ALL ABOUT MURMURATIONS

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1. Introduction

These notes are a rough transcript of my talk at the conference on Analytic Number Theory and Related Topics at RIMS in October 2023, reporting on joint work with Jonathan Bober, Min Lee, and David Lowry-Duda. See [BBLLD23] for a more formal account of our work. I thank RIMS for their hospitality and the organizers, Yu Yasufuku and Maki Nakasuji, for the invitation, financial support, and their (near-)infinite patience.

1.1. **Murmurations?** The murmurations craze is a phenomenon that has taken the analytic number theory world by storm over the past year. Consider the following graph, which shows, for each prime $p \in [2, 7919]$, the average of

$$a_p = p + 1 - \#E(\mathbb{F}_p),$$

where E ranges over elliptic curves of conductor $N \in [7500, 10000]$ (up to isogeny), of fixed rank (blue = rank 0, red = rank 1).

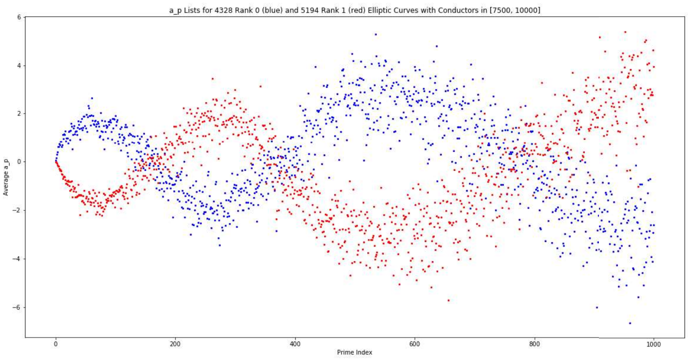


Figure 1

This pattern was discovered in 2022 by Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov [HLOP22] using *machine learning algorithms*. They dubbed it "murmurations of elliptic curves". (A *murmuration* is a wave-like pattern made by flocks of certain species of birds, notably starlings.)

1.2. **The plan.** The plan for the rest of the paper is as follows:

- In §2 I will give some background on elliptic curves over $\mathbb{Q}$ that explains why it isn't surprising that there is *some* pattern like this.
- Then in §3 we will see the specific features of the murmuration pattern that make it surprising and defied explanation for a while. I will try to demystify the phenomenon and explain what's actually going on through a series of reductions.
- Finally, in §4 I will conclude with some results that we can prove and a few words about the proofs.

2. Background

2.1. **Elliptic curves over $\mathbb{Q}$ (and their torsion).** First, we have Mordell–Weil theorem, which describes the structure of the group of rational points of an elliptic curve:

Theorem (Mordell, 1922; generalized by Weil in 1929)**.** *The group of rational points of an elliptic curve $E/\mathbb{Q}$ is finitely generated: $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}}$.*

This was refined by Mazur, who classified all possibilities for the torsion part:

Theorem (Mazur, 1978)**.** *$E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of the following groups:*

$$\mathbb{Z}/m\mathbb{Z} \text{ for } m \in \{1, \ldots, 10, 12\}; \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z} \text{ for } m \in \{1, \ldots, 4\}.$$

Each of these groups occurs for infinitely many $E/\mathbb{Q}$.

Moreover, there is an efficient algorithm to compute the torsion part of any given elliptic curve using the Lutz–Nagell theorem. A key step in the proof is the following:

Theorem (Lutz–Nagell, 1937)**.** *If $E/\mathbb{Q}$ has good reduction at a prime p then the reduction map $E(\mathbb{Q})_{\text{tors}} \to E(\mathbb{F}_p)$ is injective.*

Loosely speaking, this says we can "see" the torsion of $E(\mathbb{Q})$ in the reduced curves $E(\mathbb{F}_p)$. (That is not to say that the groups $E(\mathbb{F}_p)$ determine $E(\mathbb{Q})$, though there is a sense in which that is true up to isogeny.)

