.

For integers N such that $\sigma^*(N) = MN + 1$ with $M > 1$ and $\omega(N) = r$,

1. Subbarao proved that $M \geq 3$ must be odd, $r \geq 16$, and $10^{20} < N < (r-1)^{2^{r-1}}$.
2. Hasanalizade [11] proved that $N > ((\log 3)M3^{M-1})^{3^M}$ and $\omega(N) > 51^{M/3} - 1$.
3. Hasanalizade also proved that $N > ((\log 2)(AM^2 - 1)2^{AM^2-1}/3)^{2^{AM^2-1}}$ and $\omega(N) > 1578^{AM^2/9}/2$, where $A = 0.998\cdots$ when 3 divides N.

Subbarao also proved that if $\psi(N) = MN + 1$ with $M > 1$ and $3 \mid N$, then $\omega(N) \geq 185$.

We prove the following upper bounds for M.

Theorem 3. *If $\sigma^*(N) = MN \pm 1$, then $M < 18.87067 \log\log\log N_1$ for $N_1 \geq 19$. Moreover, if $\psi(N) = MN \pm 1$, then $M < 15.52051 \log\log\log N$ for $N \geq 19$.*

Theorem 4. *If $\sigma^*(N) = MN \pm 1$ and $\omega(N_1) \geq 4$, then $M < 19.40333 \log\log\omega(N_1)$. Moreover, if $\psi(N) = MN \pm 1$ and $\omega(N) \geq 4$, then $M < 15.72775 \log\log\omega(N)$.*

Our upper bounds are eventually stronger than known bounds in the sense of being at least of triple-exponential and double-exponential order of M for N and $\omega(N)$ respectively.

2. Explicit sieve estimates

We write the summatory function of an arithmetic function f for $M_f(x) = \sum_{n \leq x} f(n)$. For a set U of primes, we put

$$P_U(x) = \prod_{p \in U, p \leq x} \left(1 - \frac{1}{p}\right)^{-1}, S_U(x) = \sum_{p \in U, p \leq x} \frac{1}{p}, \theta_U(x) = \sum_{p \in U, p \leq x} \log p,$$

and $\pi_U(x) = \sum_{p \in U, p \leq x} 1$ to be the number of primes in U below x.

Given an integer a, we call a set U of primes *a-self-repulsive* if for any two primes p and q in U, we have $q \not\equiv a \pmod{p}$.

Studies of 1-self-repulsive sets of primes have been begun by Golomb [8], who observed that if N is an integer such that $\gcd(N, \varphi(N)) = 1$ and U be the set of prime factors of N, then, U must be 1-self-repulsive. Indeed, we can easily see that if $\gcd(N, \varphi^*(N)) = 1$ and U be the set of prime factors of N, then, U must be 1-self-repulsive.

More generally, letting $\varphi_a(N) = \prod_{p^e || N} (p - a) p^{e-1}$, we can easily see that if $\gcd(N, \varphi_a(N)) = 1$, then N is squarefree, $\gcd(N, a) = 1$, and the set of prime factors of N must be a-self-repulsive.

Using Brun-Selberg upper bound sieve, Meijer [15], who used the term G-sequence to mean 1-self-repulsive set, proved that there exist some absolute constants c_1 and c_2 such that, if U is a 1-self-repulsive set of primes, then

$$\pi_U(x) P_U(x) \leq \frac{c_1 x}{\log x} \tag{2.1}$$

and

$$P_U(x) \leq c_2 \log\log x \tag{2.2}$$

for $x \geq 3$.

Our purpose of this section is to prove the following explicit estimate for $\pm$-self-repulsive sets.

