CONJECTURES OF DIOPHANTINE EQUATIONS ON PIATETSKI-SHAPIRO SEQUENCES

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Abstract. Let $\alpha > 1$ be a non-integral real number. Let $\mathrm{PS}(\alpha)$ be the set of positive integers of the form $\lfloor n^\alpha \rfloor$ for some $n \in \mathbb{N}$. In this article, we discuss the equation $x + y = z$, where $(x, y, z) \in \mathrm{PS}(\alpha)^3$. The author conjectures that for almost all or all $2 < \alpha < 3$ the equation $x + y = z$ has infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$. In this article, we aim to present heuristic and numerical evidence of the conjecture.

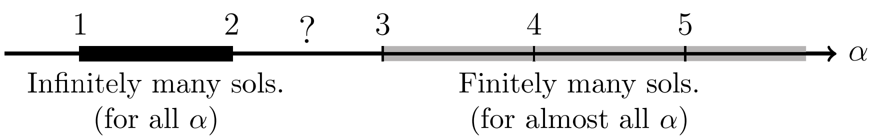
1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. We say that an integral sequence $(a_n)_{n\in\mathbb{N}}$ is a Piatetski-Shapiro sequence if there exists a non-integral $\alpha > 1$ such that $a_n = \lfloor n^\alpha \rfloor$ for all $n \in \mathbb{N}$. Further, we define $\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$. In this article, we discuss the following equation:

$$x + y = z, \tag{1.1}$$

where $(x, y, z) \in \mathrm{PS}(\alpha)^3$. The equation (1.1) is an extension of Fermat's equation ($x^n + y^n = z^n$) from the integral powers to real powers of integers. By Fermat's last theorem, the equation (1.1) does not have any solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$ if $\alpha \geq 3$ and $\alpha \in \mathbb{N}$.

By using the result given by Frantzikinakis and Wierdl [FW09, Proposition 5.1], for every $\alpha \in (1, 2)$, there are infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$ to (1.1). Further, the author [Sai] showed that for almost all $\alpha > 3$ in the sense of the 1-dimensional Lebesgue measure, the equation (1.1) has at most finitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$. We describe these results in the following figure.



For all real numbers s, t with $1 < s < t$, we define

$$\mathcal{A}(s,t) = \{\alpha \in [s,t] \colon (1.1) \text{ has infinitely many solutions } (x,y,z) \in \mathrm{PS}(\alpha)^3\}. \tag{1.2}$$

Matsusaka and the author [MS21, Theorem 1.1] showed that for all real numbers s, t with $2 < s < t$, the Hausdorff dimension of $\mathcal{A}(s,t)$ is not less than $1/s^3$. In particular, $\mathcal{A}(s,t)$ is dense in $[s,t]$. However, we do not know whether $\mathcal{A}(2,3)$ has positive Lebesgue measure or not. Here, the author conjectures the following.

Conjecture 1.1. *For almost all or all $\alpha \in (2,3)$, the equation* (1.1) *would have infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$.*

The goal of this article is to give heuristic and numerical observations on Conjecture 1.1. This problem was first proposed by Glasscock [Gla17]. He investigated the equation

$$y = ax + b, \tag{1.3}$$

where a and b are fixed real numbers satisfying that (1.3) has infinitely many solutions $(x, y) \in \mathbb{N}^2$. In [Gla17, Gla20], he showed that for almost all $\alpha > 1$

(1) if $\alpha < 2$, then (1.3) has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$;
(2) if $\alpha > 2$, then (1.3) has at most finitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$.

The author also contributed to the solvability of (1.3) on $\mathrm{PS}(\alpha)$. In [Sai22], if we additionally assume $0 \le b < a$, then the author showed that for all $1 < \alpha < 2$, we find infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$ to (1.3). Thus, he improved (1) of Glasscock's result from "for almost all" to "for all" in the case $1 < \alpha < 2$. Moreover, in the same article, the author showed that for all $2 < s < t$, the Hausdorff dimension of the set

$$\{\alpha \in [s, t] \colon \text{(1.3) has infinitely many } (x, y) \in \mathrm{PS}(\alpha)^2\}$$

is coincident with $2/s$. Without the assumption $0 \le b < a$, it is unknown whether his improvements are true or not.

