

# On the order estimation of the double $L$ -function

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## Abstract

Double  $L$ -function is a generalization of classical Dirichlet  $L$ -function to two variable functions. We give upper bounds of double  $L$ -functions, and Our upper bounds are explicit in conductor aspect. This article a review of the author's talk given at the "RIMS workshop 2023 Analytic Number Theory and Related Topics".

## §1. Introduction

Let  $s = \sigma + it$  be a complex variable. The study of moments of the Riemann zeta-function is an active research area in analytic number theory. Classically, it is known that

$$\int_1^T |\zeta(1/2 + it)|^2 dt = T \log T - T(\log(2\pi) - 2\gamma + 1) + E. \quad (1)$$

In 1949, Atkinson [2] obtained the precise formula of the error term  $E$  in (1). First, Atkinson divided the product of two Riemann zeta-functions into

$$\zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2),$$

where

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} (m+n)^{s_2}}.$$

By using the Poisson summation formula, he proved a meromorphic continuation of  $\zeta_2(s_1, s_2)$  to a certain domain. Nowadays  $\zeta_2(s_1, s_2)$  is called the double zeta-function and is widely studied. So Atkinson's work is the first result of analytic aspects of  $\zeta_2(s_1, s_2)$ . It is known that mean squares are closely related to the Lindelöf hypothesis. So the double zeta-function is not only a generalization of the Riemann zeta-function, but also has a correspondence to the moment problem.

In this direction, we want to know the magnitude of the double zeta-function. This was first done by Ishikawa and Matsumoto [8] and they showed the upper bounds of the double zeta-function. Their result was improved by several works by Kiuchi and Tanigawa [10],

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Kiuchi, Tanigawa and Zhai [11] and Banerjee, Minamide and Tanigawa [3]. In the latest work of [3], they showed

$$\zeta_2(1/2 + it_1, 1/2 + it_2) \ll |t_1|^{\mu(\frac{1}{2})+\varepsilon} |t_2|^{\mu(\frac{1}{2})+\varepsilon}$$

for  $|t_2|^{\frac{\frac{1}{2}-\mu(\frac{1}{2})}{1+\mu(\frac{1}{2})}} \ll |t_1| \ll |t_2|^{\frac{1+\mu(\frac{1}{2})}{\frac{1}{2}-\mu(\frac{1}{2})}}$ , where

$$\mu(\sigma) := \inf\{c \geq 0 \mid \zeta(\sigma + it) \ll |t|^c\}. \quad (2)$$

Since it is conjectured that  $\mu(1/2) = 0$  (the Lindelöf Hypothesis). So [3] implies that under the Lindelöf Hypothesis, the double zeta-function can become small when the condition (2) holds. Unconditionally, Bourgain [4] showed that  $\mu(1/2) \leq 13/84$ .

## §2. Main results

Let  $\chi$  be a Dirichlet character modulo  $q$  and let  $L(s, \chi)$  denote the corresponding Dirichlet  $L$ -function defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This series converges absolutely for  $\sigma > 1$  and can be extended analytically to all  $\mathbb{C}$ -plane, except for a simple pole at  $s = 1$  if  $\chi$  is the principal character. For the case of Dirichlet  $L$ -functions, the order on the critical line is also conjectured (the generalized Lindelöf Hypothesis). It asserts that for  $L(s, \chi)$  and for all  $\varepsilon > 0$ , it holds that

$$L(1/2 + it, \chi) \ll_{\varepsilon} ((|t| + 1)q)^{\varepsilon}.$$

As a generalization of Dirichlet  $L$ -functions, we define double  $L$ -functions. Let  $\chi_1, \chi_2$  be Dirichlet characters modulo  $q$ . Then the corresponding double Dirichlet  $L$ -function is defined by

$$L_2(s_1, s_2; \chi_1, \chi_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)}{m^{s_1}(m+n)^{s_2}}. \quad (3)$$

The double  $L$ -function can be continued meromorphically to  $\mathbb{C}^2$ , and if  $\chi_1, \chi_2$  are both non-principal, then  $L_2(s_1, s_2; \chi_1, \chi_2)$  is entire on  $\mathbb{C}^2$ . We prove the following result.

