AN APPLICATION OF MELLIN-BARNES TYPE INTEGRALS TO THE MEAN SQUARES OF DIRICHLET-HURWITZ-LERCH L-FUNCTIONS

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Abstract. Complete asymptotic expansions associated with the mean squares, in the discrete and continuous forms, of Dirichlet-Hurwitz-Lerch L-functions are presented (Theorems 1 and 2), together with their outlined proofs.

1. Introduction

Throughout the paper, $s=\sigma+it$, u and v are complex variables, α and λ real parameters with $\alpha \geq 0$, χ any Dirichlet character modulo (arbitrary) $q \geq 1$, and $\overline{\chi}$ the complex conjugate of χ. We frequently use the notation $e(s)=e^{2\pi is}$, $e_q(s)=e(s/q)=e^{2\pi is/q}$, denote by ι the principal character modulo $q \geq 1$, and write $X_c(l)=X(c+l)$ $(c,l \in \mathbb{Z})$ for any Dirichlet character X.

The Dirichlet-Hurwitz-Lerch L-function $L_{\chi_c}(s,\alpha,\lambda)$ is defined by

$$L_{\chi_c}(s,\alpha,\lambda)=\sum_{l=0}^{\infty}{}' \frac{\chi_c(l)e_f\{(\alpha+l)\lambda\}}{(\alpha+l)^s} \qquad (\mathrm{Re}(s)=\sigma>1), \tag{1.1}$$

and its meromorphic continuation over the whole s-plane. The primed summation symbols hereafter indicate omission of the impossible terms of the form $1/0^s$ (if they occur). This reduces if $(q,\chi)=(1,\iota)$ to the Lerch zeta-function $\psi(s,\alpha,\lambda)$=$e(\alpha\lambda)\phi(s,\alpha,\lambda)$, and further if $(q,\lambda)=(1,0)$ to the Hurwitz zeta-function $\zeta(s,\alpha)$, while if $(q,\alpha)=(1,0)$ to the exponential zeta-function $\zeta_\lambda(s)$, if $(\alpha,\lambda)=(0,0)$ to the (shifted) Dirichlet L-function $L_{\chi_c}(s)$, and hence if $(q,\chi)=(1,\lambda)$ and $(\alpha,\lambda)=(0,0)$ to the Riemann zeta-function $\zeta(s)$.

A more flexible definition of the Dirichlet-Hurwitz-Lerch L-function, for any real α and λ, and for any integer c, asserts

$$L^*_{\chi_c}(s,\alpha,\lambda)=\sum_{-\alpha<l\in\mathbb{Z}} \frac{\chi_c(l)e_q\{(\alpha+l)\lambda\}}{(\alpha+l)^s} \qquad (\mathrm{Re}(s)=\sigma>1),, \tag{1.2}$$

for which several results have recently been shown by Noda and the author [13]. Let $\Gamma(s)$ denote the gamma function, and $G_\chi=\sum_{h=0}^{q-1}\chi(h)e_q(h)$ Gauß' sum. We can show:

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Theorem -2 ([13, Theorem 5]). *For any real α and λ, any integer c, and any primitive character χ modulo $q \geq 1$, we have*

$$L^*_{\chi_c}(s,\alpha,\lambda) = e_q\{\lambda(\alpha-c)\}\frac{G_\chi \Gamma(s)}{q^s(2\pi)^{1-s}}\Big\{\chi(-1)e^{\pi i(1-s)/2}L^*_{\overline{\chi}}\big(1-s,\lambda,-(\alpha-c)\big) + e^{-\pi i(1-s)/2}L^*_{\overline{\chi}}\big(1-s,-\lambda,\alpha-c\big)\Big\}.$$

Next let B_k $(k=0,1,\ldots)$ be the Bernoulli numbers (cf [3, p.35, 1.13.(1)]), and τ a complex parameter in the sector $|\arg\tau|<\pi/2$. Then the celebrated formulae of Euler and Ramanujan for specific values of $\zeta(s)$ assert respectively that

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!}B_{2k} \qquad (k=1,2,\ldots),$$

and for any integer $k \neq 0$,

$$\zeta(2k+1) + 2\sum_{l=1}^{\infty}\frac{l^{-2k-1}e^{-2\pi l\tau}}{1-e^{-2\pi l\tau}} + (2\pi)^{2k+1}\sum_{j=0}^{k+1}\frac{(-1)^j B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}\tau^{2k+1-2j}$$
$$= (-1)^k\tau^{2k}\bigg\{\zeta(2k+1) + 2\sum_{l=1}^{\infty}\frac{l^{-2k-1}e^{-2\pi l/\tau}}{1-e^{-2\pi l/\tau}}\bigg\}.$$

