On generalization of duality formulas for the Arakawa-Kaneko type zeta functions

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Abstract: Kaneko and Tsumura introduced the Arakawa-Kaneko type zeta function $\eta(-k_1,\dots,-k_r;s_1,\dots,s_r)$ for non-negative integers $k_1,\dots,k_r$ and complex variables $s_1,\dots,s_r$. Recently, Yamamoto showed that, by using the multiple integral expression, $\eta(u_1,\dots,u_r;s_1,\dots,s_r)$ can be extended to an analytic function of $2r$ variables. Also, he showed that the function $\eta(u_1,\dots,u_r;s_1,\dots,s_r)$ satisfies a duality formula. In this note, by using the a generalization of non-strict multi-indexed polylogarithm, we define a kind of the Arakawa-Kaneko type zeta function, and show that this function satisfies a certain duality formula. This note is based on the author's talk at RIMS conference.

1 Introduction

In [1], Arakawa and Kaneko introduced the function

$$\xi(\mathbf{k}_r;s)=\xi(k_1,\dots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{\mathbf{k}_r}(1-e^{-t})}{e^t-1}dt$$

for $\mathbf{k}_r=(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, which is called the Arakawa-Kaneko zeta function. Here,

$$\mathrm{Li}_{\mathbf{k}_r}(z)=\mathrm{Li}_{k_1,\dots,k_r}(z)=\sum_{0<m_1<\cdots<m_r}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}\quad(|z|<1)$$

is the multiple polylogarithm. They also showed that the function $\xi(\mathbf{k}_r;s)$ can be continued analytically to the whole plane $\mathbb{C}$. As a relative of $\xi(\mathbf{k}_r;s)$, in [8], Kaneko and Tsumura introduced the function

$$\eta(\mathbf{k}_r;s)=\eta(k_1,\dots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{\mathbf{k}_r}(1-e^{t})}{1-e^t}dt$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, $\mathbf{k}_r\in\mathbb{Z}_{\geq 1}^r$ or $\mathbf{k}_r\in\mathbb{Z}_{\leq 0}^r$, and showed that the function $\eta(\mathbf{k}_r;s)$ can be continued analytically to the whole plane $\mathbb{C}$. The function $\eta(\mathbf{k}_r;s)$ is considered to be a twin sibling of the function $\xi(\mathbf{k}_r;s)$. It is shown that the values of these functions at non-positive integers can be expressed in terms of the multi-poly-Bernoulli numbers

$$\frac{\mathrm{Li}_{\mathbf{k}_r}(1-e^{-t})}{1-e^{-t}}=\sum_{m=0}^\infty B_m^{(\mathbf{k}_r)}\frac{t^m}{m!},\quad \frac{\mathrm{Li}_{\mathbf{k}_r}(1-e^{-t})}{e^{t}-1}=\sum_{m=0}^\infty C_m^{(\mathbf{k}_r)}\frac{t^m}{m!},$$

and that the values of these functions at positive integers are closely related to the multiple zeta values

$$\zeta(\mathbf{k}_r)=\zeta(k_1,\dots,k_r)=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

for $\mathbf{k}_r\in\mathbb{Z}_{\geq 1}^r$ with $k_r\geq 2$ (for details, see [1,8]).

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Analogously, for $\mathbf{u}_r = (u_1, \ldots, u_r) \in \mathbb{C}^r$ and $d \in \{1, \ldots, r\}$, Kaneko and Tsumura introduced the multi-indexed poly-Bernoulli numbers by the generating function

$$\frac{\mathrm{Li}^{ш}_{\mathbf{u}_r}(1-e^{-t_1-\cdots-t_r}, \ldots, 1-e^{-t_r})}{(1-e^{-t_1-\cdots-t_r})\cdots(1-e^{-t_d-\cdots-t_r})} = \sum_{m_1,\ldots,m_r=0}^{\infty} B^{(\mathbf{u}_r),(d)}_{\mathbf{m}_r} \frac{t_1^{m_1}}{m_1!}\cdots\frac{t_r^{m_r}}{m_r!}, \tag{1}$$

and defined the related Arakawa-Kaneko type zeta function for $-\mathbf{k}_r = (-k_1, \ldots, -k_r) \in \mathbb{Z}^r_{\le 0}$ by

$$\eta(-\mathbf{k}_r; \mathbf{s}_r) = \prod_{j=1}^{r} \frac{1}{\Gamma(s_j)} \int_0^{\infty} t_1^{s_1-1}\cdots t_r^{s_r-1} \frac{\mathrm{Li}^{ш}_{-\mathbf{k}_r}(1-e^{t_1+\cdots+t_r}, \ldots, 1-e^{t_r})}{(1-e^{t_1+\cdots+t_r})\cdots(1-e^{t_r})} dt_1\cdots dt_r. \tag{2}$$