Glasssock successfully discovered that $\alpha = 2$ is a threshold for the infiniteness or finiteness of solutions to (1.3) on $\mathrm{PS}(\alpha)$. Further, he proposed the following interesting question.

Question 1.2 ([Gla17, Question 6])**.** *Does there exist $\mathscr{G} > 1$ with the property that for almost every or all $\alpha > 1$, the equation $x + y = z$ has infinitely many solutions $(x, y, z) \in \mathrm{PS}(\alpha)^3$ or not, according as $\alpha < \mathscr{G}$ or $\alpha > \mathscr{G}$?*

The result of Matsusaka and the author [MS21, Theorem 1.1] leads to a negative answer to the "all" version of Question 1.2. The "almost all" version is still open, which is directly connected with Conjecture 1.1

Notation 1.3. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. For all intervals $I \subset \mathbb{R}$, let $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. We write $f(x) = g(x) + o(h(x))$ if $(f(x) - g(x))/h(x) \to 0$ as $x \to \infty$, where $h(x)$ is a positive-valued function.

2. Heuristic observations

For every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$s(\alpha, N) = \{(\lfloor p^\alpha \rfloor, \lfloor q^\alpha \rfloor) \colon p, q \in \mathbb{N},\ p \le q \le N,\ \lfloor p^\alpha \rfloor + \lfloor q^\alpha \rfloor = \lfloor r^\alpha \rfloor \text{ for some } r \in \mathbb{N}\}.$$

In this section, by heuristic calculation, we propose the following conjecture. If Conjecture 2.1 was true for every $2 < \alpha < 3$, then Conjecture 1.1 would be also true.

Conjecture 2.1. *For almost all or all $\alpha \in (2, 3)$, we would have*

$$\#s(\alpha, N) = \left(\frac{C(\alpha)}{\alpha(3-\alpha)} + o(1) \right) N^{3-\alpha} \quad (\text{as } N \to \infty),$$

where $C(\alpha) := \int_0^1 (1 + x^\alpha)^{1/\alpha - 1}\, dx$.

Yoshida [Yos, Corollary 1.2] gave a quantitative result on the number of solutions (1.1). For all $1 < \alpha < (\sqrt{21} + 4)/5 = 1.7165\cdots$, he showed that

$$\#\{(\ell, m, n) \in \mathbb{N}^3 \colon \ell < x, \lfloor \ell^\alpha \rfloor + \lfloor m^\alpha \rfloor = \lfloor n^\alpha \rfloor\} = (\beta\alpha^{-\beta}\zeta(\beta) + o(1))x^{\alpha(\beta-1)+1} \tag{2.1}$$

as $x \to \infty$, where $\beta = 1/(\alpha - 1)$ and $\zeta(s)$ denotes the Riemann zeta function. Note that the left-hand side of (2.1) is slightly different from $\#s(\alpha, N)$.

Heuristic evidence of Conjecture 2.1. By the definition of $s(\alpha, N)$, it follows that

$$\#s(\alpha,N)=\sum_{1\le q\le N}\sum_{1\le p\le q}\#[A,B)_{\mathbb{Z}},$$

where $A=(\lfloor p^\alpha\rfloor+\lfloor q^\alpha\rfloor)^{1/\alpha}$ and $B=(\lfloor p^\alpha\rfloor+\lfloor q^\alpha\rfloor+1)^{1/\alpha}$. We now assume that

$$\#[A,B)_{\mathbb{Z}}=B-A. \tag{2.2}$$

By $0<B-A<1$, the assumption (2.2) never holds. We ignore this problem to calculate heuristically, which is not a small gap.

By ignoring the integral parts and applying Taylor's expansion, we see that

$$B-A\sim\frac{1}{\alpha}(p^\alpha+q^\alpha)^{1/\alpha-1}=\frac{q^{2-\alpha}}{\alpha}\left(1+\left(\frac{p}{q}\right)^\alpha\right)^{1/\alpha-1}\frac{1}{q}.$$

Remark that we do not give the precise definition of "$X\sim Y$". The symbol means that X and Y share the almost same magnitude. By Riemann sum, we obtain

$$\lim_{q\to\infty}\sum_{1\le p\le q}\left(1+\left(\frac{p}{q}\right)^\alpha\right)^{1/\alpha-1}\frac{1}{q}=\int_0^1(1+x^\alpha)^{1/\alpha-1}\,dx=C(\alpha).$$