Let $L^*_{\chi_a}(s,\alpha,\mu)$ and $L^*_{\psi_b}(s,\beta,\nu)$ for any real α, β, μ and ν, and any integers a and b be the Dirichlet-Hurwitz-Lerch L-functions (defined by (1.2)), attached to any (shifted) primitive characters χ_a and ψ_b modulo $f \geq 1$ and $g \geq 1$ respectively. Then we can further show:

Theorem -1 ([13, Theorem 4]). *There exist various character analogues of Euler's formula for $L^*_{\chi_a}(s,\alpha,\mu)$, as well as of Ramanujan's formula connecting specific values of $L^*_{\chi_a}(s,\alpha,\mu)$ and $L^*_{\psi_b}(s,\beta,\nu)$ with any primitive characters of (possibly) different moduli.*

The observation above suggests that the following empirical 'theorem' seems to be true!

Theorem 0. *It is worth pursuing the functional (or arithmetical) nature of a class of Dirichlet-Hurwitz-Lerch L-functions.*

2. Asymptotics for the discrete mean square

Let $\varphi(n)$ denote Euler's totient function, $\mu(n)$ Möbius' function, and write, for any $n \in \mathbb{Z}$, the shifted factorial of s as

$$(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)} = \begin{cases} s(s+1)\cdots(s+n-1) & \text{if } n \geq 0, \\ \dfrac{1}{(s-1)(s-2)\cdots(s-|n|)} & \text{if } n < 0. \end{cases}$$

The chief concern in this section is the asymptotic expansions for the discrete mean square

$$\varphi(q)^{-1}\sum_{\chi(\bmod q)}\big|L_{\chi_c}(\sigma+it,\alpha,\lambda)\big|^2, \tag{2.1}$$

averaged over all characters χ modulo $q \geq 1$.

We give here an overview of the results related to (2.1). Atkinson [1] first established a precise asymptotic formula for the error term $E(T)$ of the mean square $\int_0^T|\zeta(1/2+it)|^2dt$ in terms of an innovative dissection method applied to the product $\zeta(u)\zeta(v)$. Heath-Brown [5] derived, irrelevant to [1], an asymptotic series for $\sum_{\chi(\bmod q)}|L_\chi(1/2)|^2$ (at the central point) as $q\to+\infty$. Motohashi [16] obtained, when $q=p$ is a prime, an asymptotic formula for $(p-1)^{-1}\sum_{\chi(\bmod p)}|L_\chi(1/2+it)|^2$ as $p\to+\infty$ with the error term $O(p^{-3/2})$, based Atkinson's dissection method. Matsumoto and the author [10] established a (ramified) asymptotic expansion for $\varphi(q)^{-1}\sum_{\chi(\bmod q)}|L_\chi(\sigma+it)|^2$ as $q\to+\infty$, in the stripe $0<\sigma<1$, which further implies, when $q=p$ is a prime, a complete asymptotic expansion for $(p-1)^{-1}\sum_{\chi(\bmod p)}|L_\chi(\sigma+it)|^2$ as $p\to+\infty$ through the set of primes, in the same region of σ above, based on Atkinson's dissection method. They [11] derived, taking the limit $\sigma+it\to1^-$ of the result above, a complete asymptotic expansion for $(p-1)^{-1}\sum_{\chi(\bmod p),\,\chi\neq\iota}|L_\chi(1)|^2$ as $p\to+\infty$ through the set of primes. The author [7] gave a quite transparent treatment of the same discrete mean squares by joining Atkinson's dissection method to the Mellin-Barnes type integrals, which appropriate to the relevant settings. The reader is to be referred, e.g. to [9, Sect. 2] for a more detailed history.