Here, for $\mathbf{u}_r, \mathbf{z}_r = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with $|z_j| < 1$ $(j = 1, \ldots, r)$,

$$\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r) = \mathrm{Li}^{ш}_{u_1,\ldots,u_r}(z_1, \ldots, z_r) = \sum_{l_1,\ldots,l_r=1}^{\infty} \frac{z_1^{l_1}\cdots z_r^{l_r}}{l_1^{u_1}\cdots(l_1+\cdots+l_r)^{u_r}}$$

is the multiple polylogarithm of ш-type. In [8], Kaneko and Tsumura showed that the duality formula

$$B^{(-\mathbf{k}_r),(r)}_{\mathbf{n}_r} = B^{(-\mathbf{n}_r),(r)}_{\mathbf{k}_r} \quad (\mathbf{k}_r, \mathbf{n}_r \in \mathbb{Z}^r_{\ge 0})$$

holds, which is a generalization of that for poly-Bernoulli numbers $B_n^{(-k)}$. Recently, in [13], Yamamoto showed that the function $\eta(-\mathbf{k}_r; \mathbf{s}_r)$ can be extended to an analytic function of $2r$ variables $\eta(\mathbf{u}_r; \mathbf{s}_r)$. Also, in [13], he conjectured that

$$\eta(\mathbf{k}_r; \mathbf{n}_r) \in \mathcal{Z}$$

holds for $\mathbf{k}_r, \mathbf{n}_r \in \mathbb{Z}_{\ge 1}$. Here,

$$\mathcal{Z} = \langle\ \zeta(\mathbf{k}_r) \mid \mathbf{k}_r \in \mathbb{Z}^r_{\ge 1}, k_r \ge 2\ \rangle_{\mathbb{Q}}$$

is the $\mathbb{Q}$-linear space spanned by all multiple zeta values for admissible indices $\mathbf{k}_r$. This conjecture was solved by Brown [3], and later by Ito and Sato [7]. In particular, Ito and Sato showed that

$$\eta(\mathbf{k}_r; \mathbf{n}_r) \in \mathcal{Z}_{k_1+\cdots+k_r+n_1+\cdots+n_r}$$

holds. Here, for $k \in \mathbb{Z}_{\ge 0}$,

$$\mathcal{Z}_k = \langle\ \zeta(\mathbf{k}_r) \mid \mathbf{k}_r \in \mathbb{Z}^r_{\ge 1}, k_r \ge 2, k_1 + \cdots + k_r = k\ \rangle_{\mathbb{Q}}$$

is the $\mathbb{Q}$-linear subspace of $\mathcal{Z}$ spanned by all multiple zeta values for admissible indices of weight k. As an analogue of $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r)$, Baba, Nakasuji and Sakata introduced the multiple polylogarithm $\mathrm{Li}^{ш,\star}_{\mathbf{u}_r}(\mathbf{z}_r)$ and multi-indexed poly-Bernoulli numbers $\mathbb{B}^{\star,(\mathbf{u}_r)}_{\mathbf{n}_r}$ as follows.

Definition 1.1 ([2])**.** For $\mathbf{u}_r, \mathbf{z}_r \in \mathbb{C}^r$ with $|z_j| < 1$ $(j = 1, \ldots, r)$, the multiple polylogarithm $\mathrm{Li}^{ш,\star}_{\mathbf{u}_r}(\mathbf{z}_r)$ and multi-indexed poly-Bernoulli numbers $\mathbb{B}^{\star,(\mathbf{u}_r)}_{\mathbf{n}_r}$ are defined by

$$\mathrm{Li}^{ш,\star}_{\mathbf{u}_r}(\mathbf{z}_r) = \sum_{l_1=1,l_2,\ldots,l_r=0}^{\infty} \frac{z_1^{l_1}\cdots z_r^{l_r}}{l_1^{u_1}\cdots(l_1+\cdots+l_r)^{u_r}},$$

$$\frac{\mathrm{Li}^{ш,\star}_{\mathbf{u}_r}(1-e^{-t_1-\cdots-t_r}, \ldots, 1-e^{-t_r})}{(1-e^{-t_1-\cdots-t_r})\cdots(1-e^{-t_r})} = \sum_{m_1,\ldots,m_r=0}^{\infty} \mathbb{B}^{\star,(\mathbf{u}_r)}_{\mathbf{m}_r} \frac{t_1^{m_1}}{m_1!}\cdots\frac{t_r^{m_r}}{m_r!},$$

respectively.

Note that $\mathrm{Li}^{ш,\star}_{\mathbf{u}_r}(\mathbf{z}_r)$ is a ш-type analogue of the non-strict multiple polylogarithm

$$\mathrm{Li}^{\star}_{\mathbf{k}_r}(z)=\sum_{0<m_1\le\cdots\le m_r}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}\ (|z|<1),$$

since

$$\mathrm{Li}^{\star}_{\mathbf{k}_r}(z)=\mathrm{Li}^{ш,\star}_{\mathbf{k}_r}(z,\ldots,z)$$

holds. In [2], Baba, Nakasuji and Sakata showed various relations among $\mathbb{B}^{\star,(\mathbf{u}_r)}_{\mathbf{n}_r}$. Inspired by their results, we consider the following polylogarithm, Bernoulli numbers and Arakawa-Kaneko type η function.

Definition 1.2. For $\mathbf{u}_r,\mathbf{z}_r\in\mathbb{C}^r$ with $|z_j|<1$ $(j=1,\ldots,r)$ and $\mathbf{a}_r=(a_1,\ldots,a_r)\in\mathbb{Z}^r_{\ge0}$ with $a_1\ge1$, we define $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r;\mathbf{a}_r)$ by

$$\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r;\mathbf{a}_r)=\sum_{\substack{l_j=a_j\\ j=1,\ldots,r}}^{\infty}\frac{z_1^{l_1}\cdots z_r^{l_r}}{l_1^{u_1}\cdots(l_1+\cdots+l_r)^{u_r}}. \tag{3}$$