2.2. **Elliptic curves over $\mathbb{Q}$ (and their ranks).** It seems natural to ask whether the same is true for rank, i.e. does the rank have any influence on $\#E(\mathbb{F}_p)$ for primes p? That is precisely the subject of the BSD conjecture:

Conjecture (Birch and Swinnerton-Dyer, 1965)**.** *Suppose $E/\mathbb{Q}$ has rank r and discriminant Δ. Then there exists $C_E > 0$ such that*

$$\prod_{\substack{p \leq x \\ p \nmid \Delta}} \frac{\#E(\mathbb{F}_p)}{p} = \big(C_E + o(1)\big)(\log x)^r \quad \text{as } x \to \infty.$$

If the BSD conjecture is true then

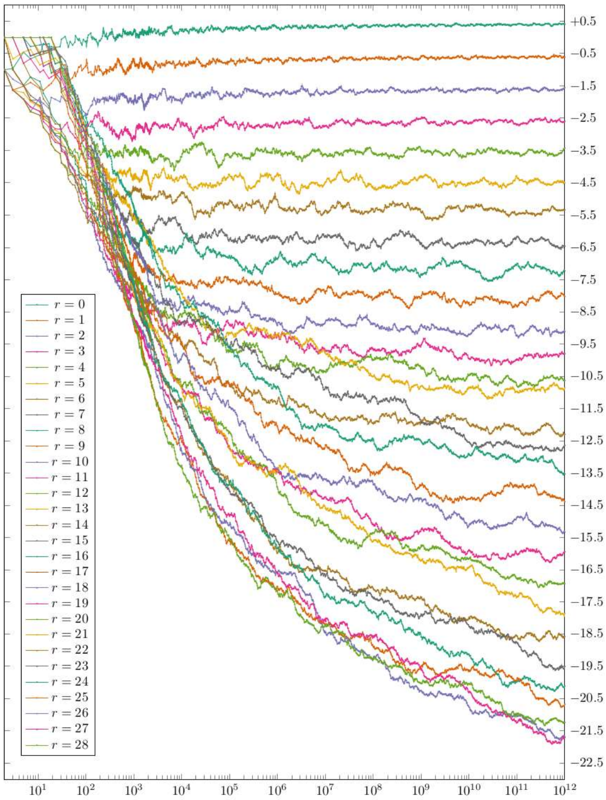
$$\lim_{x \to \infty} \frac{\sum_{\substack{p \leq x \\ p \nmid \Delta}} \frac{\#E(\mathbb{F}_p) - (p + \frac{1}{2} + r)}{p}}{\sum_{\substack{p \leq x \\ p \nmid \Delta}} \frac{1}{p}} = 0.$$

Equivalently, recalling that $a_p = p + 1 - \#E(\mathbb{F}_p)$:

$$\lim_{x \to \infty} \frac{\sum_{p \leq x} \frac{a_p}{p}}{\sum_{p \leq x} \frac{1}{p}} = \frac{1}{2} - r.$$

Thus, the rank is not visible in $\#E(\mathbb{F}_p)$ for an individual p, but it does show up as a bias in its average value, contributing r extra points to each curve on average.

Here is a plot from Andrew Sutherland illustrating this for some curves with ranks varying from 0 to 28.



The data supports the existence of a bias, but it is also clear that the convergence is very slow (note the log scale of the horizontal axis!), particularly when the rank is large.

3. Explaining the murmuration phenomenon

3.1. **The murmuration pattern is ~~not~~ surprising.** According to BSD, larger rank makes a_p more negative, so we should expect there to be a *negative correlation* between them. However, looking again at Figure 1, that is not always the case. The correlation starts out in the expected direction for small primes, but then switches signs! And then switches again!

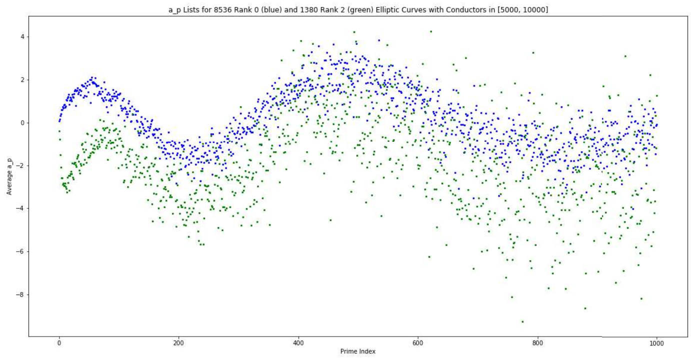
3.2. **It's really about root numbers!** For $E/\mathbb{Q}$ of conductor N, the *root number* ϵ is the sign of the functional equation of its complete L-function:

$$\Lambda(s,E) := (2\pi)^{-s}\Gamma(s)\prod_p \frac{1}{1-a_p p^{-s}+\chi_0(p)p^{1-2s}} = \epsilon N^{1-s}\Lambda(2-s,E),$$

where χ_0 is the trivial character mod N.