Theorem 5. *Let U be an ± 1-self-repulsive set of primes. Then, for $x > e^{73}$, we have*

$$\pi_U(x) < \frac{8e^{\gamma}x\left(1+\frac{1}{\log x}\right)\left(1+\frac{1}{2\log^3 x}\right)}{P_U(x)\log x\left(1-\frac{\log\log x-8\gamma}{\log x}\right)^2\left(1-\frac{\log\log x}{\log x}\right)}. \tag{2.3}$$

We use the following notations:

1. Let x be a positive number and A be a set of integers contained in an interval of length at most x.
2. For each prime p, let Ω_p be a set of residue classes modulo p and $\rho(p)$ denote the number of residue classes in Ω_p.
3. $Z(A, w, \Omega)$ denote the number of integers in A that do not belong to Ω_p for any prime $p \leq w$.
4. $F = G + O^*(H)$ means that $|F - G| \leq H$
5. $\gcd(n, U) = 1$ means that no prime in U divides n.
6. Let $g(m)$ be the multiplicative function supported only on the squarefree integers m defined by $g(p) = \rho(p)/(p - \rho(p))$ for each prime p and
$$M_g(z) = \sum_{n \leq z} g(n).$$

In particular, if U is self-repulsive, then we take $\Omega_p = \{0, 1 \pmod p\}$ for primes p in U, $\Omega_p = \{0 \pmod p\}$ for primes p outside U, and A to be the set of positive integers below x to obtain

$$\pi_U(x) \leq Z(A, w, \Omega) + w \tag{2.4}$$

for any real w.

Instead of Brun-Selberg sieve, we use the large sieve method as in [7], [27], and [28]. As mentioned in the Introduction, Theorem 7.14 of [12] immediately gives the following estimate:

Lemma 6. *Assume that $\rho(p) < p$ for any prime p. Then, for any $w \geq 1$ we have*

$$Z(A, w, \Omega) \leq \frac{x + w^2}{M_g(w)}. \tag{2.5}$$

So that, our concern is to obtain a lower estimate for $M_g(x)$ with $\rho(n) = \rho_U(n)$ the multiplicative function supported on squarefree integers defined by $\rho(p) = 2$ for primes p in U and $\rho(p) = 1$ for primes p outside U. Our argument is based on the solution of Exercise 1.27 of [16]. Here we only give the digest of a proof for each lemma.

Lemma 7. *For a multiplicative function $f(n)$ over positive integers, let $M_{f,U}(x) = \sum_{n \leq x, \gcd(n,U)=1} f(n)$. In particular, we have $M_f(x) = M_{f,1}(x) = \sum_{n \leq x} f(n)$. If*

$f(n)$ always takes nonnegative value, then

$$M_{f,U}(x) \geq \frac{M_f(x)}{\prod_{p \in U} \sum_{e \geq 0} f(p^e)}. \tag{2.6}$$

Proof. Let U_0 be the set of primes in U below x. Now the lemma can be proved by induction of the number of primes in U_0. □

Lemma 8. *For $y \geq 60$,*

$$\sum_{m \leq y} \frac{\tau(y)}{y} > \frac{\log^2 y}{2} + 2\gamma \log y + 0.4. \tag{2.7}$$

Proof. Theorem 1.2 of [1] gives that for all $w \geq 9995$,

$$\sum_{n \leq w} \tau(n) = w \log w + (2\gamma - 1)w + \Delta(w) \tag{2.8}$$

with $|\Delta(w)| \leq 0.764 w^{1/3} \log w$.

Now the lemma follows using partial summation and the approximate value $2\gamma - 1 + \int_1^\infty \Delta(t)t^{-2}dt = \gamma^2 - 2\gamma_1 = 0.478809\cdots$ (see Lemma 1 of [23]), where $\gamma_1 = -0.072815\cdots$ is the first Stieltjes constant.

We note that in Corollary 2.2 of [1] and Lemma 3.3 of [20], the constant term B_0 is erroneously given as $\gamma^2 - \gamma_1$, which should be $\gamma^2 - 2\gamma_1$ as in [23]. □

Now we would like to show the following lower bound for $M_g(y)$.