**Theorem 1.** *Let  $(s_1, s_2) \in \mathbb{C}^2$  with  $\sigma_1 + \sigma_2 > 0$  and  $0 < \sigma_2 < 1$ . Suppose that  $\chi_1, \chi_2$  are primitive Dirichlet characters modulo  $q$ . Then if  $\chi_2(-1) = 1$  we have*

$$\begin{aligned} L_2(s_1, s_2; \chi_1, \chi_2) &= \frac{2(2\pi)^{s_2-1}\tau(\chi_2)\Gamma(1-s_2)}{q^{s_2}} \sum_{mn \leq \frac{q|t_2|}{2\pi}} \frac{\chi_1(m)\overline{\chi_2}(n) \sin(\frac{\pi s_2}{2} + \frac{2\pi mn}{q})}{m^{s_1}n^{1-s_2}} \\ &\quad + O\left(q^{\frac{1}{2}}(q|t_2|)^{\delta+\varepsilon}\right), \end{aligned}$$

and if  $\chi_1(-1) = -1$  we have

$$L_2(s_1, s_2; \chi_1, \chi_2) = \frac{2(2\pi)^{s_2-1} \tau(\chi_2) \Gamma(1-s_2)}{iq^{s_2}} \sum_{mn \leq \frac{q|t_2|}{2\pi}} \frac{\chi_1(m) \overline{\chi_2}(n) \cos(\frac{\pi s_2}{2} + \frac{2\pi mn}{q})}{m^{s_1} n^{1-s_2}} \\ + O\left((1+|t_1+t_2|)q^{\frac{3}{2}+\varepsilon} (q|t_2|)^{-\min\{1, \frac{\sigma_1+\sigma_2}{2}\}+\varepsilon}\right) + O\left(q^{\frac{1}{2}}(q|t_2|)^{\delta+\varepsilon}\right),$$

where  $\tau(\chi_2)$  is the Gauss sum,  $\delta = \max\{0, 1-\sigma_1-\sigma_2\}$  and implicit constants are independent on  $q, t_1$  and  $t_2$ .

**Remark 2.** It is not necessary to restrict the conductor of the characters to the same one, but for simplicity, we fix the same conductor.

The above formulas can be regarded as approximation formulas for double  $L$ -functions. Since  $2(2\pi)^{s-1} \Gamma(1-s) \sin(\pi s/2), 2(2\pi)^{s-1} \Gamma(1-s) \cos(\pi s/2) \asymp t^{\frac{1}{2}-\sigma}$ , Theorem 1 implies the following result.

**Theorem 3.** Let  $(s_1, s_2) \in \mathbb{C}^2$  with  $\sigma_1 + \sigma_2 > 0, 0 < \sigma_2 < 1$  and  $|t_2| \geq 2$ . Suppose that  $\chi_1, \chi_2$  are primitive Dirichlet characters modulo  $q$ . Then if  $\chi_2(-1) = 1$  we have

$$L_2(s_1, s_2; \chi_1, \chi_2) \ll (q|t_2|)^{\frac{1}{2}+\delta+\varepsilon},$$

and if  $\chi_2(-1) = -1$  we have

$$L_2(s_1, s_2; \chi_1, \chi_2) \ll (q|t_2|)^{\frac{1}{2}+\delta+\varepsilon} + (1+|t_1+t_2|)q^{\frac{3}{2}+\varepsilon} (q|t_2|)^{-\min\{1, \frac{\sigma_1+\sigma_2}{2}\}},$$

where  $\delta = \max\{0, 1-\sigma_1-\sigma_2\}$  and the implicit constants are independent on  $q, t_1$  and  $t_2$ .