Also, for $\mathbf{k}_r\in\mathbb{Z}^r,\mathbf{n}_r\in\mathbb{Z}^r_{\ge0},\sigma\in\mathfrak{S}_r,\mathbf{a}_r\in\mathbb{Z}^r_{\ge0}$ with $a_1,a_{\sigma^{-1}(1)}\ge1$ and $\mathbf{b}_r=(b_1,\ldots,b_r)\in\mathbb{Z}^r_{\ge0}$ with $b_1,b_{\sigma^{-1}(1)}\ge1$, we define $B^{(-\mathbf{k}_r)}_{\mathbf{n}_r}(\sigma;\mathbf{a}_r;\mathbf{b}_r)$ by

$$\begin{aligned}&\prod_{j=1}^{r}e^{(a_j-1)(t_j+\cdots+t_r)}\frac{\mathrm{Li}^{ш}_{\mathbf{k}_r}(1-e^{-t_{\sigma(1)}-t_{\sigma(1)+1}-\cdots-t_r},\ldots,1-e^{-t_{\sigma(r)}-t_{\sigma(r)+1}-\cdots-t_r};\mathbf{b}_r)}{(1-e^{-t_{\sigma(1)}-t_{\sigma(1)+1}-\cdots-t_r})^{b_1}\cdots(1-e^{-t_{\sigma(r)}-t_{\sigma(r)+1}-\cdots-t_r})^{b_r}}\\&=\sum_{m_1,\ldots,m_r=0}^{\infty}B^{(\mathbf{k}_r)}_{\mathbf{m}_r}(\sigma;\mathbf{a}_r;\mathbf{b}_r)\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_r^{m_r}}{m_r!}.\end{aligned} \tag{4}$$

Furthermore, for $\mathbf{u}_r,\mathbf{s}_r\in\mathbb{C}^r$ with $\mathrm{Re}(u_j),\mathrm{Re}(s_j)>0$, $\sigma\in\mathfrak{S}_r,\mathbf{a}_r\in\mathbb{Z}^r_{\ge0}$ with $a_1,a_{\sigma^{-1}(1)}\ge1$ and $\mathbf{b}_r\in\mathbb{Z}^r_{\ge0}$ with $b_1,b_{\sigma^{-1}(1)}\ge1$ and $a_{\sigma(j)}+b_j\ge1$, we define $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$ by

$$\begin{aligned}&\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)\\&=\prod_{j=1}^{r}\frac{1}{\Gamma(s_j)}\int_{(0,\infty)^r}\prod_{j=1}^{r}t_j^{s_j-1}e^{(1-a_j)(t_j+\cdots+t_r)}\frac{\mathrm{Li}^{ш}_{\mathbf{u}_r}(1-e^{t_{\sigma(1)}+t_{\sigma(1)+1}+\cdots+t_r},\ldots,1-e^{t_{\sigma(r)}+t_{\sigma(r)+1}+\cdots+t_r};\mathbf{b}_r)}{(1-e^{t_{\sigma(1)}+t_{\sigma(1)+1}+\cdots+t_r})^{b_1}\cdots(1-e^{t_{\sigma(r)}+t_{\sigma(r)+1}+\cdots+t_r})^{b_r}}\prod_{j=1}^{r}dt_j.\end{aligned} \tag{5}$$

The Bernoulli numbers $B^{(-\mathbf{k}_r)}_{\mathbf{n}_r}(\sigma;\mathbf{a}_r;\mathbf{b}_r)$ are one of the generalizations of $B^{(-\mathbf{k}_r),(r)}_{\mathbf{n}_r}$. Also, the Arakawa-Kaneko type η function $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$ are one of the generalizations of $\eta(\mathbf{u}_r;\mathbf{s}_r)$ (for details, see Section 4). The aim of this paper is to show that the function $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$ satisfies a certain duality formula. Also, we show that we can write the special values of $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$ at positive integers in terms of multiple zeta values, and at non-positive integers in terms of $B^{(-\mathbf{k}_r)}_{\mathbf{n}_r}(\sigma;\mathbf{a}_r;\mathbf{b}_r)$.

The paper is organized as follows. In Section 2, we review some notations and the known results. In Section 3, we show analytic continuations of $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r;\mathbf{a}_r)$ and $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$. Also, we show that the function $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$ satisfies a certain duality formula. In Section 4, we consider some special values of $\eta(\mathbf{u}_r;\mathbf{s}_r;\sigma;\mathbf{a}_r;\mathbf{b}_r)$.

2 Preliminaries

In this section, we recall some notations and the known results. First of all, we review the properties $\eta(\mathbf{u}_r;\mathbf{s}_r)$. As we state in Section 1, the function $\eta(\mathbf{k}_r;s)$ is first defined by Kaneko and Tsumura [8]. In the article, they gave the following conjecture, which was already proved.

Theorem 2.1 (cf. [8, p. 37])**.** For $k, n \in \mathbb{Z}_{\geq 1}$,

$$\eta(k;n) = \eta(n;k) \tag{6}$$

holds.

This theorem was first proved by Yamamoto [13], who showed the more general case. He showed the following results.