Lemma 9. *For $y > e^{30}$, we have*

$$M_g(y) > P_U(y) e^{-\gamma} \left(\frac{\log y}{2} + 2\gamma + \frac{0.1}{\log y} \right). \tag{2.9}$$

Proof. We put $\Omega_U(n)$ be the number of prime factors in U of n counted with multiplicity, $\tau_U(n)$ be the number of divisors of n composed of primes in U, and $\mathrm{rad}(n) = \prod_{p|n} p$ be the product of distinct prime divisors of n.

We put V to be the set of integers composed only of primes in U. Then, we see that

$$\begin{aligned} \sum_{n \leq y} g(n) &= \sum_{n \leq y} \mu^2(n) \prod_{p|n, p \in U} \frac{2}{p-2} \prod_{p|n, p \notin U} \frac{1}{p-1} \\ &\geq \sum_{\mathrm{rad}\, k \leq y} \frac{2^{\Omega_U(k)}}{k} \\ &\geq \sum_{k \leq y} \frac{\tau_U(k)}{k} = \sum_{m \leq y} \left(\frac{1}{m} \sum_{d \leq y/m, d \in V} \frac{1}{d} \right), \end{aligned} \tag{2.10}$$

where we observe that $2^{\Omega_U(k)} \geq \tau_U(k)$. Now the lemma follows using Lemma 7 and Theorem 7 of [24]. $\square$

Now we shall prove Theorem 5. Lemma 6 immediately gives

$$Z(A, y, \Omega) \leq \frac{x+y^2}{M_g(y)} < \frac{e^{\gamma}(x+y^2)}{P_U(y)\left(\frac{\log y}{2} + 2\gamma + \frac{0.12}{\log y}\right)}. \tag{2.11}$$

With the aid of Theorem 5.9 of [6], we have

$$\frac{P_U(x)}{P_U(y)} \leq \prod_{y<p\leq x} \frac{p}{p-1} < \frac{\log x}{\log y}\left(1 + \frac{1}{5\log^3 y}\right)^2 \tag{2.12}$$

(but Ramaré's zero density estimate in [21], on which Dusart's estimates in [6] are based, is objected by [2]. Corollary 11.2 in [2] can instead be used to obtain Dusart's estimates), and therefore

$$Z(A, y, \Omega) < \frac{e^{\gamma}(x+y^2)\log x}{P_U(x)(\frac{\log^2 y}{2} + 2\gamma \log y + 0.12)}\left(1 + \frac{1}{5\log^3 y}\right)^2. \tag{2.13}$$

Taking $y = \sqrt{x/\log x}$ (we note that $y > e^{30}$ since we have assumed that $x > e^{73}$), we have

$$Z(A, y, \Omega) < \frac{8e^{\gamma}x\left(1 + \frac{1}{\log x}\right)\left(1 + \frac{0.49}{\log^3 x}\right)}{P_U(x)\log x\left(1 - \frac{\log\log x - 8\gamma}{\log x}\right)^2\left(1 - \frac{\log\log x}{\log x}\right)}. \tag{2.14}$$

Now Theorem 5 immediately follows from (2.4).

3. Proofs of Theorems

Here we only give the proof of Theorem 1. We put U to be the set of prime factors p of N such that p^2 does not divide N, so that $N_1 = \prod_{p\in U} p$. As we noted in the last section, U must be 1-self-repulsive if $M\varphi^*(N) = N \pm 1$ and (-1)-self-repulsive if $N = M\sigma^*(N) \pm 1$.

Assume that N is a positive integer satisfying $M\varphi^*(N) = N \pm 1$ for some integer $M \geq 2$. Let x_1 be the largest prime factor of N_1. We note that $P_U(x_1) = \prod_{p\in U} p/(p-1) = N_1/\varphi(N_1)$ and $\theta_U(x_1) = \sum_{p\in U} \log p = \log N_1$.