When  $\chi_1, \chi_2$  are both primitive, we succeed in extending the region where the order estimates hold to  $\sigma_1 + \sigma_2 > 0$  if  $0 < \sigma_2 < 1$ . Moreover, when  $0 < \sigma_2 < 1$ , our bounds are better than [13], even including the conductor aspect when  $q \ll |t_2|^{2\theta_2(\eta)-\varepsilon}$  if  $\chi_2(-1) = 1$ , and when  $q \ll |t_2|^{\theta_2(\eta)-\varepsilon} / |t_1+t_2|$  if  $\chi_2(-1) = -1$ , where  $\theta_2(\sigma) = \inf\{\xi \geq 0 \mid L(\sigma+it, \chi_2) \ll |t|^\xi\}$ .

We omit the proofs of Theorem 1 and Theorem 3. For details, see [15]. We describe ideas of the proofs in the next section.

### §3. Confluent hypergeometric functions

We consider the Kummer's equation

$$z \frac{d^2 w}{dz^2} + (c-z) \frac{dw}{dz} - aw = 0. \quad (4)$$

It is known that Kummer's confluent hypergeometric function

$${}_1F_1(a, c; x) = \sum_{n=0}^{\infty} \frac{a(a+1) \dots (a+n-1)}{c(c+1) \dots (c+n-1)} \frac{x^n}{n!},$$

and Tricomi's confluent hypergeometric function

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} {}_1F_1(a-c+1, 2-c; x)$$

are standard solution of (4).

From [14], [9], an expression in terms of  $\Psi(a, c; x)$  is known for  $\zeta_2(s_1, s_2)$ . Moreover, in 2011, Komori, Matsumoto and Tsumura [12] showed that the similar expression holds for  $L_2(s_1, s_2; \chi_1, \chi_2)$ . In this work, we prove the following expression of  $L_2(s_1, s_2; \chi_1, \chi_2)$  which is slightly different from [12].

**Lemma 4.** *If  $\chi_1, \chi_2$  are both primitive, then for  $\sigma_1 + \sigma_2 > 0$  we have*

$$L_2(s_1, s_2; \chi_1, \chi_2) = \frac{\varepsilon(\chi_2)}{\sqrt{q}} \sum_{m, n=1}^{\infty} \frac{\chi_1(m) \overline{\chi_2}(n)}{m^{s_1+s_2-1}} \\ \times \left\{ e^{\frac{\pi i}{2} \kappa_2} \Psi(1, 2 - s_2; 2\pi i m n / q) + e^{-\frac{\pi i}{2} \kappa_2} \Psi(1, 2 - s_2; -2\pi i m n / q) \right\},$$

where  $\varepsilon(\chi_2) = \frac{\tau(\chi_2)}{\sqrt{q}^{i\kappa_2}}$  with denoting  $\kappa_2$  as  $\chi_2(-1) = (-1)^{\kappa_2}$ .

Then, we can transform Toricomi's confluent hypergeometric function to the incomplete gamma function of second kind. By using the classical formula

$$x^a e^{-x} \Psi(1, a + 1; x) = \Gamma(a, x),$$

we can calculate  $L_2(s_1, s_2; \chi_1, \chi_2)$  because we can apply the estimates of integrals introduced by Hardy and Littlewood [6].

## §4. Other double $L$ -functions

We describe a similar argument for different other double  $L$ -functions. For Dirichlet characters  $\chi_1, \chi_2$ , the corresponding double  $L$ -functions which is different from (3) is defined by

$$\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_1(m) \chi_2(m+n)}{m^{s_1} (m+n)^{s_2}}.$$

This double  $L$ -function was first introduced by Goncharov [5], and Akiyama and Ishikawa [1]. In [1], they showed that even though  $\chi_1, \chi_2$  are both primitive,  $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$  is not entire on  $\mathbb{C}^2$ . Indeed, they showed that the true singularities of  $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$  are given by

$$\begin{cases} s_1 + s_2 = 0, -2, -4, -6, -8, \dots & \text{if } \chi_1 \chi_2(-1) = 1, \\ s_1 + s_2 = 1, -1, -3, -5, -7, \dots & \text{if } \chi_1 \chi_2(-1) = -1. \end{cases}$$