Theorem 2.2 ([13, Lemma 2.1])**.** For $\mathbf{u}_r, \mathbf{z}_r \in \mathbb{C}^r$ with $|z_j| < 1$ $(j = 1, \ldots, r)$ and sufficiently small $\varepsilon \in \mathbb{R}_{>0}$, the multiple polylogarithm of ш-type

$$\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r) = \sum_{l_1,\ldots,l_r=1}^{\infty} \frac{z_1^{l_1} \cdots z_r^{l_r}}{l_1^{u_1} \cdots (l_1 + \cdots + l_r)^{u_r}}$$

has the integral expression

$$\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r) = \prod_{j=1}^{r} \frac{\Gamma(1-u_j)}{2\pi i e^{\pi i u_j}} \int_{(\mathcal{C}_\varepsilon)^r} \prod_{j=1}^{r} \frac{x_j^{u_j-1} z_j}{e^{x_j+\cdots+x_r} - z_j} dx_j. \tag{7}$$

Here, $\mathcal{C}_\varepsilon$ denotes the contour which goes from $+\infty$ to ε along the real axis, goes round counterclockwise along the circle around the origin of radius ε (let $C(0;\varepsilon)$ be this circle), and then goes back to $+\infty$ along the real axis. By (7), $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r)$ can be continued analytically to the region $(\mathbf{u}_r, \mathbf{z}_r) \in \mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$.

Theorem 2.3 ([13, Definition 2.3, Theorem 2.5])**.** For $\mathbf{u}_r, \mathbf{s}_r \in \mathbb{C}^r$ with $\mathrm{Re}(s_j) > 0$, the function

$$\eta(\mathbf{u}_r;\mathbf{s}_r) = \prod_{j=1}^{r} \frac{1}{\Gamma(s_j)} \int_0^\infty t_1^{s_1-1} \cdots t_r^{s_r-1} \frac{\mathrm{Li}^{ш}_{\mathbf{u}_r}(1 - e^{t_1+\cdots+t_r}, \ldots, 1 - e^{t_r})}{(1 - e^{t_1+\cdots+t_r}) \cdots (1 - e^{t_r})} dt_1 \cdots dt_r \tag{8}$$

is defined and has the integral expression

$$\eta(\mathbf{u}_r;\mathbf{s}_r) = \prod_{j=1}^{r} \frac{\Gamma(1-u_j)\Gamma(1-s_j)}{(2\pi i)^2 e^{\pi i (u_j+s_j)}} \int_{(\mathcal{C}_\varepsilon)^{2r}} \prod_{j=1}^{r} \frac{x_j^{u_j-1} t_j^{s_j-1}}{e^{x_j+\cdots+x_r} + e^{t_j+\cdots+t_r} - 1} dx_j dt_j. \tag{9}$$

By (9), $\eta(\mathbf{u}_r;\mathbf{s}_r)$ can be continued analytically to the region $(\mathbf{u}_r, \mathbf{s}_r) \in \mathbb{C}^{2r}$. Furthermore, $\eta(\mathbf{u}_r;\mathbf{s}_r)$ satisfies the duality formula

$$\eta(\mathbf{u}_r;\mathbf{s}_r) = \eta(\mathbf{s}_r;\mathbf{u}_r). \tag{10}$$

To give the analytic continuation of $\eta(\mathbf{u}_r;\mathbf{s}_r)$, the following lemma is essential. For sufficiently small $\varepsilon \in \mathbb{R}_{>0}$, put $D_\varepsilon = \{\alpha \in \mathbb{C} | -\varepsilon \leq \mathrm{Im}(\alpha) \leq \varepsilon, -\varepsilon \leq \mathrm{Re}(\alpha)\}$.

Lemma 2.4 ([13, Lemma 2.4])**.** For $x, t \in D_\varepsilon$ and sufficiently small $\varepsilon \in \mathbb{R}_{>0}$,

$$\left| \frac{e^{\frac{1}{2}x} e^{\frac{1}{2}t}}{e^x + e^t - 1} \right| \tag{11}$$

is bounded.

Remark 2.5. Kawasaki and Ohno gave a combinatorial proof of Theorem 2.1 (for details, see[9]).

For the values of $\eta(\mathbf{u}_r;\mathbf{s}_r)$ at positive integers, Ito showed the following theorem.

Theorem 2.6 ([6, Theorem C.12], [7, Theorem 4.3])**.** For $\mathbf{k}_r, \mathbf{n}_r \in \mathbb{Z}_{\geq 1}$ and $d \in \{1, \ldots, r\}$, we have

$$\eta(\mathbf{k}_r;\mathbf{n}_r) \in \mathcal{Z}_{k_1+\cdots+k_r+n_1+\cdots+n_r}.$$

To show Theorem 2.6, we review some properties of the hyperlogarithm. Note that, though we use the same terminology "hyperlogarithm", the following definition and results obtained by Ito are generalizations of known results.

Definition 2.7 (cf. [11])**.** Let $a_0 \in \mathbb{R}$, a_{n+1} be a variable with $a_0 < a_{n+1}$ and $a_1, \ldots, a_n$ be variables with $a_j \in \mathbb{C} \setminus (a_0, a_{n+1})$ for each point, $a_1 \neq a_0$ and $a_n \neq a_{n+1}$. For them, the hyperlogarithm is defined by

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0 < t_1 < \cdots < t_n < a_{n+1}} \prod_{j=1}^{n} \frac{dt_j}{t_j - a_j}.$$

We can show the following lemma by definition.