We now proceed to state our first main result. For this, let $\langle x\rangle=x-\lfloor x\rfloor$ denote the fractional part of $x\in\mathbb{R}$, and define the (exceptional) set $E\subset\mathbb{C}$ as

$$E=\bigl\{s\in\mathbb{C}\bigm|\operatorname{Re}s=1-n/2\text{ or }s=1-n\quad(n=0,1,\ldots)\bigr\}.\tag{2.2}$$

Theorem 1. *Let $c,q\in\mathbb{Z}$ and $\alpha,\lambda\in\mathbb{R}$ be arbitrary with $q\geq1$ and $\alpha\geq0$. Then for any integer $N\geq0$, in the region $-N+1<\sigma<N+1$ except the points $\sigma+it\in E$, we have*

$$\begin{aligned}\varphi(q)^{-1}&\sum_{\chi(\bmod q)}\bigl|L_{\chi_c}(\sigma+it,\alpha,\lambda)\bigr|^2\\&=q^{-2\sigma}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)k^{2\sigma}\zeta\Bigl(2\sigma,\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle\Bigr)\\&\quad+2q^{-2\sigma}\varphi(q)\Gamma(2\sigma-1)\operatorname{Re}\Bigl\{\zeta_\lambda(2\sigma-1)\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\Bigr\}\\&\quad+2q^{-2\sigma}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)\operatorname{Re}\bigl\{S_{c,q}(\sigma+it,\sigma-it;\alpha,\lambda;k)\bigr\},\end{aligned}\tag{2.3}$$

where k runs through all positive divisors of q, and $S_{c,q}$ is given by

$$\begin{aligned}S_{c,q}(u;v;\alpha,\lambda;k)=&\sum_{n=0}^{N-1}\frac{(-1)^n(u)_n}{n!}\zeta_\lambda(u+n)\zeta\Bigl(v-n,\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle\Bigr)k^{v-n}\\&+T_{c,q,N}(u,v;\alpha,\lambda;k).\end{aligned}$$

Here $T_{c,q,N}$ is the reminder expressed by the Mellin-Barnes type integral in (2.9) below, and bounded above as

$$\begin{aligned}&T_{c,q,N}(\sigma+it;\sigma-it;\alpha,\lambda;k)\\&\quad=\begin{cases}O\bigl\{k^{\sigma-N}(|t|+1)^{2N+1/2-\sigma}\bigr\}&\text{if }-N+1<\sigma<N,\\O\bigl\{k^{\sigma-N}(|t|+1)^{(3N+1-\sigma)/2+\varepsilon}\bigr\}&\text{if }N\leq\sigma<N+1\end{cases}\end{aligned}$$

for any $\varepsilon>0$, where the implied O-constants depend at most on σ, q, N and ε.

Remark. The presence of the error bounds above is reasonable, since the n-th indexed term in the asymptotic series is of order

$$\ll \begin{cases} k^{\sigma-n}(|t|+1)^{2n+1/2-\sigma} & \text{if } -n+1<\sigma<n, \\ k^{\sigma-n}(|t|+1)^{(3n+1-\sigma)/2+\varepsilon} & \text{if } n\le\sigma<n+1. \end{cases}$$

Let $\delta(x)$ is equal to 1 or 0 according to $x\in\mathbb{Z}$ or otherwise, and $\gamma_j(\alpha,\lambda)$ $(j=0,1,\dots)$ the j-th generalized Euler-Stieltjes constants defined by

$$\psi(s,\alpha,\lambda)=\frac{\delta(\lambda)}{s-1}+\sum_{j=0}^{\infty}\gamma_j(\alpha,\lambda)(s-1)^j$$

centered at $s=1$, where $\gamma_j=\gamma_j(0,0)=\gamma_j(1,0)$ $(j=0,1,\dots)$ are the classical Euler-Stieltjes constant (cf. [3, p.34, 1.12.(17)]). The asymptotic expansions on the exceptional set E (see (2.2)) can then be deduced from Theorem 1 by taking appropriate limits, e.g., the following formulae are valid.

Corollary 1.1. *Under the same settings as in Theorem 1, we have:*

i) *letting* $\sigma\to 1/2$,

$$\begin{aligned}
&\varphi(q)^{-1}\sum_{\chi(\bmod q)}\Bigl|L_{\chi_c}\Bigl(\frac12+it,\alpha,\lambda\Bigr)\Bigr|^2 \\
&=q^{-1}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)k\gamma_0\Bigl(\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle,0\Bigr) \\
&\quad+\frac{\varphi(q)}{q}\Biggl[\log q+\sum_{p|q}\frac{\log p}{p-1}+\mathrm{Re}\Bigl\{\zeta_\lambda'(0)-\zeta_\lambda(0)\frac{\Gamma'}{\Gamma}\Bigl(\frac12+it\Bigr)\Bigr\}+\gamma_0\Biggr] \\
&\quad+2q^{-1}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)\mathrm{Re}\Bigl\{S_{c,q}\Bigl(\frac12+it,\frac12-it;\alpha,\lambda;k\Bigr)\Bigr\};
\end{aligned}$$