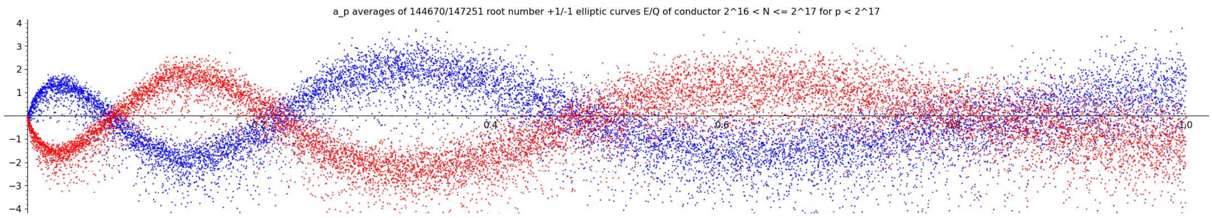
If the BSD conjecture is true then $\epsilon = (-1)^r$. Moreover, a folklore conjecture states that almost all $E/\mathbb{Q}$ (in any natural ordering) have rank 0 or 1. So a correlation with r is really a correlation with ϵ.

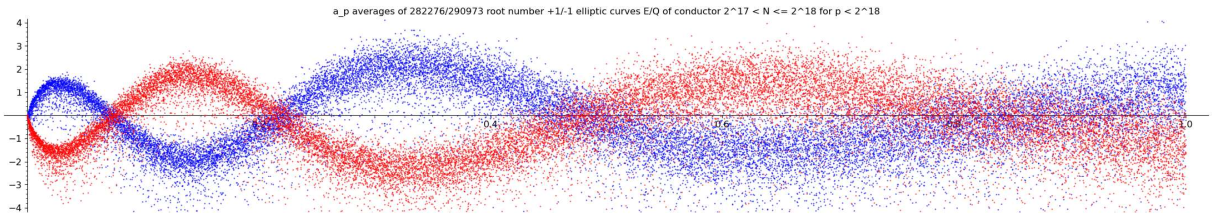
Below is a plot from [HLOP22] similar to Figure 1, but with ranks 0 and 2. Although it is rather noisy, one can see that the rank 2 graph has the same shape as the rank 0 graph, just shifted down by 2, in agreement with the a_p bias predicted by BSD. Thus the oscillatory pattern in the murmuration is entirely due to the correlation with the root number, which is then superimposed onto the (constant) BSD bias.



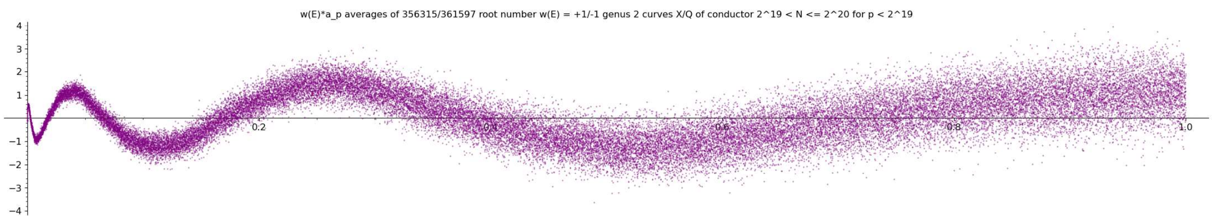
blue = rank 0, green = rank 2

3.3. **It's really about $\frac{p}{N}$!** The next observation, due to Jonathan Bober, is that the prime p should be scaled relative to the conductor of the curve, N. In this way, we can consider families of curves with conductors on vastly different scales, and one observes the same pattern. This is illustrated by two plots from Andrew Sutherland below. Note that the two plots use disjoint ranges of p and N individually, but with p/N approximately the same in each.