Lemma 2.8. [7, Lemma 4.4] Let the notation be the same as above. For $c \in \mathbb{R}_{>0}$, we have

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = I(ca_0; ca_1, \ldots, ca_n; ca_{n+1}).$$

Also, for $a_0 < x, a_1, \ldots, a_n \in \mathbb{C} \setminus (a_0, x)$ with $a_1 \neq a_0, a_n \neq x$ and $b \in \mathbb{C} \setminus (a_0, x]$, we have

$$\int_{a_0}^{x} \frac{1}{y - b} I(a_0; a_1, \ldots, a_n; y) dy = I(a_0; a_1, \ldots, a_n, b; x).$$

Note that, for $\mathbf{k}_r \in \mathbb{Z}_{\geq 1}^r$ and $\mathbf{z}_r \in \mathbb{C}^r$ with $|z_j| < 1$ $(j = 1, \ldots, r)$, we have

$$\mathrm{Li}^{\text{ш}}_{\mathbf{k}_r}(\mathbf{z}_r) = (-1)^r I(0; z_1^{-1}, \overbrace{0, \ldots, 0}^{k_1 - 1}, \ldots, z_r^{-1}, \overbrace{0, \ldots, 0}^{k_r - 1}; 1). \tag{12}$$

To obtain Theorem 2.6, the following lemmas are essential.

Lemma 2.9 ([7, Lemma 4.5], cf. [4, Theorem 2.1], [5, Lemma 2.2], [12, Lemma 3.3.30])**.** For $a_0 \in \mathbb{R}$, $a_{n+1} = a_{n+1}(x)$ with $a_0 < a_{n+1}(x)$ and $a_1 = a_1(x), \ldots, a_n = a_n(x)$ with $a_j(x) \in \mathbb{C} \setminus (a_0, a_{n+1}(x))$ for each point x, $a_1 \neq a_0, a_n \neq a_{n+1}$, we have

$$\frac{\partial}{\partial x} I(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{j=1}^{n} (\epsilon_j(\mathbf{a}) - \epsilon_{j-1}(\mathbf{a})) I(a_0; a_1, \ldots, \widehat{a_j}, \ldots, a_n; a_{n+1}).$$

Here, for $j = 0, \ldots, n$ and $\mathbf{a} = (a_1, \ldots, a_n)$,

$$\epsilon_j(\mathbf{a}) = \begin{cases} 0 & (a_{j+1} = a_j), \\ \frac{\frac{\partial}{\partial x}(a_{j+1} - a_j)}{a_{j+1} - a_j} & (a_{j+1} \neq a_j), \end{cases}$$

and $(a_1, \ldots, \widehat{a_j}, \ldots, a_n) = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$.

By Lemma 2.9, we can transform hyperlogarithms. For example, consider transforming the hyperlogarithm

$$I(0; x_2^{-1}, x_1^{-1}; 1).$$

Since

$$\frac{\partial}{\partial x_2} I(0; x_2^{-1}, x_1^{-1}; 1) = \frac{1}{x_2 - x_1} I(0; 1; x_1) + \left(\frac{1}{x_2} - \frac{1}{x_2 - x_1} \right) I(0; 1; x_2),$$

we obtain

$$I(0; x_2^{-1}, x_1^{-1}; 1) = I(0; x_1; x_2) I(0; 1; x_1) + I(0; 1, 0; x_2) - I(0; 1, x_1; x_2).$$

Ito called this transforming process *variable removing*, and we use the same terminology.

Lemma 2.10 ([7, Lemma 4.6])**.** For $x \in \mathbb{R}$ and $x_1, \ldots, x_r \in \mathbb{C}$ with $0 < x < |x_j| < 1$ $(j = 1, \ldots, r)$, define $V(x_1, \ldots, x_r; x)$ as the $\mathbb{Q}$-linear space by

$$V(x_1, \ldots, x_r; x) = \left\langle I(0; a_1, \ldots, a_n; 1) I(0; b_1, \ldots, b_l; x) \;\middle|\; \begin{matrix} a_j \in \{0,1,x_j^{-1}\},\ a_1 \neq 0,\ a_n \neq 1, \\ b_j \in \{0,1,x_j\},\ b_1 \neq 0 \end{matrix} \right\rangle_{\mathbb{Q}}.$$

Then, for $a_1, \ldots, a_n \in \{0, 1, x^{-1}, x_1^{-1}, \ldots, x_r^{-1}\}$ with $a_1 \neq 0, a_1 \neq 1$, we have

$$I(0; a_1, \ldots, a_n; 1) \in V(x_1, \ldots, x_r; x).$$

Also, for $b_1, \ldots, b_l \in \{0, 1, x, x_1, \ldots, x_r\}$ with $b_1 \neq 0, b_l \neq x$, we have

$$I(0; b_1, \ldots, b_l; x) \in V(x_1, \ldots, x_r; x).$$

Lemma 2.11 ([7, Proposition 4.8])**.** Let $n, l \in \mathbb{Z}_{\geq 0}, a_j \in \{0, 1, x_1^{-1}, \ldots, x_r^{-1}\}$ $(j = 1, \ldots, n)$ with $a_1 \neq 0, a_n \neq 1$, $b_j \in \{0, 1, x_1, \ldots, x_r\}$ $(j = 1, \ldots, l)$ with $b_1 \neq 0, b_l \neq x_r$, and $c_j \in \{0, 1\}$ $(j = 2, \ldots, r)$. Suppose that for all $j \in \{1, \ldots, r\}$, there exists some $v \in \{1, \ldots, n\}$ such that $x_j^{-1} = a_v$ or $x_j = b_v$. Then we have

$$\int_{0<x_r<\cdots<x_1<1} I(0; a_1, \ldots, a_n; 1) I(0; b_1, \ldots, b_l; x_r) \frac{dx_1}{x_1} \prod_{j=2}^{r} \frac{dx_j}{x_j - c_j} \in \mathcal{Z}_{n+l+r}.$$

These lemmas are also needed in Section 4.