Therefore, we can expect that

$$\#s(\alpha,N)\sim\frac{1}{\alpha}\sum_{1\le q\le N}\sum_{1\le p\le q}(p^\alpha+q^\alpha)^{1/\alpha-1}\sim\frac{C(\alpha)}{\alpha}\sum_{1\le q\le N}q^{2-\alpha}\sim\frac{C(\alpha)}{\alpha(3-\alpha)}N^{3-\alpha}.$$

□

By the above calculation, we should consider the set of $(p,q)\in\mathbb{N}^2$ such that $[A,B)_{\mathbb{Z}}$ is non-empty. However, the interval $[A,B)$ would be too short to be investigated.

3. Numerical observations

In this section, we give numerical evidence on Conjectures 1.1 and 2.1. The author calculates $\#s(\alpha,N)$ for $N=50000$ and $\alpha=2.00+0.05i$ $(i=1,2,\dots,19)$ by using Mathematica. The used program will be described at the end of this article. Mathematica outputs the following.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\#s(\alpha,5\cdot10^5)$	13035	7754	4763	2739	1653	1060	647	356	247	153

α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	-
$\#s(\alpha,5\cdot10^5)$	93	69	41	22	17	7	4	0	1	-

From the table, we do not guess the existence of a threshold like Glasscock's result on (1.3). Let us note that $N=50000$ may be still small. Indeed by numerical calculation, for every $\alpha\in\{2.8,\ 2.85,\ 2.9,\ 2.95\}$, we obtain

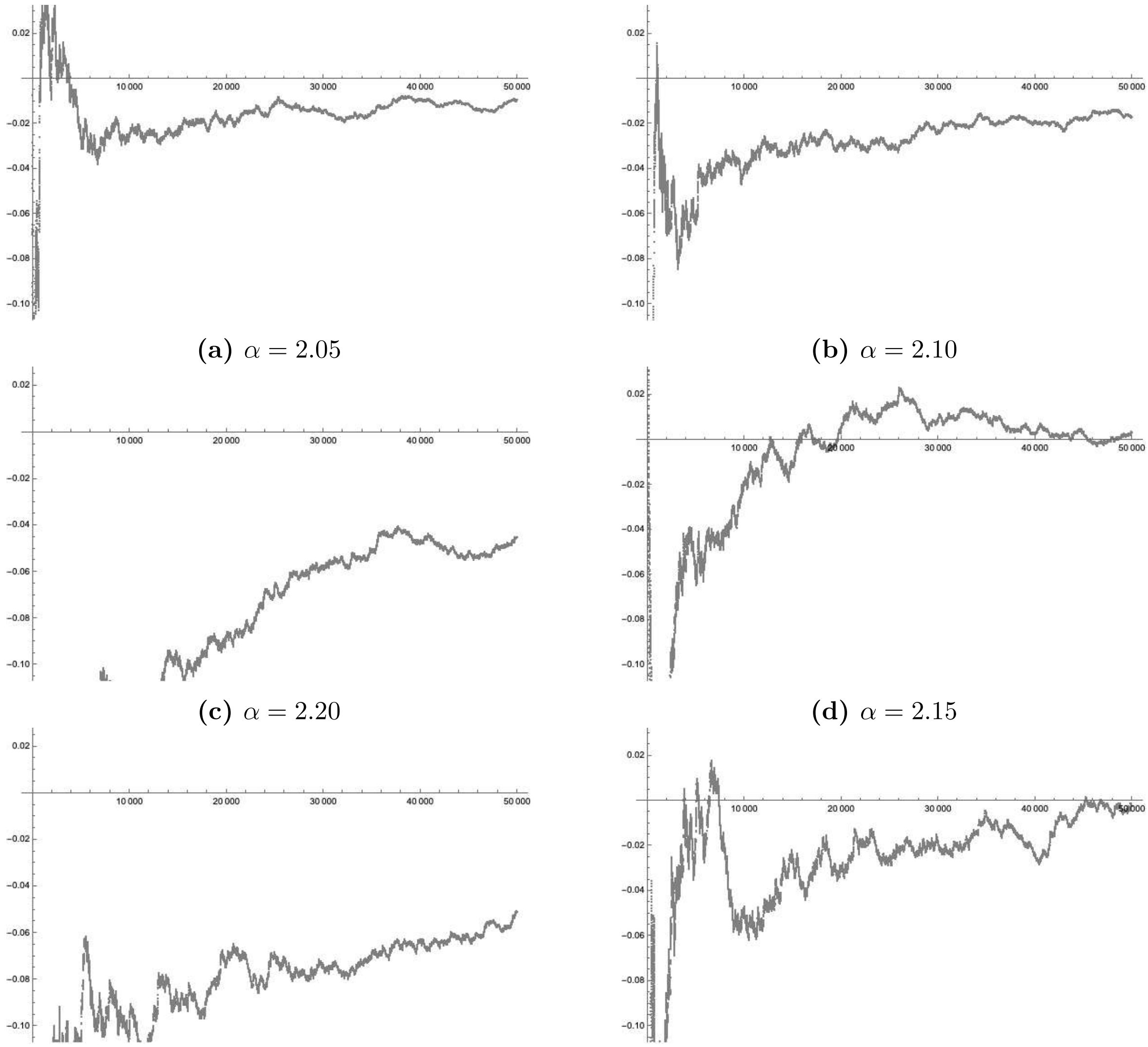
$$\frac{C(\alpha)}{\alpha(3-\alpha)}50000^{3-\alpha}\le\frac{C(2.8)}{2.8(3-2.8)}50000^{3-2.8}=13.70\cdots. \tag{3.1}$$