ii) *letting* $\sigma\to 1$,

$$\begin{aligned}
&\varphi(q)^{-1}\sum_{\chi(\bmod q)}\bigl|L_{\chi_c}(1+it,\alpha,\lambda)\bigr|^2 \\
&=q^{-2}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)k^2\zeta\Bigl(2,\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle\Bigr) \\
&\quad+\frac{\varphi(q)}{q^2}\Biggl[-\delta(\lambda)\Bigl\{2\,\mathrm{Re}\,\frac{1}{it}\frac{\Gamma'}{\Gamma}(1+it)+\frac{1}{t^2}\Bigr\}+\frac{\mathrm{Im}\,\gamma_0(0,\lambda)}{t}\Biggr] \\
&\quad+2q^{-2}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)\mathrm{Re}\bigl\{S_{c,q}(1+it,1-it;\alpha,\lambda;k)\bigr\}.
\end{aligned}$$

Here k runs through all positive divisors of q, p through all prime divisors of q, and $S_{c,q}$ gives an asymptotic series as in Theorem 1.

The formula in Theorem 1 does not asserts (in a strict sense) a complete asymptotic expansion in the descending order of q itself; however it gives, if $q=p$ is a prime, a (true) complete asymptotic expansion in the descending order of p, since $S_{c,p}(u,v;\alpha,\lambda;1)$ can be computed explicitly.

Corollary 1.2. *Under the same settings as in Theorem 1, in the region* $-N+1<\sigma<N+1$ *except the points* $\sigma+it\in E$*, we have*

$$(2.4)\qquad (p-1)^{-1}\sum_{\chi(\bmod p)}\bigl|L_{\chi_c}(\sigma+it,\alpha,\lambda)\bigr|^2$$
$$=(1+p^{-2\sigma})\zeta(2\sigma,\alpha)-p^{-2\sigma}\zeta\Bigl(2\sigma,\frac{\alpha}{p}+\Bigl\langle-\frac{c}{p}\Bigr\rangle\Bigr)-p^{-2\sigma}\bigl|\psi(\sigma+it,\alpha,\lambda)\bigr|^2$$
$$+2p^{1-2\sigma}\Gamma(2\sigma-1)\operatorname{Re}\Bigl\{\zeta_\lambda(2\sigma-1)\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\Bigr\}$$
$$+2p^{-2\sigma}\operatorname{Re}\bigl\{S_{c,p}(\sigma+it,\sigma-it;\alpha,\lambda;p)\bigr\},$$

whose limiting case $\sigma\to 1/2$ *asserts*

$$(2.5)\qquad (p-1)^{-1}\sum_{\chi(\bmod p)}\Bigl|L_{\chi_c}\Bigl(\frac12+it,\alpha,\lambda\Bigr)\Bigr|^2$$
$$=\bigl(1+p^{-1}\bigr)\gamma_0(\alpha,0)-p^{-1}\gamma_0\Bigl(\frac{\alpha}{p}+\Bigl\langle-\frac{c}{p}\Bigr\rangle,0\Bigr)$$
$$-2\operatorname{Re}\Bigl\{\zeta_\lambda'(0)+\zeta_\lambda(0)\frac{\Gamma'}{\Gamma}\Bigl(\frac12+it\Bigr)\Bigr\}+\gamma_0+\log p-p^{-1}\Bigl|\psi\Bigl(\frac12+it,\alpha,\lambda\Bigr)\Bigr|^2$$
$$+2p^{-1}\operatorname{Re}\Bigl\{S_{c,p}\Bigl(\frac12+it;\frac12-it;\alpha,\lambda;p\Bigr)\Bigr\}.$$

Here the term $S_{c,p}$*, both in (2.4) and (2.5), gives a complete asymptotic expansion in the descending order of* p *as* $p\to+\infty$ *through the set of primes.*