3 Analytic continuations of $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r; \mathbf{a}_r)$ and $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$

In this section, we show analytic continuations of $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r; \mathbf{a}_r)$ and $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$.

Lemma 3.1 ([13, p. 2])**.** For each $\mathbf{z}_r \in (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$, there exists a neighborhood K of $\mathbf{z}_r$ and $\varepsilon_0 \in \mathbb{R}_{>0}$ such that $e^{x_j + \cdots + x_r} - z'_j \neq 0$ for $j = 1, \ldots, r$ whenever $\mathbf{z}_r \in K, 0 < \varepsilon < \varepsilon_0$ and $x_1, \ldots, x_r \in \mathcal{C}_\varepsilon$.

Lemma 3.2. For $\mathbf{u}_r, \mathbf{z}_r \in \mathbb{C}^r$ with $|z_j| < 1$, $\mathbf{a}_r \in \mathbb{Z}_{\geq 0}^r$ with $a_1 = 1$ and sufficiently small $\varepsilon \in \mathbb{R}_{>0}$, we have

$$\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r; \mathbf{a}_r) = \prod_{j=1}^{r} \frac{1}{(e^{2\pi i u_j} - 1)\Gamma(u_j)} \int_{(\mathcal{C}_\varepsilon)^r} \prod_{j=1}^{r} \frac{x_j^{u_j - 1} e^{(1-a_j)(x_j + \cdots + x_r)} z_j^{a_j}}{e^{x_j + \cdots + x_r} - z_j} dx_j. \tag{13}$$

In particular, $\mathrm{Li}^{ш}_{\mathbf{u}_r}(\mathbf{z}_r; \mathbf{a}_r)$ can be continued analytically to the region $(\mathbf{u}_r, \mathbf{z}_r) \in \mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$.

Remark 3.3. In [10, Theorem 3.17], Komori defined

$$\zeta(\boldsymbol{\xi}, \mathbf{d}, \mathbf{b}, \mathbf{s}) = \prod_{j=1}^{N} \frac{1}{\Gamma(s_j)} \prod_{t \in S} \frac{1}{e^{2\pi i t(\mathbf{s})} - 1}$$
$$\times \int_{\widehat{\Sigma}} \frac{e^{(b_{11} + \cdots + b_{1R} - d_1) z_1} \cdots e^{(b_{N1} + \cdots + b_{NR} - d_N) z_N} z_1^{s_1 - 1} \cdots z_N^{s_N - 1}}{(e^{z_1 b_{11} + \cdots + z_N b_{N1}} - e^{\xi_1}) \cdots (e^{z_1 b_{1R} + \cdots + z_N b_{NR} - e^{\xi_r}})} dz_1 \wedge \cdots \wedge dz_r$$

for $N, R \in \mathbb{Z}_{\geq 1}, \boldsymbol{\xi} = (\xi_1, \ldots, \xi_R) \in (\mathbb{C}/2\pi i\mathbb{Z})^R, \mathbf{d} = (d_1, \ldots, d_N), \mathbf{s} = (s_1, \ldots, s_N) \in \mathbb{C}^N, \mathbf{b} = (b_{nr})_{1 \leq n \leq N, 1 \leq r \leq R} \in \mathbb{C}^{N \times R}, S$: a set of linear functionals on $\mathbb{C}^N$ and $\widehat{\Sigma}$: a union of smooth surfaces. Also, he gives the analytic continuation of $\zeta(\boldsymbol{\xi}, \mathbf{a}, \mathbf{b}, \mathbf{s})$. Lemma 3.2 is the case $N = R = r, b_{ij} = 1, d_j = r - \sum_{l=1}^{j}(1 - a_l)$ and $S = \{t_j : \mathbb{C}^N \to \mathbb{C}^N | t_j(s_1, \ldots, s_r) = s_j\}$.

Proposition 3.4. For $\mathbf{u}_r, \mathbf{s}_r \in \mathbb{C}^r, \sigma \in \mathfrak{S}_r, \mathbf{a}_r \in \mathbb{Z}_{\geq 0}^r$ with $a_1, a_{\sigma^{-1}(1)} \geq 1$ and $\mathbf{b}_r \in \mathbb{Z}_{\geq 0}^r$ with $b_1, b_{\sigma^{-1}(1)} \geq 1$ and $a_{\sigma(j)} + b_j \geq 1$, the function $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$ has the integral expression

$$\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$$
$$= \prod_{j=1}^{r} \frac{1}{(e^{2\pi i u_j} - 1)(e^{2\pi i s_j} - 1)\Gamma(u_j)\Gamma(s_j)} \int_{\mathcal{C}^{2r}} \prod_{j=1}^{r} t_j^{s_j - 1} x_j^{u_j - 1} \frac{e^{(1-a_{\sigma(j)})(t_{\sigma(j)} + t_{\sigma(j)+1} + \cdots + t_r)} e^{(1-b_j)(x_j + \cdots + x_r)}}{e^{t_{\sigma(j)} + t_{\sigma(j)+1} + \cdots + t_r} + e^{x_j + \cdots + x_r} - 1} dt_j dx_j. \tag{14}$$

In particular, the function $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$ can be continued analytically to the region $(\mathbf{u}_r, \mathbf{s}_r) \in \mathbb{C}^{2r}$. Furthermore, by the integral expression (14), we obtain a duality formula

$$\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r) = \eta(\mathbf{s}_r; \mathbf{u}_r; \sigma^{-1}; \mathbf{b}_r; \mathbf{a}_r). \tag{15}$$

To prove Proposition 3.4, we need the following lemma.