We begin by proving that $N_1/\varphi(N_1) < 15.68996 \log\log\log N_1$. Let $x_0 = e^{73}$. We discuss three cases: (i) $x_1 \leq x_0$, (ii) $x_1 > x_0$, $\theta_U(x_1) \geq x_1/\log\log x_1$, and (iii) $x_1 > x_0$, $\theta_U(x_1) < x_1/\log\log x_1$. In the case (iii), we put x_2 be the largest number x such that $\theta_U(x) \geq x/\log\log x$ and $x_3 = \theta_U(x_1)$. Then we settle four subcases. (a) $x_3 > x_2$ and $x_2 \leq x_0$, (b) $x_3 > x_2 > x_0$, (c) $x_3 \leq x_2 \leq x_0$, and (d) $x_3 \leq x_2$ and $x_2 > x_0$.

3.1. **Case (i).** putting p_1 to be the largest prime such that $\prod_{p\leq p_1} p \leq N_1$, the Corollary of Theorem 8 in [24] gives that

$$\frac{N_1}{\varphi(N_1)} \leq P(p_1) < \frac{e^\gamma}{2}\left(\log p_1 + \frac{1}{\log p_1}\right) < 15.15486 \log\log p_1, \tag{3.1}$$

where the last inequality follows from the fact that $p_1 \leq x_1 \leq x_0$. If $p_1 > 500000$, then Theorem 1 of [2] gives that $p_1 < 1.0268\theta(p_1) < 1.0268 \log N_1$ and we obtain $N_1/\varphi(N_1) < 15.56102 \log\log\log N_1$, which is more than we desired. If $p_1 < 500000$ and $N_1 > 3704$, then we have $P(p_1) < 11.68731 < 15.68996 \log\log\log N_1$. If $N_1 = 19$ or $23 \leq N_1 \leq 3703$, then we can confirm $N_1/\varphi(N_1) < 7.34789 \log\log\log N_1$ by calculation.

3.2. **General remarks for Cases (ii) and (iii).** Assume that $x_1 > x_0$. As we have seen in the last section, U must be 1-self-repulsive. Let x be a real number such that $x_0 \leq x \leq x_1$ and $\theta_U(x) \geq x/\log\log x$. Observing that $\pi_U(x) \geq \theta_U(x)/\log x > x/(\log x \log\log x)$, Theorem 5 immediately gives that

$$P_U(x) < \frac{8e^\gamma\left(1+\frac{1}{\log x}\right)\left(1+\frac{1}{2\log^3 x}\right)}{\left(1-\frac{\log\log x - 8\gamma}{\log x}\right)^2\left(1-\frac{\log\log x}{\log x}\right)} \log\log x. \tag{3.2}$$

Hence, (3.2) gives that

$$P_U(x) < 8e^\gamma \delta(\log x) \log\log\theta_U(x), \tag{3.3}$$

where

$$\delta(t) = \frac{\left(1+\frac{1}{t}\right)\left(1+\frac{1}{2t^3}\right)}{\left(1-\frac{\log t - 8\gamma}{t}\right)^2\left(1-\frac{\log t}{t}\right)\left(1-\frac{1.01011\log\log t}{t\log t}\right)}. \tag{3.4}$$

For $t > 73$, we can see that

$$\begin{aligned}\delta(t) &< 1 + \frac{3\log t - 7.75695}{t} + \frac{(3\log t - 7.75695)^2}{2(1-0.07007)t^2} \\ &< 1 + \frac{3\log t - 7.55957}{t}.\end{aligned} \tag{3.5}$$

3.3. **Case (ii).** Taking $x = x_1$, we have $P_U(x_1) = N_1/\varphi^*(N_1)$ and $\theta_U(x_1) = \log N_1$ as we noted above. Hence, (3.3) together with (3.5) yield that

$$\begin{aligned}\frac{N_1}{\varphi^*(N_1)} &< 8e^\gamma\left(1+\frac{3\log\log x_1 - 7.55957}{\log x_1}\right)\log\log\log N_1 \\ &< 15.28538 \log\log\log N_1.\end{aligned} \tag{3.6}$$