We now proceed to outline of the proof Theorem 1. For this, we set

$$R(u,v;\lambda)=\Gamma(u+v-1)\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)},$$

use a (modified) Möbius' inversion

$$\sideset{}{'}\sum_{\substack{l=0\\(c+l,q)=1}}^{\infty}\frac{1}{(\alpha+l)^s}=q^{-s}\sum_{k|q}\mu\Bigl(\frac{q}{k}\Bigr)k^s\zeta\Bigl(s,\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle\Bigr)\qquad(\sigma>1),$$

and write

$$\Gamma\binom{\alpha_1,\dots\alpha_m}{\beta_1,\dots,\beta_n}=\frac{\prod_{h=1}^m\Gamma(\alpha_h)}{\prod_{k=1}^n\Gamma(\beta_k)}$$

for $\alpha_h, \beta_k \in \mathbb{C}$ ($h = 1, \ldots, m; k = 1, \ldots, n$). The dissection formula, for $\mathrm{Re}(u) > 1$ and $\mathrm{Re}(v) > 1$,

$$\begin{aligned}
&\varphi(q)^{-1} \sum_{\chi(\bmod q)} L_{\chi_c}(u, \alpha, \lambda) L_{\overline{\chi}_c}(v, \alpha, -\lambda) \\
&\quad = q^{-u-v} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) k^{u+v} \zeta\Big(u+v, \frac{\alpha k}{q} + \Big\langle -\frac{ck}{q} \Big\rangle\Big) + f_{c,q}(u, v; \alpha, \lambda) \\
&\qquad + f_{c,q}(v, u; \alpha, -\lambda),
\end{aligned}$$

is crucial in proving Theorem 1, where $f_{c,q}$ is a variant of Euler's double zeta-function (see [1] for the case of $\zeta(u)\zeta(v)$). This further splits into

$$f_{c,q}(u, v; \alpha, \lambda) = q^{-u-v} \varphi(q) R(u, v; \lambda) + g_{c,q}(u, v; \alpha, \lambda),$$

where $g_{c,q}$ is given by

$$g_{c,q}(u, v; \alpha, \lambda) = q^{-u-v} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) S_{c,q}(u, v; \alpha, \lambda; k)$$

with $S_{c,q}$ being expressed as the Mellin-Barnes type integral of the form

$$\begin{aligned}
S_{c,q}(u, v; \alpha, \lambda; k) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma\binom{u+s, -s}{u} \zeta_\lambda(-s) \\
&\qquad \times \zeta\Big(u+v+s, \frac{\alpha k}{q} + \Big\langle -\frac{ck}{q} \Big\rangle\Big) k^{u+v+s} ds,
\end{aligned} \tag{2.6}$$

where the path $\mathcal{C}$ separates the poles of the integrand at $s = 1-u-v$ and $s = -1+m$ ($m = 0, 1, \ldots$) from those at $s = -u-n$ ($n = 0, 1, \ldots$).

We proceed further along the lines above, moving appropriately the path $\mathcal{C}$ to the left, and eventually obtain the formula (2.8) below, which yields, upon setting $(u, v) = (\sigma+it, \sigma-it)$, various *complete* asymptotic expansions for the discrete mean square (2.1). Let (σ) for any $\sigma \in \mathbb{R}$ denote the vertical straight path from $\sigma - i\infty$ to $\sigma + i\infty$, and define the (exceptional) set $\widetilde{E} \subset \mathbb{C}^2$ as

$$\widetilde{E} = \Big\{(u, v) \in \mathbb{C}^2 \;\Big|\; u+v = 2-n \text{ or } u = 1-n \text{ or } v = 1-n \;(n = 0, 1, 2, \ldots)\Big\}. \tag{2.7}$$

We can then show the following formula. For any integer $N \geq 0$, in the region $-N+1 < \mathrm{Re}(u) < N+1$ and $-N+1 < \mathrm{Re}(v) < N+1$ except the points $(u, v) \in \widetilde{E}$, we have

$$\begin{aligned}
&\varphi(q)^{-1} \sum_{\chi(\bmod q)} L_{\chi_c}(u, \alpha, \lambda) L_{\overline{\chi}_c}(v, \alpha, -\lambda) \\
&\quad = q^{-u-v} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) k^{u+v} \zeta\Big(u+v, \frac{\alpha k}{q} + \Big\langle -\frac{ck}{q} \Big\rangle\Big) + q^{-u-v} \varphi(q) \big\{R(u, v; \lambda) \\
&\qquad + R(v, u; -\lambda)\big\} + q^{-u-v} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) \big\{S_{c,q}(u, v; \alpha, \lambda; k) + S_{c,q}(v, u; \alpha, -\lambda; k)\big\},
\end{aligned} \tag{2.8}$$