Lemma 3.5 (cf. [13, Lemma 2.4])**.** For $x, t \in D_\varepsilon$ and sufficiently small $\varepsilon \in \mathbb{R}_{>0}$,

$$\left| \frac{e^{\frac{4r-1}{4r}x} e^{\frac{1}{4r}t}}{e^x + e^t - 1} \right| \tag{16}$$

can be bounded by a constant which does not depend on ε.

Note that, by putting $\sigma = \mathrm{id}, \mathbf{a}_r = \mathbf{b}_r = (1, \ldots, 1)$ in (15), we can obtain (10).

4 Values of $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$ for some particular case

In this section, we consider the values of $\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r)$ for some particular case. For values at non-positive integers, we have the following proposition.

Proposition 4.1. For $\mathbf{k}_r \in \mathbb{Z}^r, -\mathbf{n}_r = (-n_1, \ldots, -n_r) \in \mathbb{Z}_{\leq 0}^r, \sigma \in \mathfrak{S}_r, \mathbf{a}_r \in \mathbb{Z}_{\geq 0}^r$ with $a_1, a_{\sigma^{-1}(1)} \geq 1$ and $\mathbf{b}_r \in \mathbb{Z}_{\geq 0}^r$ with $b_1, b_{\sigma^{-1}(1)} \geq 1$ and $a_j + b_j \geq 1$, we have

$$\eta(\mathbf{k}_r; -\mathbf{n}_r; \sigma; \mathbf{a}_r; \mathbf{b}_r) = B_{\mathbf{n}_r}^{(\mathbf{k}_r)}(\sigma; \mathbf{a}_r; \mathbf{b}_r). \tag{17}$$

For the values at positive integers, we have the following proposition. For simplicity, we put

$$\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r; 1, \ldots, 1) = \eta^\star(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{a}_r)$$

for $\mathbf{a}_r \in \{0, 1\}^r$ with $a_1 = 1$, and

$$\eta(\mathbf{u}_r; \mathbf{s}_r; \sigma; 1, \ldots, 1; \mathbf{b}_r) = \eta^{\star\star}(\mathbf{u}_r; \mathbf{s}_r; \sigma; \mathbf{b}_r)$$

for $\mathbf{b}_r \in \{0, 1\}^r$ with $b_1 = 1$. Note that, with this notation, Theorem 15 can be written as

$$\eta^\star(\mathbf{k}_r; \mathbf{n}_r; \sigma; \mathbf{a}_r) = \eta^{\star\star}(\mathbf{n}_r; \mathbf{k}_r; \sigma^{-1}; \mathbf{a}_r) \tag{18}$$

with $\mathbf{a}_r \in \mathbb{Z}_{\geq 0}^r$ with $(a_1, a_{\sigma^{-1}(1)}) = (1, 1)$.

Theorem 4.2. For $\mathbf{k}_r, \mathbf{n}_r \in \mathbb{Z}_{\geq 1}^r, \sigma \in \mathfrak{S}_r, \mathbf{a}_r = \in \{0, 1\}^r$ with $(a_1, a_{\sigma^{-1}(1)}) = (1, 1)$, we have

$$\eta^\star(\mathbf{k}_r; \mathbf{n}_r; \sigma; \mathbf{a}_r), \eta^{\star\star}(\mathbf{k}_r; \mathbf{n}_r; \sigma; \mathbf{a}_r) \in \mathcal{Z}_{k_1 + \cdots + k_r + n_1 + \cdots + n_r}.$$

To show Theorem 4.2, we need the variable removing in Section 2.

Example 4.3. For the case $r=2, n_1=n_2=k_1=k_2=1, \sigma=\mathrm{id}, a_1=1, a_2=0$ in Theorem 15 (or (18)), we have

$$\eta^\star(1,1;1,1;\mathrm{id};1,0)=\eta^{\star\star}(1,1;1,1;\sigma^{-1};1,0).$$

Now we calculate these values explicitly by variable removing. By changing variable $1-e^{-t_j-\cdots-t_r}=x_j$, we have

$$\begin{aligned}
\eta^\star(1,1;1,1;\mathrm{id};1,0)&=\int_{(0,\infty)^2}\frac{\mathrm{Li}^{ш,\star}_{1,1}(1-e^{t_1+t_2},1-e^{t_2})}{1-e^{t_1+t_2}}dt_1dt_2\\
&=\int_{(0,\infty)^2}\frac{\mathrm{Li}^{ш}_{1,1}(1-e^{t_1+t_2},1-e^{t_2})}{1-e^{t_1+t_2}}dt_1dt_2+\int_{(0,\infty)^2}\frac{\mathrm{Li}_2(1-e^{t_1+t_2})}{1-e^{t_1+t_2}}dt_1dt_2\\
&=\int_{0<x_2<x_1<1}\frac{\mathrm{Li}^{ш}_{1,1}\left(\frac{1}{1-x_1^{-1}},\frac{1}{1-x_2^{-1}}\right)}{x_1(x_2-1)}dx_1dx_2+\int_{0<x_2<x_1<1}\frac{\mathrm{Li}_2\left(\frac{1}{1-x_1^{-1}}\right)}{x_1(x_2-1)}dx_1dx_2\\
&=\int_{0<x_2<x_1<1}\frac{I(0;x_2^{-1},x_1^{-1};1)}{x_1(x_2-1)}dx_1dx_2-\int_{0<x_2<x_1<1}\frac{I(0;x_1,1;x_1)}{x_1(x_2-1)}dx_1dx_2.
\end{aligned}$$