3.4. **Cases (iii-a) and (iii-b).** Since $x_3 > x_2$, partial summation gives

$$\begin{aligned}S_U(x_1) - S_U(x_2) &= \frac{\theta_U(x_2)}{x_2\log x_2} - \frac{\theta_U(x_1)}{x_1\log x_1} + \int_{x_2}^{x_1}\frac{\theta_U(t)(1+\log t)}{t^2\log^2 t}dt \\ &< \log\log\log x_3 - \log\log\log x_2 + \frac{1}{\log x_2 \log\log x_2} + \frac{1}{\log x_2},\end{aligned} \tag{3.7}$$

where we see that $\theta_U(t) \le x_3$ for $t \le x_1$, and therefore

$$\frac{P_U(x_1)}{P_U(x_2)} < \frac{\log\log x_3}{\log\log x_2} \exp\left(\frac{1.233076}{\log x_0}\right). \tag{3.8}$$

In the case (a), then, with the aid of the Corollary of Theorem 8 in [24] and we can obtain $N_1/\varphi(N_1) = P_U(x_1) < 15.41303 \log\log\log N_1$, which is more than desired. In the other case (b), then, taking $x = x_2$ in (3.3), we can obtain $N_1/\varphi(N_1) = P_U(x_1) < 15.54576 \log\log\log N_1$ with the aid of (3.5) as desired.

3.5. **Cases (iii-c) and (iii-d).** If $x_3 < x_2$, then we have

$$\begin{aligned} S_U(x_1) - S_U(x_2) &< \frac{1}{\log x_2 \log\log x_2} + x_3 \int_{x_2}^{x_1} \frac{1+\log t}{t^2 \log^2 t} dt \\ &< \frac{1}{\log x_2 \log\log x_2} + \frac{1}{\log x_2}. \end{aligned} \tag{3.9}$$

In the case (c), we proceed like in the case (a) to obtain $N_1/\varphi(N_1) = P_U(x_1) < 15.63054 \log\log\log N_1$. In the case (d), we proceed like in the case (b) to obtain $N_1/\varphi(N_1) = P_U(x_1) < 15.76514 \log\log\log N_1$.

3.6. **Conclusion.** Hence, we have $N_1/\varphi(N_1) < 15.76514 \log\log\log N_1$ in any case and conclude that

$$M \le \frac{N+1}{\varphi^*(N)} \le \frac{1}{N} + \frac{N_1}{\varphi(N_1)} \prod_{p^2 | N} \frac{p^2}{p^2-1} < 19.44947 \log\log\log N_1. \tag{3.10}$$

Moreover, if $M\varphi(N) = N \pm 1$, then $N = N_1$ and therefore $M = (N\pm 1)/\varphi(N) < 15.76515 \log\log\log N$, which completes the proof of Theorem 1.

We can prove Theorem 3 in a quite similar way with $x_0 = e^{95}$ instead of e^{73}.

3.7. **Proofs of Theorems 2 and 4.** Proofs of other Theorems are similar to proofs of Theorems 1 and 3 but needs some modification. Let $x_0 = e^{72}$ and $r = \omega(N_1) \ge 4$. We discuss three cases: (i) $x_1 \le x_0$, (ii) $x_1 > x_0$, $\pi_U(x_1) \ge x_1/(\log x_1 \log\log x_1)$, and (iii) $x_1 > x_0$, $\pi_U(x_1) > x_1/(\log x_1 \log\log x_1)$. Moreover, in the case (iii), we put x_2 be the largest number x such that $\pi_U(x) \ge x/(\log x \log\log x)$ and settle four subcases. (a) $r \log r > x_2$ and $x_2 \le x_0$, (b) $r\log r > x_2 > x_0$, (c) $r \log r \le x_2 \le x_0$, and (d) $r \log r \le x_2$ and $x_2 > x_0$.

Then we can prove Theorem 2. Moreover, we can prove Theorem 4 in a quite similar way with $x_0 = e^{93}$ instead of e^{72}.

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