Thus, it is too small to guess whether Conjecture 1.1 is true or not. In the case $\alpha=2.95$, if we chose $N\ge1.671\times10^{59}$, then the most left-hand side of (3.1) would be greater than 1000.

On the other hand, we expect that Conjecture 2.1 is true for small $\alpha > 2$. Indeed, for every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$\delta(\alpha, N) = \#s(\alpha, N) \cdot \frac{\alpha(3-\alpha)}{C(\alpha)} N^{\alpha-3} - 1.$$

If $\delta(\alpha, N) \to 0$ as $N \to \infty$, then Conjecture 2.1 would be true for α. The following figure describes $\{(N, \delta(\alpha, N)) \mid N = 1, 2, \dots, 50000 \text{ and } \delta(\alpha, N) \in [-0.10, 0.02]\}$ for every $\alpha \in \{2.05,\ 2.10,\ 2.15,\ 2.20,\ 2.25,\ 2.30\}$.



(a) $\alpha = 2.05$ **(b)** $\alpha = 2.10$

(c) $\alpha = 2.20$ **(d)** $\alpha = 2.15$

(e) $\alpha = 2.25$ **(f)** $\alpha = 2.30$

Figure 1. Graphs of $\delta(\alpha, 50000)$

We also give the following table of $\delta(\alpha, 5\cdot 10^5)10^2$ to exhibit the calculation more clearly. Remark that it is not $\delta(\alpha, 5 \cdot 10^5)$, but $\delta(\alpha, 5 \cdot 10^5) \times 10^2$.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\delta(\alpha, 5\cdot 10^5)10^2$	-0.970	-1.73	0.303	-4.55	-5.10	-0.258	-0.781	-0.116	-1.43	-2.73

α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	-
$\delta(\alpha, 5\cdot 10^5)10^2$	-6.79	7.62	-2.07	-21.2	-11.2	-48.9	-61.7	-100	-90.3	-

Thus, $\delta(\alpha, N)$ seems small for several α's. This supports Conjecture 2.1. Moreover, we observe that almost all values in the table are negative. Thus, we expect that almost every (or every) $\alpha \in (2,3)$ would satisfy that

$$\#s(\alpha, N) \leq \frac{C(\alpha)}{\alpha(3-\alpha)} N^{3-\alpha}$$

for sufficiently large N. At last, we give a program to calculate $\#s(\alpha, N)$ in Mathematica.

```
ps[alpha_,n_]:= Floor[n^alpha];
................................................................................
For[alpha = 2.05, alpha <= 2.95, alpha = alpha + 0.05,
  sols = {}; gamma = 1/alpha;
For[p = 1, p <= 50000, p = p + 1, For[q = p, q <= 50000, q = q + 1,
    sum = ps[alpha,p] + ps[alpha,q]; r = Ceiling[sum^gamma];
  If[sum == ps[alpha,r],
    sols = Join[sols, {{p,q}}];
];];];
Print[{alpha, Length[sols]}]; Print[sols];]
```

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