For $\mathrm{Li}^{ш}_{1,1}(x_2,x_1)=I(0;x_2^{-1},x_1^{-1};1)$, by variable removing, we obtain

$$\begin{aligned}
I(0;x_2^{-1},x_1^{-1};1)&=I(0;x_1;x_2)I(0;1;x_1)+I(0;1,0;x_2)-I(0;1,x_1;x_2),\\
I(0;1,x_1,1;x_1)&=I(0;1,1,1;x_1)-I(0;1,0,1;x_1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\eta^\star(1,1;1,1;\mathrm{id};1,0)=&\int_{0<x_2<x_1<1}\frac{I(0;x_1;x_2)I(0;1;x_1)+I(0;1,0;x_2)-I(0;1,x_1;x_2)}{x_1(x_2-1)}dx_1dx_2\\
&-\int_{0<x_2<x_1<1}\frac{I(0;x_1,1;x_1)}{x_1(x_2-1)}dx_1dx_2\\
&=\int_0^1\frac{I(0;x_1,1;x_1)I(0;1;x_1)+I(0;1,0,1;x_1)-I(0;1,x_1,1;x_1)}{x_1}dx_1\\
&-\int_0^1\frac{I(0;x_1,1;x_1)I(0;1;x_1)}{x_1}dx_1\\
&=\int_0^1\frac{I(0;1,0,1;x_1)-I(0;1,x_1,1;x_1)}{x_1}dx_1\\
&=\int_0^1\frac{2I(0;1,0,1;x_1)-I(0;1,1,1;x_1)}{x_1}dx_1\\
&=\zeta(1,1,2)+2\zeta(2,2).
\end{aligned}$$

For $\eta^{\star\star}(1,1;1,1;\mathrm{id}^{-1};1,0)$, a similar calculation shows

$$\eta^{\star\star}(1,1;1,1;\mathrm{id}^{-1};1,0)=2\zeta(1,1,2)+2\zeta(1,3).$$

Therefore, we obtain

$$\zeta(1,1,2)+2\zeta(2,2)=2\zeta(1,3)+2\zeta(1,1,2).$$

Example 4.4. For $r=2, k_1=k_2=1, n_1=1, n_2=2, \sigma=(1,2)\in\mathfrak{S}_2$ and $a_1=a_2=1$, we have

$$\begin{aligned}
\eta^\star(1,1;1,2;\sigma;1,1)&=-2\zeta(1,4)+\zeta(2,3)+4\zeta(1,1,3),\\
\eta^{\star\star}(1,2;1,1;\sigma^{-1};1,1)&=\zeta(1,2,2)+\zeta(2,3)+3\zeta(1,1,3)+4\zeta(1,4)-\zeta(3,2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&-2\zeta(1,4)+\zeta(2,3)+4\zeta(1,1,3)=\zeta(1,2,2)+\zeta(2,3)+3\zeta(1,1,3)+4\zeta(1,4)-\zeta(3,2)\\
&\to 6\zeta(1,4)-\zeta(3,2)-\zeta(1,1,3)+\zeta(1,2,2)=0.
\end{aligned}$$

Remark 4.5. In the same way, for the identity permutation id and $\sigma = (1,2)$, we have

$$\begin{aligned}
&\eta^\star(1;4;\mathrm{id};1) = \eta^{\star\star}(4;1;\mathrm{id};1)\ (\Leftrightarrow \eta(1;4) = \eta(4;1))\\
&\ \to \zeta(5) + \zeta(1,4) + \zeta(2,3) + \zeta(3,2) + \zeta(1,1,3) + \zeta(1,2,2) + \zeta(2,1,2) - 3\zeta(1,1,1,2) = 0,\\
&\eta^\star(2;3;\mathrm{id};1) = \eta^{\star\star}(3;2;\mathrm{id};1)\ (\Leftrightarrow \eta(2;3) = \eta(3;2))\\
&\ \to 2\zeta(1,4) + \zeta(2,3) + \zeta(3,2) + \zeta(1,2,2) + \zeta(2,1,2) - 2\zeta(1,1,1,2) = 0,\\
&\eta^\star(1,1;1,2;\mathrm{id};1,1) = \eta^{\star\star}(1,2;1,1;\mathrm{id};1,1)\ (\Leftrightarrow \eta(1,1;1,2) = \eta(1,2;1,1))\\
&\ \to 2\zeta(1,4) - \zeta(3,2) + 3\zeta(1,1,3) + \zeta(1,2,2) = 0,\\
&\eta^\star(1,1;2,1;\mathrm{id};1,1) = \eta^{\star\star}(2,1;1,1;\mathrm{id};1,1)\ (\Leftrightarrow \eta(1,1;2,1) = \eta(2,1;1,1))\\
&\ \to -2\zeta(1,4) - \zeta(2,3) - 3\zeta(1,1,3) + \zeta(2,1,2) = 0,\\
&\eta^\star(1,1;2,1;\sigma;1,1) = \eta^{\star\star}(2,1;1,1;\sigma^{-1};1,1)\\
&\ \to 4\zeta(1,4) - 2\zeta(2,3) - \zeta(3,2) + \zeta(1,1,3) + 3\zeta(1,2,2) = 0.
\end{aligned}$$

Combining these with Example 4.4, we obtain at most 6 linearly independent relations.

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