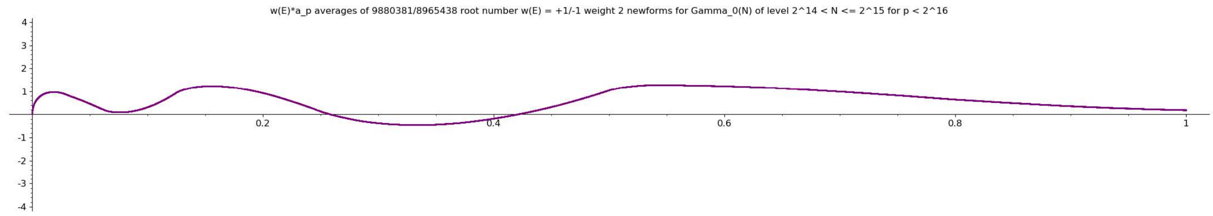




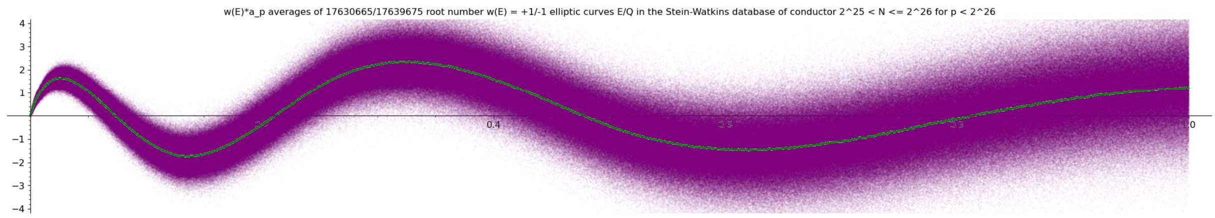
3.4. **Myriad more murmurations (it's not really about elliptic curves!)** More experiments by He, Lee, Oliver, Pozdnyakov, and Sutherland have demonstrated correlations between root numbers and a_p values in many natural families of L-functions. In the plots below, the red data (root number -1) has been flipped and combined with the blue data (root number 1).



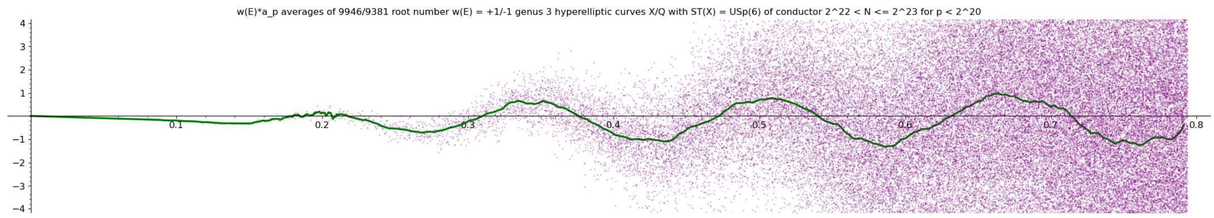
abelian surfaces with generic Sato–Tate group



self-dual holomorphic newforms of weight 2



elliptic curves from the Stein–Watkins database



genus 3 curves with generic Sato–Tate group

As these examples illustrate, the murmuration phenomenon is not confined to elliptic curves, or even to degree 2 L-functions.

In the plot from the Stein–Watkins database, we see that the pictures can become noisier with very large data sets, but that can be remedied with an additional local average over the prime, which picks out the smooth curve in the middle.

A more extreme example is the plot for genues 3 curves. That is a very thin family of degree 6 L-functions, with a data set that is likely far from complete. Nevertheless, the local averaging picks out what is presumably a smooth murmuration curve.

3.5. **It's really about a phase transition in the 1-level density!** Let $\mathscr{F}$ be a family of objects with L-functions $\Lambda_f(s)$ (defined in some sensible way). Suppose each $f \in \mathscr{F}$ has a well-defined conductor N_f, and set $\mathscr{F}(N) = \{f \in \mathscr{F} : N_f = N\}$.

The Katz–Sarnak philosophy predicts that for large N, the low-lying zeros of $\Lambda_f(s)$ for $f \in \mathscr{F}(N)$ behave statistically like eigenvalues of matrices drawn from a certain random matrix ensemble associated to $\mathscr{F}$. An example statistic of interest is the 1-*level density*:

$$E(\mathscr{F}; \phi) = \lim_{N\to\infty} \frac{1}{\#\mathscr{F}(N)} \sum_{f\in\mathscr{F}(N)} \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log N}{2\pi}\right),$$

where γ_f runs through zeros of $\Lambda_f(\frac{1}{2} + it)$ and ϕ is a suitable test function. It measures the distribution of low-lying zeros of $\Lambda_f(s)$ *on average* over elements of $\mathscr{F}$ of large conductor.