where $S_{c,q}$ is expressed as

$$\begin{aligned}S_{c,q}(u,v;\alpha,\lambda;k)=\sum_{n=0}^{N-1}\frac{(-1)^n(u)_n}{n!}\zeta_\lambda(u+n)\zeta\Bigl(v-n,\frac{\alpha k}{q}+\Bigl\langle\frac{ck}{q}\Bigr\rangle\Bigr)k^{v-n}\\+T_{c,q,N}(u,v;\alpha,\lambda;k),\end{aligned}$$

and $T_{c,q,N}$ is given by the Mellin-Barnes type integral

$$\begin{aligned}T_{c,q,N}(u,v;\alpha,\lambda;k)=\frac{1}{2\pi i}\int_{(\sigma_N)}\Gamma\binom{u+s,-s}{u}\zeta_\lambda(-s)\\\times\zeta\Bigl(u+v+s,\frac{\alpha k}{q}+\Bigl\langle-\frac{ck}{q}\Bigr\rangle\Bigr)k^{u+v+s}ds\end{aligned}\tag{2.9}$$

with σ_N satisfying $-\operatorname{Re}(u)-N<\sigma_N<\min(-\operatorname{Re}(u)-N+1,-1,-\operatorname{Re}(u+v))$. Theorem 1 is in fact a direct consequence of (2.8) upon setting $(u,v)=(\sigma+it,\sigma-it)$.

3. Asymptotic expansions for the continuous mean square

The chief concern in this section is the asymptotic expansions for the continuous mean square

$$\int_0^1\bigl|L_{\chi_c}(\sigma+it,\alpha+q\xi,\lambda)\bigr|^2d\xi.\tag{3.1}$$

We give here an overview of the results on (3.1), mainly when $(q,\chi)=(1,\iota)$, $(\alpha,\lambda)=(1,0)$, i.e. the case of the continuous mean square of $\zeta(s,1+\xi)$. Koksma-Lekkerkerker [14] initiated the study into the direction to obtain the asymptotic bound $O(\log t)$ as $t\to+\infty$ on the critical line $\sigma=1/2$. Subsequent research are made by Gallagher [4], Balasubramanian [2], Rane [17], Klush [15], Zhang [18][19]. Matsumoto and the author [11] established, for the case above, a complete asymptotic expansion in the descending order of $\operatorname{Im}s=t$ as $t\to+\infty$, by means of Atkinson's dissection method. The author [6] derived a complete asymptotic expansion when $(q,\chi)=(1,\iota)$ and $\alpha=1$, i.e. for the case of the continuous mean square of $\phi(s,1+\xi,\lambda)$, by means of Atkinson's dissection method enhanced by Mellin-Barnes type integrals. The author [8] established a complete asymptotic expansion, when $(q,\chi)=(1,\iota)$, for the multiple mean square of the form

$$\int_0^1\cdots\int_0^1\bigl|\phi(s,\alpha+\xi_1+\cdots+\xi_m,\lambda)\bigr|^2d\xi_1\cdots d\xi_m\quad(m=1,2,\ldots)$$

in the descending order of $\operatorname{Im}s=t$ as $t\to\pm\infty$, by means of Atkinson's dissection method, enhanced by Mellin-Barnes type integrals. These are further manipulated with several properties of (generalized) hypergeometric functions. The reader is to be referred, e.g. to [9, Sect. 3] for a more detailed history.

We proceed to state our second main result. We set, for any $(u,v)\in\mathbb{C}^2\setminus\widetilde{E}$ (see (2.7)),

$$\mathcal{R}_{\chi_c}(u,v;\lambda)=\Gamma(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)}\sum_{a,b=0}^{q-1}\chi_c(1+a+b)\overline{\chi}_c(b)\psi\Bigl(u+v-1,\frac{1+a}{q},\lambda\Bigr).$$

Theorem 2. *Let $c, q \in \mathbb{Z}$ and $\alpha, \lambda \in \mathbb{R}$ be arbitrary with $q \geq 1$ and $\alpha \geq 0$. Then for any integer $N \geq 0$, in the region $-N+1 < \sigma < N+1$ except the points $\sigma + it \in E$ (see (2.2)), we have*