In 2002, Ishikawa [7] considered

$$H(x) = \sum_{\substack{m_1 m_2 \leq x \\ m_1 < m_2}} \chi_1(m_1) \chi_2(m_2) = \sum_{n \leq x} h(n),$$

where

$$h(n) = \sum_{\substack{n_1 n_2 = n \\ n_1 < n_2}} \chi_1(n_1) \chi_2(n_2)$$

and obtained the Riesz mean of  $h(n)$ . We should remark that he considered the multiple  $L$ -function of general depth, but for simplicity, we omit details.

If we can obtain sharper upper bounds of  $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$ , it might be possible to obtain an asymptotic formula for  $H(x)$ . However, we cannot apply our method to  $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$  when  $\chi_1\chi_2(-1) = -1$  because in this case, the series in Lemma 4 does not converge. Moreover, when  $\chi_1\chi_2(-1) = 1$ , there is no pole of  $\mathcal{L}_2(s_1, s_2; \chi_1, \chi_2)$  in the domain  $\sigma > 0$ . So even though we obtain tight upper bound, the main term does not appear. We expect that it is difficult to extend our method to the domain  $\sigma_1, \sigma_2 < 0$ . Therefore further studies are required to obtain an asymptotic formula for  $H(x)$ .

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## References

- [1] S. Akiyama and H. Ishikawa, On analytic continuation of multiple  $L$ -functions and related zeta-functions, analytic number theory (Beijing/Kyoto, 1999), Dev. Math. **6** (2002), 1–16.
- [2] F. V. Atkinson, The mean value of the Riemann zeta function, Acta Math. **81** (1949), 353–376.
- [3] D. Banerjee, M. Minamide, Y. Tanigawa, Bounds of double zeta-functions and their applications, Pacific J. Math. **304** (2020), no. 1, 15–41.
- [4] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, J. Amer. Math. Soc. **30** (2017), no. 1, 205–224.
- [5] A. B. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Letters **5** (1998), 497–516.
- [6] G.H. Hardy, J.E. Littlewood, The zeros of Riemann’s zeta-function on the critical line, Math. Z. **10** (1921) 283–317.
- [7] H. Ishikawa, A multiple character sum and a multiple  $L$ -function, Arch. Math. (Basel) **79** (2002), no.6, 439–448.
- [8] H. Ishikawa, K. Matsumoto, On the estimation of the order of Euler-Zagler multiple zeta-functions, Illinois J. Math. **47** (2003) 1141–1166.
- [9] M. Katsurada, Asymptotic expansions of the mean values of Dirichlet  $L$ -functions II; in Analytic number theory and related topics, K. Nagasaka (ed.) (World Scientific, 1993), pp. 61–71.
- [10] I. Kiuchi, Y. Tanigawa, Bounds for double zeta-functions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **5** (2006), no. 4, 445–464.

- [11] I. Kiuchi, Y. Tanigawa, W. Zhai, Analytic properties of double zeta-functions, *Indag. Math. (N.S.)* **21** (2011), no. 1-2, 16–29.
- [12] Y. Komori, K. Matsumoto, H. Tsumura, Functional equations for double  $L$ -functions and values at non-positive integers, *Int. J. Number Theory* **7** (2011), no.6, 1441—1461.
- [13] K. Matsumoto, Y. Tanigawa, The analytic continuation and the order estimate of multiple Dirichlet series, *J. Théor. Nombres Bordeaux* **15** (2003), 267–274.
- [14] Y. Motohashi, A note on the mean value of the zeta and  $L$ -functions I, *Proc. Japan Acad., Ser. A* **61** (1985) 222–224.
- [15] Y. Toma, Bounds for double  $L$ -functions, [arXiv:2312.01911](https://arxiv.org/abs/2312.01911).