A *density theorem* is a statement of the form

$$E(\mathscr{F}; \phi) = \int_{\mathbb{R}} W_{\mathscr{F}}(x)\phi(x)\, dx$$

for some function (or measure) $W_{\mathscr{F}}$. According to the Katz–Sarnak philosophy, we should expect such a statement to hold for a certain $W_{\mathscr{F}}$ determined by the symmetry type of $\mathscr{F}$.

Example. For fixed $k \in 2\mathbb{Z}_{>0}$, let $H_k^{\pm}(N)$ denote the set of self-dual newforms of weight k and squarefree conductor N (= Hecke eigenbasis for $S_k^{\text{new}}(\Gamma_0(N))$) with root number $\epsilon_f = \pm 1$.

The family H_k^+ (resp. H_k^-) is expected to have symmetry type SO(even) (resp. SO(odd)), for which

$$W_{\text{SO(even)}}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x},\ W_{\text{SO(odd)}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x).$$

For this family we have a density theorem for a restricted class of test functions:

Theorem (Iwaniec, Luo, Sarnak, 2000)**.** *Let ϕ be a Schwartz function whose Fourier transform $\hat{\phi}$ has support contained in $(-2,2)$. Then, assuming GRH,*

$$E(H_k^+;\phi) = \int_{\mathbb{R}} \phi(x) W_{\text{SO(even)}}(x)\,dx.$$

and

$$E(H_k^-;\phi) = \int_{\mathbb{R}} \phi(x) W_{\text{SO(odd)}}(x)\,dx.$$

To relate this back to murmurations, recall the definition of 1-level density:

$$\lim_{N\to\infty} \frac{1}{\#\mathscr{F}(N)} \sum_{f\in\mathscr{F}(N)} \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log N}{2\pi}\right) = \int_{\mathbb{R}} W_{\mathscr{F}}(x)\phi(x)\,dx = \int_{\mathbb{R}} \hat{\phi}(y)\widehat{W}_{\mathscr{F}}(y)\,dy.$$

In the family from the example we have

$$\widehat{W}_{\text{SO(even)}}(y) = \delta_0(y) + \frac{\mathbf{1}_{[-1,1]}(y)}{2}, \widehat{W}_{\text{SO(odd)}}(y) = \delta_0(y) + \frac{2-\mathbf{1}_{[-1,1]}(y)}{2}.$$

Note that these functions agree on $(-1,1)$, so an important feature of the theorem is that it applies to some functions with support outside of $[-1,1]$, and can therefore distingush the SO(even) and SO(odd) symmetry types.

The theorem is proved by appealing to the "explicit formula" from analytic number theory, which relates the sums over zeros of $\Lambda_f(s)$ to sums of the coefficients $\lambda_f(p)$ over primes:

$$\sum_{\gamma_f} \phi\left(\frac{\gamma_f \log N}{2\pi}\right) \leftrightsquigarrow \sum_p \frac{\lambda_f(p)\log p}{\sqrt{p}} \hat{\phi}\left(\frac{\log p}{\log N}\right)$$

Note in particular that if $\hat{\phi}$ is supported on $[-\theta,\theta]$ then only primes $p \le N^{\theta}$ appear. Thus, the moral of the story is that H_k^+ and H_k^- cannot be distinguished from each other using only $\lambda_f(p)$ values for $p \le N^{1-\varepsilon}$.

Murmurations arise from $p \asymp N$, which is the transition range for the 1-level density, $\theta = 1$, where $\widehat{W}_{\text{SO(even)}}$ and $\widehat{W}_{\text{SO(odd)}}$ are discontinuous. The murmuration function is a different scaling limit that zooms in on the discontinuity. In other words, murmurations are an elaborate instance of the Gibbs phenomenon!