$$(3.2)\quad \int_0^1 \left|L_{\chi_c}(\sigma+it, \alpha+q\xi, \lambda)\right|^2 d\xi$$
$$= -\frac{q^{-2\sigma}}{1-2\sigma} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) k^{2\sigma} \Big(\frac{\alpha k}{q} + \Big\langle -\frac{ck}{q} \Big\rangle\Big)^{1-2\sigma} + 2q^{-2\sigma} \operatorname{Re}\big\{\mathcal{R}_{\chi_c}(\sigma+it, \sigma-it; \lambda)\big\}$$
$$- 2q^{-2\sigma} \operatorname{Re}\big\{\mathcal{S}_{\chi_c,N}(\sigma+it, \sigma-it; \alpha, \lambda) + \mathcal{T}_{\chi_c,N}(\sigma+it, \sigma-it; \alpha, \lambda)\big\},$$

where $\mathcal{S}_{\chi_c,N}$ and $\mathcal{T}_{\chi_c,N}$ are given by

$$\mathcal{S}_{\chi_c,N}(u, v; \alpha, \lambda) = \sum_{a,b=0}^{q-1} \chi_c(1+a+b)\overline{\chi}_c(b)\mathcal{S}_N\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right),$$
$$\mathcal{T}_{\chi_c,N}(u, v; \alpha, \lambda) = \sum_{a,b=0}^{q-1} \chi_c(1+a+b)\overline{\chi}_c(b)\mathcal{T}_N\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right)$$

with

$$(3.3)\qquad \mathcal{S}_N(u, v; x, y, \lambda) = \sum_{n=0}^{N-1} \frac{(u)_n y^{n+1-v}}{(1-v)_{n+1}} e(-y\lambda)\psi(u+n; x+y, \lambda),$$

$$(3.4)\qquad \mathcal{T}_N(u, v; x, y, \lambda) = \frac{(u)_N y^{N+1-v}}{(1-v)_N} {\sum_{l=0}^{\infty}}' \frac{e\{(x+l)\lambda\}}{(x+l)^{u+v-1}} \int_{x+l}^{\infty} \frac{\eta^{u+v-2}}{(y+\eta)^{u+N-1}} d\eta.$$

Note that the last expression converges absolutely for $\operatorname{Re}(u) > -N+1$ and $\operatorname{Re}(v) < N+1$. This further asserts

$$(3.5)\quad \mathcal{T}_N(u, v; x, y, \lambda)$$
$$= y^{N+1-v} \Bigg[\sum_{k=1}^{K} \frac{(-1)^{k-1}(2-u+v)_{k-1}(u)_{N-k}}{(1-v)_N} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+l)^{k+1-u+v}(x+y+l)^{u+N-k}}$$
$$+ \frac{(-1)^K(2-u+v)_K(u)_{N-K}}{(1-v)_N} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+l)^{u+v-1}} \int_{x+l}^{\infty} \frac{y^{u+v-K-2}}{(y+\eta)^{u+N-K}} d\eta \Bigg],$$

which gives upon $(u, v) = (\sigma+it, \sigma-it)$ the asymptotic expansion in the descending order of $\operatorname{Im} s = t$ as $t \to \pm\infty$.

We proceed to outline the proof of Theorem 2. The dissection formula, for $\operatorname{Re}(u) > 1$ and $\operatorname{Re}(v) > 1$,

$$L_{\chi_c}(u, \alpha, \lambda)L_{\overline{\chi}_c}(v, \alpha, -\lambda)$$
$$= q^{-u-v} \sum_{k|q} \mu\Big(\frac{q}{k}\Big) k^{u+v} \zeta\Big(u+v, \frac{\alpha k}{q} + \Big\langle -\frac{ck}{q} \Big\rangle\Big) + f_{\chi_c}(u, v; \alpha, \lambda) + f_{\overline{\chi}_c}(v, u; \alpha, -\lambda)$$

is crucial in proving Theorem 2, where f_{χ_c} (or $f_{\overline{\chi}_c}$) is a variant of Euler's double zeta-function (see [1] for the case of $\zeta(u)\zeta(v)$). This further splits into

$$f_{\chi_c}(u,v;\alpha,\lambda)=q^{-u-v}\mathcal{R}_{\chi_c}(u,v;\lambda)+g_{\chi_c}(u,v;\alpha,\lambda),$$

where

$$g_{\chi_c}(u,v;\alpha,\lambda)=q^{-u-v}\sum_{a,b=0}^{q-1}\chi_c(1+a+b)\overline{\chi}_c(b)g\biggl(u,v;\frac{1+a}{q},\frac{\alpha+b}{q},\lambda\biggr)$$

with

$$g(u,v;x,y,\lambda)=\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma\binom{u+s,-s}{u}\psi(-s,x,\lambda)\zeta(u+v+s,y)ds,$$

where $\mathcal{C}$ is the same contour as in (2.6).