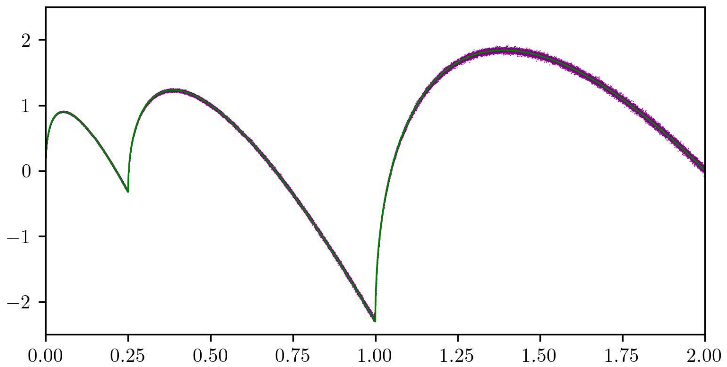
4. Some things we can prove

4.1. **Zubrilina's breakthrough.** The first proven result is due to Nina Zubrilina:

Theorem (Zubrilina, [Zub23])**.** *Fix $k \in 2\mathbb{Z}_{>0}$. Then there is a continuous function $M_k : \mathbb{R}_{>0} \to \mathbb{R}$ such that, for any fixed $y \in \mathbb{R}_{>0}$ and $\delta \in (0,1)$,*

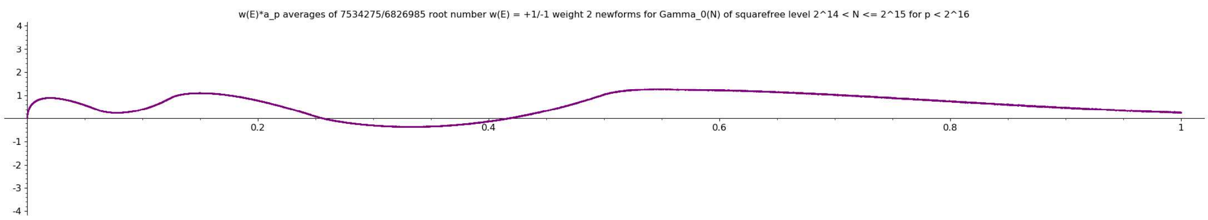
$$\lim_{\substack{p \text{ prime} \\ p \to \infty}} \frac{\sum_{\substack{N \in [p/y, p/y+p^\delta] \cap \mathbb{Z} \\ N \text{ squarefree}}} \sum_{f \in H_k(N)} \epsilon_f \lambda_f(p) \sqrt{p}}{\sum_{\substack{N \in [p/y, p/y+p^\delta] \cap \mathbb{Z} \\ N \text{ squarefree}}} \sum_{f \in H_k(N)} 1} = M_k(y).$$

A comparison of Zubrilina's theorem to numerical data is shown below.



M_2 versus numerics for $|N - 2^{18}| < 2^{10}$ and $p < 2^{19}$

Although this looks rather different from the corresponding plot in §3.4, there is a simple explanation. That plot took p/N in a dyadic interval, so it should tend to a convolution of Zubrilina's density function M_2, and that is indeed the case:



dyadic convolution of M_2 versus numerics for $N \in [2^{14}, 2^{15}]$ and $p < 2^{16}$

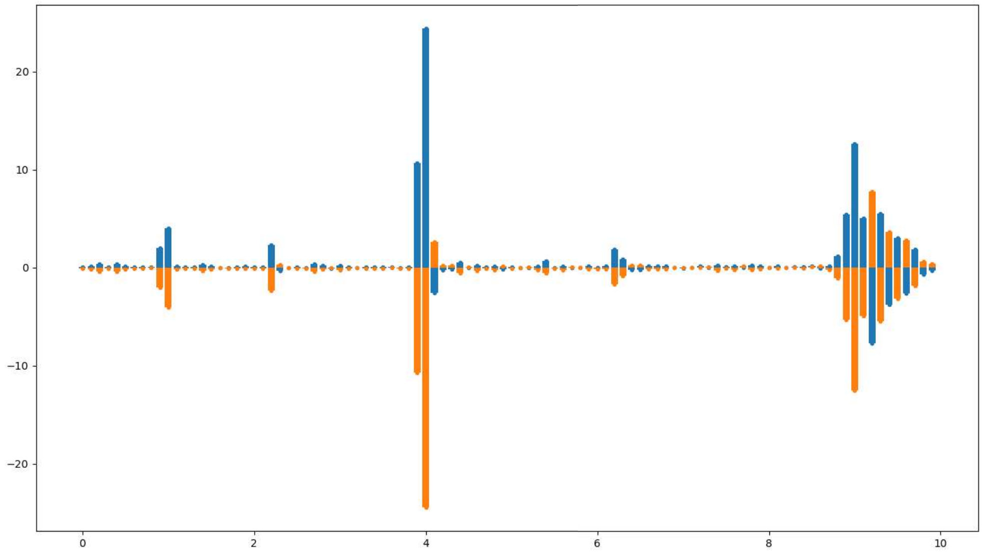
4.2. **Archimedean murmurations?** At the workshop "Murmurations in Arithmetic" at ICERM in July 2023, Sarnak asked whether there is an analogue of the murmuration phenomenon for the family of Maass forms of level 1 and Laplace eigenvalue $\lambda \to \infty$.

This is an example of a family with varying archimedean parameters. Our work treats the simpler (but morally related) family of holomorphic forms of level 1 and weight $k \to \infty$. In this family the root number ϵ_f is simply $(-1)^{\frac{k}{2}}$, so we expect to see biases in $\lambda_f(p)$ depending on $k \bmod 4$.

In analogy with the arithmetic scaling in Zubrilina's theorem, one might expect to see murmurations for p growing in proportion to the *analytic conductor*:

$$\mathcal{N}(k) := \left(\frac{\exp \psi(k/2)}{2\pi}\right)^2 = \left(\frac{k-1}{4\pi}\right)^2 + O(1).$$

Here again is some numerical data:



averages of $\lambda_f(p)\sqrt{p}$ for f of weight $k \in [2400, 3300]$ and fixed root number (blue $= +1$, orange $= -1$), collated by value of $\frac{p}{\mathcal{N}(k)}$

The plot is strikingly different from Zubrilina's, but there are some recognizable features. First, the sign pattern is unmistakable, so it looks like there is a correlation with the root number. Second, the distribution is apparently not absolutely continuous, and in fact it seems to have many point masses. On closer inspection they seem to be near squares of rational numbers.

4.3. **Another theorem!** Armed with this knowledge of the expected answer, we eventually proved the following new case of murmurations in the weight aspect:

Theorem (Bober, B., Lee, Lowry-Duda, [BBLLD23])**.** *Assume GRH for the L-functions of Dirichlet characters and modular forms. Fix $\varepsilon \in (0, \frac{1}{12})$, $\delta \in \{0, 1\}$, and a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H \in \mathbb{R}_{>0}$ with $K^{\frac{5}{6}+\varepsilon} < H < K^{1-\varepsilon}$, and set $N = \mathcal{N}(K)$. Then as $K \to \infty$, we have*

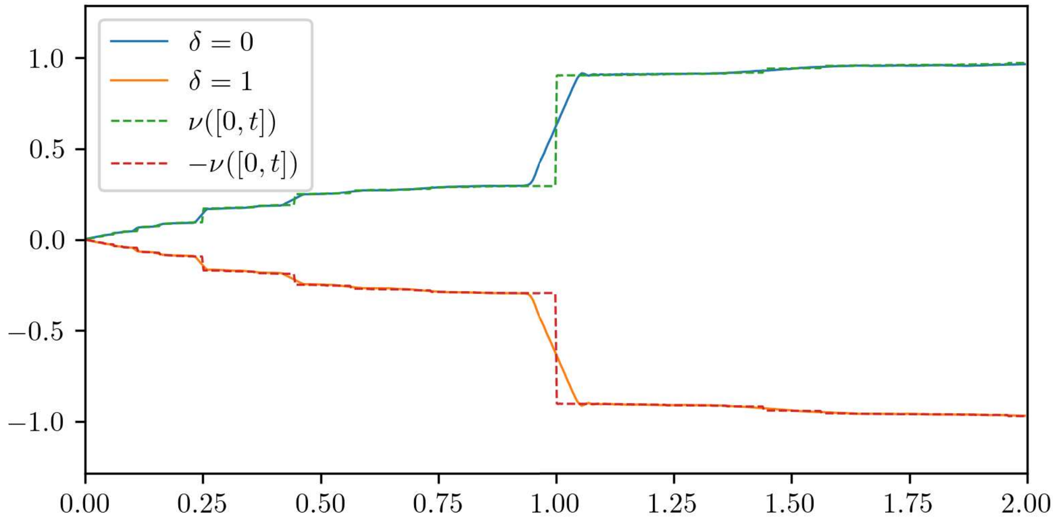
$$\frac{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} \lambda_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}} \left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1) \right),$$

where

$$\nu(E) = \frac{1}{\zeta(2)} \sum_{\substack{a,q \in \mathbb{Z}_{>0} \\ \gcd(a,q)=1 \\ (a/q)^{-2} \in E}}^{*} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{a}\right)^3 = \frac{1}{2} \sum_{t=-\infty}^{\infty} \prod_{p \nmid t} \frac{p^2 - p - 1}{p^2 - p} \cdot \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) dy,$$

and the $*$ means terms occurring at the endpoints of E are halved.