We suppose now that $\mathrm{Re}(u)>1$ and $\mathrm{Re}(v)<1$, under which the path $\mathcal{C}$ can be taken as a straight line $\mathcal{C}=(\sigma_0)$ with σ_0 satisfying $-\mathrm{Re}(u)<\sigma_0<\min(-1,1-\mathrm{Re}(u+v))$. It suffices, for the treatment of the continuous mean square (3.1), to evaluate the integral

$$\int_0^1 g_{\chi_c}(u,v;\alpha+q\xi,\lambda)d\xi=-q^{-u-v}\sum_{a,b=0}^{q-1}\chi_c(1+a+b)\overline{\chi}_c(b)\widetilde{g}\Bigl(u,v;\frac{1+a}{q},\frac{\alpha+b}{q},\lambda\Bigr),$$

say, where the relation, for any complex $s\neq 1$,

$$\int_0^1\zeta(s,y+\xi)d\xi=-\frac{y^{1-s}}{1-s}$$

(cf. [8, Lemma 2]) is used to integrate the Mellin-Barnes type expression of $g(u,v;x,y,\lambda)$. This gives

$$\widetilde{g}(u,v;x,y,\lambda)=\frac{1}{2\pi i}\int_{(\sigma_0)}\Gamma\binom{u+s,-s}{u}\psi(-s,x,\lambda)\frac{y^{1-u-v-s}}{1-u-v-s}ds.$$

Note further that the Mellin-Barnes formula, for $0<\mathrm{Re}(z)<\mathrm{Re}(w)$,

$$\frac{1}{w-z}=\frac{1}{2\pi i}\int_{(\rho)}\Gamma\binom{z+r,w,1+r,-r}{z,w+1+r}e^{\pi ir}dr$$

holds with a constant ρ satisfying $\max(-\mathrm{Re}\,z,-1)<\rho<0$ (cf. [6, Lemma 3]). This upon $z=u+s$ and $w=1-v$ is substituted into the denominator factor $1/(1-u-v-s)$ above to transform the s-integral expression of $\widetilde{g}$ as

$$\widetilde{g}(u,v;x,y,\lambda)=\frac{1}{2\pi i}\int_{(\rho_0)}\Gamma\binom{u+r,1-v,1+r,-r}{u,2-v+r}e^{\pi ir}e(-y\lambda)\psi(u+r,x,\lambda)y^{1-v+r}dr$$

with ρ_0 satisfying $\max(-\mathrm{Re}(u),-\sigma_0,-1)<\rho_0<0$. Moving the path (ρ_0) to the right appropriately, we obtain

$$\widetilde{g}(u,v;x,y,\lambda)=\mathcal{S}_N(u,v;x,y,\lambda)+\mathcal{T}_N(u,v;x,y,\lambda)$$

with the expression in (3.3) and

$$\mathcal{T}_N(u,v;x,y,\lambda)=\frac{(u)_N y^{N+1-v}}{(1-v)_{N+1}}\sum_{l=0}^{\infty}\frac{e\{(x+l)\lambda\}}{(x+y+l)^{u+N}}{}_2F_1\Bigl({u+N,1 \atop N+2-v};\frac{y}{x+y+l}\Bigr), \tag{3.6}$$

where ${}_2F_1$ denotes Gauß' hypergeometric function (cf. [3, p.59, 2.1.1(12)]). This is further transformed, through Euler's formula for ${}_2F_1$ (cf. [3, p.59, 2.1.3.(10)]), to imply (3.4). The asymptotic expansion in (3.5) for $\mathcal{T}_N$ is obtained by substituting the formula, coming from a repeated use of a contiguity relation of ${}_2F_1$ (cf. [3, p.103, 2.8.(37)]),

$$\frac{(u)_N}{(1-v)_{N+1}}{}_2F_1\Big(\begin{matrix}u+N,1\\N+2-v\end{matrix};Z\Big)=\sum_{k=1}^{K}\frac{(-1)^{k-1}(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_{N+1}}(1-Z)^{-k}$$
$$+\frac{(-1)^K(2-u-v)_K(u)_{N-K}}{(1-v)_{N+1}}(1-Z)^{-K}{}_2F_1\Big(\begin{matrix}u+N-K,1\\N+2-v\end{matrix};Z\Big)$$

with $Z=y/(x+y+l)$ into each term of the series expression of $\mathcal{T}_N$ in (3.6) to yield (3.5).

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