Once again we see a good fit with numerical data:



A comparison of $(-1)^\delta \nu([0,t])$ versus numerics for $K = 3850$, $H = 100$

4.4. **Some differences between our theorem and Zubrilina's.**

- We need the local average over primes. Sarnak [Sar23] has speculated that this is closely related to the conductor growth, and that local averaging is needed for any family with at most $O(x)$ L-functions of conductor $\leq x$.
- We need GRH.
- Our distribution:
  - is orthogonal to Lebesgue measure;
  - has no sign changes;
  - is essentially periodic as a function of $1/\sqrt{y}$.

On this last point, my vague guess is that this stems from the fact that our family is indexed by the dual of SO(2). There might be a p-adic analogue in the depth aspect. A more general philosophical explanation for the large discrepancy between our distribution and the other cases of murmurations in arithmetic families remains elusive.

4.5. **Some more things that should be possible.** Here are some open tasks that should be amenable to proof, for anyone who is interested:

- Maass forms of level 1 and eigenvalue $\lambda \to \infty$
- $H_k(N)$ for fixed N and $k \to \infty$
- all self-dual newforms, ordered by analytic conductor
- newforms of conductor p^n for fixed p and $n \to \infty$
- integer murmurations

4.6. **Proof steps.**

(1) Apply the Eichler–Selberg trace formula to get

$$\sum_{f \in H_k(1)} \lambda_f(p) = -p^{\frac{1-k}{2}} + \frac{(-1)^{\frac{k}{2}}}{\pi} \sum_{\substack{t \in \mathbb{Z} \\ t^2 < 4p}} \cos((k-1)\phi_{t,p}) L(1, \psi_{t^2-4p}),$$

where

$$\phi_{t,p} = \arcsin\left(\frac{t}{2\sqrt{p}}\right) \quad \text{and} \quad \psi_{d\ell^2}(m) = \left(\frac{d}{m/\gcd(m,\ell)}\right).$$

(2) Take a partition of unity to split the k sum into shorter ranges of length $\asymp h$ for an auxiliary parameter h. Apply Poisson summation to evaluate each individual smoothed sum. Since k runs over a fixed congruence class mod 4, this effectively concentrates the terms of the RHS around values of $\phi_{t,p}$ near 0 and $\pm\frac{\pi}{2}$. Since there are few integers t with t^2 close to $4p$, the terms for $\phi_{t,p}$ near $\pm\frac{\pi}{2}$ do not contribute to leading order.

(3) Apply the prime number theorem in arithmetic progressions to compute the sum over p for each fixed $t \in \mathbb{Z}$. This changes the sum into an integral over $\alpha := (p/N)^{-\frac{1}{2}}$, with $L(1, \psi_{t^2-4p})$ replaced by $L(1, \overline{\psi}_t)$, where $\overline{\psi}_t$ is the local average

$$\overline{\psi}_t(m) = \frac{1}{\varphi(m^2)} \sum_{\substack{a \bmod m^2 \\ (a,m)=1}} \psi_{t^2-4a}(m).$$

(4) Evaluate the sum over t at a fixed value of α, which reveals large spikes when α is near a rational number with small squarefree denominator.

(5) Apply the circle method to compute the integral over α.

References

[BBLLD23] Jonathan Bober, Andrew R. Booker, Min Lee, and David Lowry-Duda. Murmurations of modular forms in the weight aspect. arXiv:2310.07746, 2023.

[HLOP22] Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov. Murmurations of elliptic curves. arXiv:2204.10140, 2022.

[Sar23] Peter Sarnak. Letter to Drew Sutherland and Nina Zubrilina on murmurations and root numbers. `https://publications.ias.edu/sarnak/paper/2726`, 2023.

[Zub23] Nina Zubrilina. Murmurations. arXiv:2310.07681, 2023.

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